Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations

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Preface

To our wives, Masha and Marian

Interest to the so-called completely integrable systems with infinite number of degrees of freedom aroused immediately after publication of the famous series of papers by Gardner, Greene, Kruskal, Miura, and Zabusky [**75, 77, 96, 18, 66, 19**] (see also [**76**]) on striking properties of the Korteweg–de Vries (KdV) equation. It soon became clear that systems of such a kind possess a number of characteristic properties, such as infinite series of symmetries and/or conservation laws, inverse scattering problem formulation, $L-A$ pair representation, existence of prolongation structures, etc. And though no satisfactory definition of complete integrability was yet invented, a need of testing a particular system for these properties appeared.

Probably, one of the most efficient tests of this kind was first proposed by Lenard [**19**] who constructed a *recursion operator* for symmetries of the KdV equation. It was a strange operator, in a sense: being formally integro-differential, its action on the first classical symmetry (x-translation) is well-defined and produces the entire series of higher KdV equations. But applied to the scaling symmetry, it gave expressions containing terms of the type $\int u\,dx$ which had no adequate interpretation in the framework of the existing theories. And it is not surprising that P. Olver wrote "The deduction of the form of the recursion operator (if it exists) requires a certain amount of inspired guesswork..." [**80**, p. 315]: one can hardly expect efficient algorithms in the world of rather fuzzy definitions, if any.

In some sense, our book deals with the problem of how to construct a well-defined concept of a recursion operator and use this definition for particular computations. As it happened, a final solution can be explicated in the framework of the following conceptual scheme.

We start with a smooth manifold M (a space of independent variables) and a smooth locally trivial vector bundle $\pi\colon E \to M$ whose sections play the role of dependent variables (unknown functions). A partial differential equation in the bundle π is a smooth submanifold $\mathcal{E}$ in the space $J^k(\pi)$ of k-jets of π. Any such a submanifold is canonically endowed with a distribution, the Cartan distribution. Being in general nonintegrable, this distribution possesses different types of maximal integral manifolds a particular case of which are (generalized) solutions of $\mathcal{E}$. Thus we can define *geometry* of the equation $\mathcal{E}$ as geometry related to the corresponding Cartan distribution. Automorphisms of this geometry are classical symmetries of $\mathcal{E}$.

Dealing with geometry of differential equations in the above sense, one soon finds that a number of natural constructions arising in this context is in fact a finite part of more general objects existing on differential consequences of the initial equation. This leads to introduction of prolongations $\mathcal{E}^l$ of $\mathcal{E}$ and, in the limit, of the *infinite prolongation* $\mathcal{E}^\infty$ as a submanifold of the manifold $J^\infty(\pi)$ of infinite jets. Using algebraic language mainly, all finite-dimensional constructions are carried over both to $J^\infty(\pi)$ and $\mathcal{E}^\infty$ and, surprisingly at first glance, become there even more simple and elegant. In particular, the Cartan distribution on $\mathcal{E}^\infty$ becomes completely integrable (i.e., satisfies the conditions of the Frobenius theorem). Nontrivial symmetries of this distribution are called higher symmetries of $\mathcal{E}$.

Moreover, the Cartan distribution on $\mathcal{E}^\infty$ is in fact the horizontal distribution of a certain flat connection $\mathcal{C}$ in the bundle $\mathcal{E}^\infty \to M$ (the Cartan connection) and the connection form of $\mathcal{C}$ contains all vital geometrical information about the equation $\mathcal{E}$. We call this form the *structural element* of $\mathcal{E}$ and it is a form-valued derivation of the smooth function algebra on $\mathcal{E}^\infty$. A natural thing to ask is what are deformations of the structural element (or, of the equation structure on $\mathcal{E}$). At least two interesting things are found when one answers this question.

The first one is that the deformation theory of equation structures is closely related to a cohomological theory based on the *Frölicher–Nijenhuis bracket* construction in the module of form-valued derivations. Namely, if we denote by $D_1\Lambda^i(\mathcal{E})$ the module of derivations with values in i-forms, the Frölicher–Nijenhuis bracket acts in the following way:

$$[\![\cdot,\cdot]\!]^{\mathrm{fn}}\colon D_1\Lambda^i(\mathcal{E}) \times D_1\Lambda^j(\mathcal{E}) \to D_1\Lambda^{i+j}(\mathcal{E}).$$

In particular, for any element $\Omega \in D_1\Lambda^1(\mathcal{E})$ we obtain an operator

$$\partial_\Omega\colon D_1\Lambda^i(\mathcal{E}) \to D_1\Lambda^{i+1}(\mathcal{E})$$

defined by the formula $\partial_\Omega(\Theta) = [\![\Omega,\Theta]\!]^{\mathrm{fn}}$ for any $\Theta \in D_1\Lambda^i(\mathcal{E})$. Since $D_1\Lambda^*(\mathcal{E}) = \bigoplus_{i=1}^\infty D_1\Lambda^i(\mathcal{E})$ is a graded Lie algebra with respect to the Frölicher–Nijenhuis bracket and due to the graded Jacobi identity, one can see that the equality $\partial_\Omega \circ \partial_\Omega = 0$ is equivalent to $[\![\Omega,\Omega]\!]^{\mathrm{fn}} = 0$. The last equality holds, if Ω is a connection form of a flat connection. Thus, any flat connection generates a cohomology theory. In particular, natural cohomology groups are related to the Cartan connection and we call them *$\mathcal{C}$-cohomology* and denote by $H^i_{\mathcal{C}}(\mathcal{E})$.

We restrict ourselves to the vertical subtheory of this cohomological theory. Within this restriction, it can be proved that the group $H^0_{\mathcal{C}}(\mathcal{E})$ coincides with the Lie algebra of higher symmetries of the equation $\mathcal{E}$ while $H^1_{\mathcal{C}}(\mathcal{E})$ consists of the equivalence classes of infinitesimal deformations of the equation structure on $\mathcal{E}$. It is also a common fact in cohomological deformation theory [**20**] that the group $H^2_{\mathcal{C}}(\mathcal{E})$ contains obstructions to continuation of infinitesimal deformations up to formal ones. For partial differential equations, triviality of this group is, roughly speaking, the reason for existence of commuting series of higher symmetries.

The second interesting and even more important thing in our context is that the contraction operation defined in $D_1\Lambda^*(\mathcal{E})$ is inherited by the groups $H^i_{\mathcal{C}}(\mathcal{E})$. In particular, the group $H^1_{\mathcal{C}}(\mathcal{E})$ is an associative algebra with respect to this operation while contraction with elements of $H^0_{\mathcal{C}}(\mathcal{E})$ is a representation of this algebra. In effect, having a nontrivial element $\mathcal{R} \in H^1_{\mathcal{C}}(\mathcal{E})$ and a symmetry $s_0 \in H^0_{\mathcal{C}}(\mathcal{E})$ we are able to obtain a whole infinite series $s_n = \mathcal{R}^n s_0$ of new higher symmetries. This is just what is expected of recursion operators!

Unfortunately (or, perhaps, luckily) a straightforward computation of the first $\mathcal{C}$-cohomology groups for known completely integrable equations (the KdV equation, for example) leads to trivial results only, which is not surprising at all. In fact, normally recursion operators for nonlinear integrable systems contain integral (nonlocal) terms which cannot appear when one works using the language of infinite jets and infinite prolongations only. The setting can be extended by introduction of new entities — nonlocal variables. Geometrically, this is being done by means of the concept of a *covering*. A covering over $\mathcal{E}^\infty$ is a fiber bundle $\tau\colon W \to \mathcal{E}^\infty$ such that the total space W is endowed with and integrable distribution $\tilde{\mathcal{C}}$ and the differential τ_* isomorphically projects any plane of the distribution $\tilde{\mathcal{C}}$ to the corresponding plane of the Cartan distribution $\mathcal{C}$ on $\mathcal{E}^\infty$. Coordinates along the fibers of τ depend on coordinates in $\mathcal{E}^\infty$ in an integro-differential way and are called nonlocal.

Geometry of coverings is described in the same terms as geometry of infinite prolongations, and we can introduce the notions of symmetries of W (called *nonlocal* symmetries of $\mathcal{E}$), the structural element, $\mathcal{C}$-cohomology, etc. For a given equation $\mathcal{E}$, we can choose an appropriate covering and may be lucky to extend the group $H^1_{\mathcal{C}}(\mathcal{E})$. For example, for the KdV equation it suffices to add the nonlocal variable $u_{-1} = \int u\,dx$, where u is the unknown function, and to obtain the classical Lenard recursion operator as an element of the extended $\mathcal{C}$-cohomology group. The same effect one sees for the Burgers equation. For other integrable systems such coverings may be (and usually are) more complicated.

To finish this short review, let us make some comments on how recursion operators can be efficiently computed. To this end, note that the module $D(\mathcal{E})$ of vector fields on $\mathcal{E}^\infty$ splits into the direct sum $D(\mathcal{E}) = D^v(\mathcal{E}) \oplus \mathcal{C}D(\mathcal{E})$, where $D^v(\mathcal{E})$ are π-vertical fields and $\mathcal{C}D(\mathcal{E})$ consists of vector fields lying in the Cartan distribution. This splitting induces the dual one: $\Lambda(\mathcal{E}) = \Lambda^1_h(\mathcal{E}) \oplus \mathcal{C}\Lambda^1(\mathcal{E})$. Elements of $\Lambda^1_h(\mathcal{E})$ are called horizontal forms while elements of $\mathcal{C}\Lambda^1(\mathcal{E})$ are called Cartan forms (they vanish on the Cartan distribution). By consequence, we have the splitting $\Lambda^i(\mathcal{E}) = \bigoplus_{p+q=i} \mathcal{C}^p\Lambda(\mathcal{E}) \otimes \Lambda^q(\mathcal{E})$, where

$$\mathcal{C}^p\Lambda(\mathcal{E}) = \underbrace{\mathcal{C}\Lambda^1(\mathcal{E}) \wedge \dots \wedge \mathcal{C}\Lambda^1(\mathcal{E})}_{p \text{ times}}, \quad \Lambda_h^q(\mathcal{E}) = \underbrace{\Lambda_h^1(\mathcal{E}) \wedge \dots \wedge \Lambda_h^1(\mathcal{E})}_{q \text{ times}}.$$

This splitting generates the corresponding splitting in the groups of $\mathcal{C}$-cohomologies: $H_{\mathcal{C}}^i(\mathcal{E}) = \bigoplus_{p+q=i} H_{\mathcal{C}}^{p,q}(\mathcal{E})$ and nontrivial recursion operators are elements of the group $H_{\mathcal{C}}^{1,0}(\mathcal{E})$.

The graded algebra $\mathcal{C}^*\Lambda(\mathcal{E}) = \bigoplus_{p\geq 0} \mathcal{C}^p\Lambda(\mathcal{E})$ may be considered as the algebra of functions on a super differential equation related to the initial equation $\mathcal{E}$ in a functorial way. This equation is called the *Cartan* (*odd*) *covering* of $\mathcal{E}$. An amazing fact is that the symmetry algebra of this covering is isomorphic to the direct sum $H_{\mathcal{C}}^{*,0}(\mathcal{E}) \oplus H_{\mathcal{C}}^{*,0}(\mathcal{E})$. Thus, due to the general theory, to find an element of $H_{\mathcal{C}}^{p,0}(\mathcal{E})$ we have just to take a system of forms $\Omega = (\omega^1, \dots, \omega^m)$, where $\omega^j \in \mathcal{C}^p\Lambda(\mathcal{E})$ and $m = \dim \pi$, and to solve the equation $\ell_{\mathcal{E}}\omega = 0$, where $\ell_{\mathcal{E}}$ is the linearization of $\mathcal{E}$ restricted to $\mathcal{E}^\infty$. In particular, for $p = 1$ we shall obtain recursion operators, and the action of the corresponding solutions on symmetries of $\mathcal{E}$ is just contraction of a symmetry with the Cartan vector-form Ω.

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This scheme is exposed in details below. Though some topics can be found in other books (see, e.g., [**60, 12, 80, 5, 81, 101**]; the collections [**39**] and [**103**] also may be recommended), we included them in the text to make the book self-contained. We also decided to include a lot of applications in the text to make it interesting not only to those ones who deal with pure theory.

The material of the book is arranged as follows.

In Chapter 1 we deal with spaces of finite jets and partial differential equations as their submanifold. The Cartan distribution on $J^k(\pi)$ is introduced and it maximal integral manifolds are described. We describe automorphisms of this distribution (Lie–Bäcklund transformations) and derive defining relations for classical symmetries. As applications, we consider classical symmetries of the Burgers equation, of the nonlinear diffusion equation (and obtain the so-called group classification in this case), of the nonlinear Dirac equation, and of the self-dual Yang–Mills equations. For the latter, we get monopole and instanton solutions as invariant solutions with respect to the symmetries obtained.

Chapter 2 is dedicated to higher symmetries and conservation laws. Basic structures on infinite prolongations are described, including the Cartan connection and the structural element of a nonlinear equation. In the context of conservation laws, we briefly expose the results of A. Vinogradov on the $\mathcal{C}$-spectral sequence [**102**]. We give here a complete description for higher symmetries of the Burgers equation, the Hilbert–Cartan equation, and the classical Boussinesq equation.

In Chapter 3 we describe the nonlocal theory. The notion of a covering is introduced, the relation between coverings and conservation laws is discussed. We reproduce here quite important results by N. Khor′kova [**43**] on the reconstruction of nonlocal symmetries by their shadows. Several applications are considered in this chapter: nonlocal symmetries of the Burgers and KdV equation, symmetries of the massive Thirring model and symmetries of the Federbush model. In the last case, we also discuss Hamiltonian structures for this model and demonstrate the existence of infinite number of hierarchies of symmetries. We finish this chapter with an interpretation of Bäcklund transformations in terms of coverings and discuss a definition of recursion operators as Bäcklund transformations belonging to M. Marvan [**73**].

Chapter 4 starts the central topic of the book: algebraic calculus of form-valued derivations. After introduction of some general concepts (linear differential operators over commutative algebras, algebraic jets and differential forms), we define basic constructions of Frölicher–Nijenhuis and Richardson–Nijenhuis brackets [**17, 78**] and analyze their properties. We show that to any integrable derivation X with values in one-forms, i.e., satisfying the condition $[\![X, X]\!]^{\mathrm{fn}} = 0$, a complex can be associated and investigate main properties of the corresponding cohomology group. A source of examples for integrable elements is provided by *algebras with flat connections*. These algebras can be considered as a model for infinitely prolonged differential equation. Within this model, we introduce algebraic counterparts for the notions of a symmetry and a recursion operator and prove some results describing the symmetry algebra structure in the case when the second cohomology group vanishes. In particular, we show that in this case infinite series of commuting symmetries arise provided the model possesses a nontrivial recursion operator.

Chapter 5 can be considered as a specification of the results obtained in Chapter 4 to the case of partial differential equations, i.e., the algebra in question is the smooth function algebra on $\mathcal{E}^\infty$ while the flat connection is the Cartan connection. The cohomology groups arising in this case are $\mathcal{C}$-cohomology of $\mathcal{E}$. Using spectral sequence techniques, we give a complete description of the $\mathcal{C}$-cohomology for the "empty" equation, that is for the spaces $J^\infty(\pi)$ and show that elements of the corresponding cohomology groups can be understood as graded evolutionary derivations (or vector fields) on $J^\infty(\pi)$. We also establish relations between $\mathcal{C}$-cohomology and deformations of the equation structure and show that infinitesimal deformations of a certain kind (elements of $H^{1,0}_{\mathcal{C}}(\mathcal{E})$, see above) are identified with recursion operators for symmetries. After deriving defining equations for these operators, we demonstrate that in the case of several classical systems (the Burgers equation, KdV, the nonlinear Schrödinger and Boussinesq equations) the results obtained coincide with the well-known recursion operators. We also investigate the equation of isometric immersions of two-dimensional Riemannian surfaces into $\mathbb{R}^3$ (a particular case of the Gauss–Mainardi–Codazzi equations, which we call the Sym equation) and prove its complete integrability, i.e., construct a recursion operator and infinite series of symmetries.

Chapter 6 is a generalization of the preceding material to the graded case (or, in physical terms, to the supersymmetric case). We redefine all necessary algebraic construction for graded commutative algebras and introduce the notion of a *graded extension* of a partial differential equation. It is shown that all geometrical constructions valid for classical equations can be applied, with natural modifications, to graded extensions as well. We describe an approach to the construction of graded extensions and consider several illustrative examples (graded extensions of the KdV and modified KdV equations and supersymmetric extensions of the nonlinear Schrödinger equation).

Chapter 7 continues the topics started in the preceding chapter. We consider here two supersymmetric extensions of the KdV equations (one- and two-dimensional), new extensions of the nonlinear Schrödinger equation, and the supersymmetric Boussinesq equation. In all applications, recursion operators are constructed and new infinite series of symmetries, both local and nonlocal, are described.

Finally, in Chapter 8 we briefly describe the software used for computations described in the book and without which no serious application could be obtained.

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CHAPTER 1

Classical symmetries

This chapter is concerned with the basic notions needed for our exposition — those of jet spaces and of nonlinear differential equations. Our main purpose is to put the concept of a nonlinear partial differential equation (PDE) into the framework of smooth manifolds and then to apply powerful techniques of differential geometry and commutative algebra. We completely abandon analytical language, maybe good enough for theorems of existence, but not too useful in search for main underlying structures.

We describe the geometry of jet spaces and differential equations (its geometry is determined by the *Cartan distribution*) and introduce *classical symmetries* of PDE. Our exposition is based on the books [**60, 12**]. We also discuss several examples of symmetry computations for some equations of mathematical physics.

1. Jet spaces

We expose here main facts concerning the geometrical approach to jets (finite and infinite) and to nonlinear differential operators.

1.1. Finite jets. Traditional approach to differential equations consists in treating them as expressions of the form

$$F\big(x_1, \dots, x_n, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \dots\big) = 0, \tag{1.1}$$

where $x_1, \dots, x_n$ are *independent variables*, while $u = u(x_1, \dots, x_n)$ is an unknown function (*dependent variable*). Such an equation is called *scalar*, but one can consider equations of the form (1.1) with $F = (F^1, \dots, F^r)$ and $u = (u^1, \dots, u^m)$ being vector-functions. Then we speak of *systems* of PDE. What makes expression (1.1) a differential equation is the presence of partial derivatives $\partial u/\partial x_1, \dots$ in it, and our first step is to clarify this fact in geometrical terms.

To do it, we shall restrict ourselves to the situation when all functions are *smooth* (i.e., of the C^∞-class) and note that a vector-function $u = (u^1, \dots, u^m)$ can be considered as a section of the trivial bundle $\mathbf{1}_n^m\colon \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{n+m} \to \mathbb{R}^n$. Denote $\mathbb{R}^m \times \mathbb{R}^n$ by $J^0(n,m)$ and consider the graph of this section, i.e., the set $\Gamma_u \subset J^0(n,m)$ consisting of the points

$$\big\{(x_1, \dots, x_n, u^1(x_1, \dots, x_n), \dots, u^m(x_1, \dots, x_n)\big\},$$

which is an n-dimensional submanifold in $\mathbb{R}^{n+m}$.

Let $x = (x_1, \dots, x_n)$ be a point of $\mathbb{R}^n$ and $\theta = (x, u(x))$ be the corresponding point lying on Γ_u. Then the tangent plane to Γ_u passing through the point θ is completely determined by x and by partial derivatives of u at the point x. It is easy to see that the set of such planes forms an mn-dimensional space $\mathbb{R}^{mn}$ with coordinates, say, u_i^j, $i = 1, \dots, n$, $j = 1, \dots, m$, where u_i^j "corresponds" to the partial derivative of the function u^j with respect to x_i at x.

Maintaining this construction at every point $\theta \in J^0(n, m)$, we obtain the bundle $J^1(n, m) \stackrel{\text{def}}{=} \mathbb{R}^{mn} \times J^0(n, m) \to J^0(n, m)$. Consider a point $\theta_1 \in J^1(n, m)$. By doing this, we, in fact, fix the following data: values of independent variables, x, values of dependent ones, u^j, and values of all their partial derivatives at x. Assume now that a smooth submanifold $\mathcal{E} \subset J^1(n, m)$ is given. This submanifold determines "relations between points" of $J^1(n, m)$. Taking into account the above given interpretation of these points, we see that $\mathcal{E}$ may be understood as a system of relations on unknowns u^j and their partial derivatives. Thus, $\mathcal{E}$ is a first-order differential equation! (Or a system of such equations.)

With this example at hand, we pass now to a general construction.

Let M be an n-dimensional smooth manifold and $\pi \colon E \to M$ be a smooth m-dimensional vector bundle[1] over M. Denote by $\Gamma(\pi)$ the $C^\infty(M)$-module of sections of the bundle π. For any point $x \in M$ we shall also consider the module $\Gamma_{\mathrm{loc}}(\pi; x)$ of all *local* sections at x.

REMARK 1.1. We say that φ is a local section of π at x, if it is defined on a neighborhood $\mathcal{U}$ of x (the domain of φ). To be exact, φ is a section of the pull-back $\epsilon^*\pi = \pi\,|_{\mathcal{U}}$, where $\epsilon \colon \mathcal{U} \hookrightarrow M$ is the natural embedding. If $\varphi, \varphi' \in \Gamma_{\mathrm{loc}}(\pi; x)$ are two local sections with the domains $\mathcal{U}$ and $\mathcal{U}'$ respectively, then their sum $\varphi + \varphi'$ is defined over $\mathcal{U} \cap \mathcal{U}'$. For any function $f \in C^\infty(M)$ we can also define the local section $f\varphi$ over $\mathcal{U}$.

For a section $\varphi \in \Gamma_{\mathrm{loc}}(\pi; x)$, $\varphi(x) = \theta \in E$, consider its graph $\Gamma_\varphi \subset E$ and all sections $\varphi' \in \Gamma_{\mathrm{loc}}(\pi; x)$ such that

(a) $\varphi(x) = \varphi'(x)$;
(b) the graph $\Gamma_{\varphi'}$ is tangent to Γ_φ with order k at θ.

It is easy to see that conditions (a) and (b) determine an equivalence relation $\sim_x^k$ on $\Gamma_{\mathrm{loc}}(\pi; x)$ and we denote the equivalence class of φ by $[\varphi]_x^k$. The quotient set $\Gamma_{\mathrm{loc}}(\pi; x)/\sim_x^k$ becomes an $\mathbb{R}$-vector space, if we put

$$[\varphi]_x^k + [\psi]_x^k = [\varphi + \psi]_x^k,\ a[\varphi]_x^k = [a\varphi]_x^k, \quad \varphi, \psi \in \Gamma_{\mathrm{loc}}(\pi; x),\ a \in \mathbb{R}, \qquad (1.2)$$

[1] In fact, all constructions below can be carried out — with natural modifications — for an arbitrary locally trivial bundle π (and even in more general settings). But we restrict ourselves to the vector case for clearness of exposition.

while the natural projection $\Gamma_{\mathrm{loc}}(\pi;x) \to \Gamma_{\mathrm{loc}}(\pi;x)/\sim_x^k$ becomes a linear map. We denote this space by $J_x^k(\pi)$. Obviously, $J_x^0(\pi)$ coincides with $E_x = \pi^{-1}(x)$, the fiber of the bundle π over the point $x \in M$.

REMARK 1.2. The tangency class $[\varphi]_x^k$ is completely determined by the point x and partial derivatives up to order k at x of the section φ. From here it follows that $J_x^k(\pi)$ is finite-dimensional. It is easy to compute the dimension of this space: the number of different partial derivatives of order i equals $\binom{n+i-1}{n-1}$ and thus

$$\dim J_x^k(\pi) = m \sum_{i=0}^{k} \binom{n+i-1}{n-1} = m\binom{n+k}{k}. \tag{1.3}$$

DEFINITION 1.1. The element $[\varphi]_x^k \in J_x^k(\pi)$ is called the *k-jet* of the section $\varphi \in \Gamma_{\mathrm{loc}}(\pi;x)$ at the point x.

The k-jet of φ can be identified with the k-th order Taylor expansion of the section φ. From the definition it follows that it is independent of coordinate choice (in contrast to the notion of partial derivative, which depends on local coordinates).

Let us consider now the set

$$J^k(\pi) = \bigcup_{x\in M} J_x^k(\pi) \tag{1.4}$$

and introduce a smooth manifold structure on $J^k(\pi)$ in the following way. Let $\{\mathcal{U}_\alpha\}_\alpha$ be an atlas in M such that the bundle π becomes trivial over each $\mathcal{U}_\alpha$, i.e., $\pi^{-1}(\mathcal{U}_\alpha) \simeq \mathcal{U}_\alpha \times V$, where V is the "typical fiber". Choose a basis $e_1^\alpha, \dots, e_m^\alpha$ of local sections of π over $\mathcal{U}_\alpha$. Then any section of $\pi\,|_{\mathcal{U}_\alpha}$ is representable in the form $\varphi = u^1 e_1^\alpha + \dots + u^m e_m^\alpha$ and the functions $x_1, \dots, x_n$, $u^1, \dots, u^m$, where $x_1, \dots, x_n$ are local coordinates in $\mathcal{U}_\alpha$, constitute a local coordinate system in $\pi^{-1}(\mathcal{U}_\alpha)$. Let us define the functions $u_\sigma^m \colon \bigcup_{x\in\mathcal{U}_\alpha} J_x^k(\pi) \to \mathbb{R}$, where $\sigma = (\sigma_1, \dots \sigma_n)$, $|\sigma| = \sigma_1 + \dots + \sigma_n \le k$, by

$$u_\sigma^j\big([\varphi]_x^k\big) \stackrel{\text{def}}{=} \left.\frac{\partial^{|\sigma|} u^j}{\partial x_\sigma}\right|_x, \tag{1.5}$$

$\partial x_\sigma \stackrel{\text{def}}{=} (\partial x_1)^{\sigma_1} \dots (\partial x_n)^{\sigma_n}$. Then these functions, together with local coordinates $x_1, \dots, x_n$, define the mapping $f_\alpha \colon \bigcup_{x\in\mathcal{U}_\alpha} J_x^k(\pi) \to \mathcal{U}_\alpha \times \mathbb{R}^N$, where N is the number defined by (1.3). Due to computation rules for partial derivatives under coordinate transformations, the mapping

$$(f_\alpha \circ f_\beta^{-1})\,|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^N \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^N$$

is a diffeomorphism preserving the natural projection $(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^n \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Thus we have proved the following result:

PROPOSITION 1.1. *The set $J^k(\pi)$ defined by* (1.4) *is a smooth manifold while the projection $\pi_k \colon J^k(\pi) \to M$, $\pi_k \colon [\varphi]_x^k \mapsto x$, is a smooth vector bundle.*

Note that linear structure in the fibers of π_k is given by (1.2).

DEFINITION 1.2. Let $\pi\colon E \to M$ be a smooth vector bundle, $\dim M = n$, $\dim E = n + m$.

(i) The manifold $J^k(\pi)$ is called the *manifold of k-jets* for π;
(ii) The bundle $\pi_k\colon J^k(\pi) \to M$ is called the *bundle of k-jets* for π;
(iii) The above constructed coordinates $\{x_i, u_\sigma^j\}$, where $i = 1, \dots, n$, $j = 1, \dots, m$, $|\sigma| \leq k$, are called the *special* (or *adapted*) *coordinate system* on $J^k(\pi)$ associated to the trivialization $\{\mathcal{U}_\alpha\}_\alpha$ of the bundle π.

Obviously, the bundle π_0 coincides with π.

Note that tangency of two manifolds with order k implies tangency with less order, i.e., there exists a mapping $\pi_{k,l}\colon J^k(\pi) \to J^l(\pi)$, $[\varphi]_x^k \mapsto [\varphi]_x^l$, $k \geq l$. From this remark and from the definitions we obtain the commutative diagram

$$\begin{tikzcd}
J^k(\pi) \arrow[rr, "\pi_{k,l}"] \arrow[dr, "\pi_{k,s}"] \arrow[ddr, "\pi_k"'] & & J^l(\pi) \arrow[dl, "\pi_{l,s}"'] \arrow[ddl, "\pi_l"] \\
& J^s(\pi) \arrow[d, "\pi_s"] & \\
& M &
\end{tikzcd}$$

where $k \geq l \geq s$ and all arrows are smooth fiber bundles. In other words, we have

$$\pi_{l,s} \circ \pi_{k,l} = \pi_{k,s}, \quad \pi_l \circ \pi_{k,l} = \pi_k, \qquad k \geq l \geq s. \tag{1.6}$$

On the other hand, for any section $\varphi \in \Gamma(\pi)$ (or $\in \Gamma_{\mathrm{loc}}(\pi; x)$) we can define the mapping $j_k(\varphi)\colon M \to J^k(\pi)$ by setting $j_k(\varphi)\colon x \mapsto [\varphi]_x^k$. Obviously, $j_k(\varphi) \in \Gamma(\pi_k)$ (respectively, $j_k(\varphi) \in \Gamma_{\mathrm{loc}}(\pi_k; x)$).

DEFINITION 1.3. The section $j_k(\varphi)$ is called the *k-jet of the section* φ. The correspondence $j_k\colon \Gamma(\pi) \to \Gamma(\pi_k)$ is called the *operator of k-jet*.

From the definition it follows that

$$\pi_{k,l} \circ j_k(\varphi) = j_l(\varphi), \quad j_0(\varphi) = \varphi, \qquad k \geq l, \tag{1.7}$$

for any $\varphi \in \Gamma(\pi)$.

Let $\varphi, \psi \in \Gamma(\pi)$ be two sections, $x \in M$ and $\varphi(x) = \psi(x) = \theta \in E$. It is a tautology to say that the manifolds Γ_φ and Γ_ψ are tangent to each other with order $k+l$ at θ or that the manifolds $\Gamma_{j_k(\varphi)}, \Gamma_{j_k(\psi)} \subset J^k(\pi)$ are tangent with order l at the point $\theta_k = j_k(\varphi)(x) = j_k(\psi)(x)$.

DEFINITION 1.4. Let $\theta_k \in J^k(\pi)$. An *R-plane* at θ_k is an n-dimensional plane tangent to some manifold of the form $\Gamma_{j_k(\varphi)}$ such that $[\varphi]_x^k = \theta_k$.

Immediately from definitions we obtain the following result.

PROPOSITION 1.2. *Let $\theta_k \in J^k(\pi)$ be a point in a jet space. Then the fiber of the bundle $\pi_{k+1,k}\colon J^{k+1}(\pi) \to J^k(\pi)$ over θ_k coincides with the set of all R-planes at θ_k.*

For a point $\theta_{k+1} \in J^{k+1}(\pi)$, we shall denote the corresponding R-plane at $\theta_k = \pi_{k+1,k}(\theta_{k+1})$ by $L_{\theta_{k+1}} \subset T_{\theta_k}(J^k(\pi))$.

1.2. Nonlinear differential operators. Since $J^k(\pi)$ is a smooth manifold, we can consider the algebra of smooth functions on $J^k(\pi)$. Denote this algebra by $\mathcal{F}_k(\pi)$. Take another vector bundle $\pi'\colon E' \to M$ and consider the pull-back $\pi_k^*(\pi')$. Then the set of sections of $\pi_k^*(\pi')$ is a module over $\mathcal{F}_k(\pi)$ and we denote this module by $\mathcal{F}_k(\pi, \pi')$. In particular, $\mathcal{F}_k(\pi) = \mathcal{F}_k(\pi, \mathbf{1}_M)$, where $\mathbf{1}_M$ is the trivial one-dimensional bundle over M.

The surjections $\pi_{k,l}$ and π_k generate the natural embeddings $\nu_{k,l} \overset{\text{def}}{=} \pi_{k,l}^*\colon \mathcal{F}_l(\pi, \pi') \to \mathcal{F}_k(\pi, \pi')$ and $\nu_k \overset{\text{def}}{=} \pi_k^*\colon \Gamma(\pi') \to \mathcal{F}_k(\pi, \pi')$. Due to (1.6), we have the equalities

$$\nu_{k,l} \circ \nu_{l,s} = \nu_{k,s}, \quad \nu_{k,l} \circ \nu_l = \nu_k, \qquad k \geq l \geq s. \tag{1.8}$$

Identifying $\mathcal{F}_l(\pi, \pi')$ with its image in $\mathcal{F}_k(\pi, \pi')$ under $\nu_{k,l}$, we can consider $\mathcal{F}_k(\pi, \pi')$ as a filtered module,

$$\Gamma(\pi') \hookrightarrow \mathcal{F}_0(\pi, \pi') \hookrightarrow \ldots \hookrightarrow \mathcal{F}_{k-1}(\pi, \pi') \hookrightarrow \mathcal{F}_k(\pi, \pi'), \tag{1.9}$$

over the filtered algebra

$$C^\infty(M) \hookrightarrow \mathcal{F}_0(\pi) \hookrightarrow \ldots \hookrightarrow \mathcal{F}_{k-1}(\pi) \hookrightarrow \mathcal{F}_k(\pi). \tag{1.10}$$

Let $F \in \mathcal{F}_k(\pi, \pi')$. Then we have the correspondence

$$\Delta = \Delta_F\colon \Gamma(\pi) \to \Gamma(\pi'), \quad \Delta(\varphi) \overset{\text{def}}{=} j_k(\varphi)^*(F), \qquad \varphi \in \Gamma(\pi). \tag{1.11}$$

DEFINITION 1.5. A correspondence Δ of the form (1.11) is called a (*nonlinear*) *differential operator* of order[2] $\leq k$ acting from the bundle π to the bundle π'. In particular, when $\Delta(f\varphi + g\psi) = f\Delta(\varphi) + g\Delta(\psi)$ for all $\varphi, \psi \in \Gamma(\pi)$ and $f, g \in C^\infty(M)$, the operator Δ is said to be *linear*.

From (1.9) it follows that operators Δ of order k are also operators of all orders $k' \geq k$, while (1.8) shows that the action of Δ does not depend on the order assigned to this operator.

EXAMPLE 1.1. Let us show that the k-jet operator $j_k\colon \Gamma(\pi) \to \Gamma(\pi_k)$ (see Definition 1.3) is differential. To do this, recall that the total space of the pull-back $\pi_k^*(\pi_k)$ consists of points $(\theta_k, \theta_k') \in J^k(\pi) \times J^k(\pi)$ such that

[2]For the sake of briefness, we shall use the words *operator of order* k below as a synonym of the expression *operator of order* $\leq k$.

$\pi_k(\theta_k) = \pi_k(\theta'_k)$. Consequently, we may define the *diagonal section* ρ_k of the bundle $\pi_k^*(\pi_k)$ by setting $\rho_k(\theta_k) \stackrel{\text{def}}{=} (\theta_k, \theta_k)$. Obviously, $j_k = \Delta_{\rho_k}$, i.e.,

$$j_k(\varphi)^*(\rho_k) = j_k(\varphi), \qquad \varphi \in \Gamma(\pi).$$

The operator j_k is linear.

EXAMPLE 1.2. Let $\tau^*\colon T^*M \to M$ be the cotangent bundle of M and $\tau_p^*\colon \bigwedge^p T^*M \to M$ be its p-th external power. Then the *de Rham differential* d is a first order linear differential operator acting from τ_p^* to τ_{p+1}^*, $p \geq 0$.

EXAMPLE 1.3. Consider a pseudo-Riemannian manifold M with a nondegenerate metric $g \in \Gamma(S^2\tau^*)$ (by $S^q\xi$ we denote the q-th symmetric power of the vector bundle ξ). Let $g^* \in \Gamma(S^2\tau)$ be its dual, $\tau\colon TM \to M$ being the tangent bundle. Then the correspondence $\Delta_g\colon f \mapsto g^*(df, df)$ is a (nonlinear) first order differential operator from $C^\infty(M)$ to $C^\infty(M)$.

Let $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ and $\Delta'\colon \Gamma(\pi') \to \Gamma(\pi'')$ be two differential operators. It is natural to expect that their composition $\Delta' \circ \Delta\colon \Gamma(\pi) \to \Gamma(\pi'')$ is a differential operator as well. However to prove this fact is not quite simple. To do it, we need two new and important constructions.

Let $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ be a differential operator of order k. For any $\theta_k = [\varphi]_x^k \in J^k(\pi)$, let us set

$$\Phi_\Delta(\theta_k) \stackrel{\text{def}}{=} [\Delta(\varphi)]_x^0 = (\Delta(\varphi))(x). \tag{1.12}$$

Evidently, the mapping Φ_Δ is a morphism of fiber bundles[3] , i.e., the diagram

$$\begin{tikzcd} J^k(\pi) \arrow[rr, "\Phi_\Delta"] \arrow[dr, "\pi_k"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

is commutative.

DEFINITION 1.6. The map Φ_Δ is called the *representative morphism* of the operator Δ.

For example, for $\Delta = j_k$ we have $\Phi_{j_k} = \mathrm{id}_{J^k(\pi)}$. Note that there exists a one-to-one correspondence between nonlinear differential operators and their representative morphisms: one can easily see it just by inverting equality (1.12). In fact, if $\Phi\colon J^k(\pi) \to E'$ is a morphism of the bundle π to π', a section $\varphi \in \mathcal{F}(\pi, \pi')$ can be defined by setting $\varphi(\theta_k) = (\theta_k, \Phi(\theta_k)) \in J^k(\pi) \times E'$. Then, obviously, Φ is the representative morphism for $\Delta = \Delta_\varphi$.

DEFINITION 1.7. Let $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ be a k-th order differential operator. Its *l-th prolongation* is the composition $\Delta^{(l)} \stackrel{\text{def}}{=} j_l \circ \Delta\colon \Gamma(\pi) \to \Gamma(\pi_l)$.

[3]But not of vector bundles!

LEMMA 1.3. *For any k-th order differential operator Δ, its l-th prolongation is a $(k+l)$-th order operator.*

PROOF. In fact, for any point $\theta_{k+l} = [\varphi]_x^{k+l} \in J^{k+l}(\pi)$ let us set $\Phi_\Delta^{(l)} \overset{\text{def}}{=} [\Delta(\varphi)]_x^l \in J^l(\pi)$. Then the operator $\square$, for which the morphism $\Phi_\Delta^{(l)}$ is representative, coincides with $\Delta^{(l)}$. $\square$

COROLLARY 1.4. *The composition $\Delta' \circ \Delta$ of two differential operators $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ and $\Delta'\colon \Gamma(\pi') \to \Gamma(\pi'')$ of order k and k' respectively is a $(k+k')$-th order differential operator.*

PROOF. Let $\Phi_\Delta^{(k')}\colon J^{k+k'}(\pi) \to J^{k'}(\pi')$ be the representative morphism for $\Delta^{(k')}$. Then the operator $\square$, for which the composition $\Phi_{\Delta'} \circ \Phi_\Delta^{(k')}$ is the representative morphism, coincides with $\Delta' \circ \Delta$. $\square$

To finish this subsection, we shall list main properties of prolongations and representative morphisms trivially following from the definitions.

PROPOSITION 1.5. *Let $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$, $\Delta'\colon \Gamma(\pi') \to \Gamma(\pi'')$ be two differential operators of orders k and k' respectively. Then*:

(i) $\Phi_{\Delta' \circ \Delta} = \Phi_{\Delta'} \circ \Phi_\Delta^{(k')}$,

(ii) $\Phi_\Delta^{(l)} \circ j_{k+l}(\varphi) = \Delta^{(l)}(\varphi)$ *for any* $\varphi \in \Gamma(\pi)$, $l \geq 0$,

(iii) $\pi_{l,l'} \circ \Phi_\Delta^{(l)} = \Phi_\Delta^{(l')} \circ \pi_{k+l,k+l'}$, *i.e., the diagram*

$$\begin{tikzcd} J^{k+l}(\pi) \arrow[r, "\Phi_\Delta^{(l)}"] \arrow[d, "\pi_{k+l,k+l'}"'] & J^l(\pi') \arrow[d, "\pi'_{l,l'}"] \\ J^{k+l'}(\pi) \arrow[r, "\Phi_\Delta^{(l')}"] & J^{l'}(\pi') \end{tikzcd} \tag{1.13}$$

is commutative for all $l \geq l' \geq 0$.

1.3. Infinite jets. We now pass to infinite limit in all previous constructions.

DEFINITION 1.8. The *space of infinite jets* $J^\infty(\pi)$ of the fiber bundle $\pi\colon E \to M$ is the inverse limit of the sequence

$$\cdots \to J^{k+1}(\pi) \xrightarrow{\pi_{k+1,k}} J^k(\pi) \to \cdots \to J^1(\pi) \xrightarrow{\pi_{1,0}} E \xrightarrow{\pi} M,$$

i.e., $J^\infty(\pi) = \operatorname{proj\,lim}_{\{\pi_{k,l}, k \geq l\}} J^k(\pi)$.

Though $J^\infty(\pi)$ is an infinite-dimensional manifold, no topological or analytical problems arise, if one bears in mind the genesis of this manifold (i.e., the system of maps $\pi_{k,l}$) when maintaining all constructions. Below we demonstrate how this should be done, giving definitions for all necessary concepts over $J^\infty(\pi)$.

A **point** θ of $J^\infty(\pi)$ is a sequence of points $\{x, \theta_k\}_{k\geq 0}$, $x \in M, \theta_k \in J^k(\pi)$, such that $\pi_k(\theta_k) = x$ and $\pi_{k,l}(\theta_k) = \theta_l$, $k \geq l$. Let us represent any θ_k in the form $\theta_k = [\varphi_k]_x^k$. Then the Taylor expansions of any two sections, φ_k and φ_l, $k \geq l$, coincide up to the l-th term. It means that the points of $J^\infty(\pi)$ can be understood as m-dimensional formal series. But by the Whitney theorem on extensions of smooth functions [**71**], for any such a series there exists a section $\varphi \in \Gamma(\pi)$ such that its Taylor expansion coincides with this series. Hence, any point $\theta \in J^\infty(\pi)$ can be represented in the form $\theta = [\varphi]_x^\infty$.

A **special coordinate system** can be chosen in $J^\infty(\pi)$ due to the fact that if a trivialization $\{\mathcal{U}_\alpha\}_\alpha$ gives special coordinates for some $J^k(\pi)$, then these coordinates can be used for all jet spaces $J^k(\pi)$ simultaneously. Thus, the functions $x_1, \dots, x_n, \dots, u_\sigma^j, \dots$ can be taken for local coordinates in $J^\infty(\pi)$, where $j = 1, \dots, m$ and σ is an arbitrary multi-index of the form $(\sigma_1, \dots, \sigma_n)$.

A **tangent vector** to $J^\infty(\pi)$ at a point θ is defined as follows. Let $\theta = \{x, \theta_k\}$ and $w \in T_xM$, $v_k \in T_{\theta_k}J^k(\pi)$. Then the system of vectors $\{w, v_k\}_{k\geq 0}$ determines a tangent vector to $J^\infty(\pi)$ if and only if $(\pi_k)_* v_k = w$, $(\pi_{k,l})_* v_k = v_l$ for all $k \geq l \geq 0$.

A **smooth bundle** ξ over $J^\infty(\pi)$ is a system of bundles $\eta\colon Q \to M$, $\xi_k\colon P_k \to J^k(\pi)$ together with smooth mappings $\Psi_k\colon P_k \to Q$, $\Psi_{k,l}\colon P_k \to P_l$, $k \geq l \geq 0$, such that

$$\Psi_l \circ \Psi_{k,l} = \Psi_k, \quad \Psi_{k,l} \circ \Psi_{l,s} = \Psi_{k,s}, \qquad k \geq l \geq s \geq 0,$$

and all the diagrams

$$\begin{tikzcd}
P_k \arrow[r, "\Psi_{k,l}"] \arrow[d, "\xi_k"] & P_l \arrow[r, "\Psi_l"] \arrow[d, "\xi_l"] & Q \arrow[d, "\eta"'] \\
J^k(\pi) \arrow[r, "\pi_{k,l}"] & J^l(\pi) \arrow[r, "\pi_l"] & M
\end{tikzcd}$$

are commutative. For example, if $\eta\colon Q \to M$ is a bundle, then the pull-backs $\pi_k^*(\eta)\colon \pi_k^*(Q) \to J^k(\pi)$ together with the natural projections $\pi_k^*(\eta) \to \pi_l^*(\eta)$, $\pi_k^*(\eta) \to Q$ form a bundle over $J^\infty(\pi)$. We say that ξ is a *vector bundle* over $J^\infty(\pi)$, if η and all ξ_k are vector bundles and the mappings Ψ_k, $\Psi_{k,l}$ are fiber-wise linear.

A **smooth mapping** of $J^\infty(\pi)$ to $J^\infty(\pi')$, where $\pi\colon E \to M$, $\pi'\colon E' \to M'$, is defined as a system F of mappings $F_{-\infty}\colon M \to M'$, $F_k\colon J^k(\pi) \to J^{k-s}(\pi')$, $k \geq s$, where $s \in \mathbb{Z}$ is a fixed integer called the *degree of* F, such that

$$\pi_{k-r,k-s-1} \circ F_k = F_{k-1} \circ \pi_{k,k-1}, \qquad k \geq s + 1.$$

For example, if $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ is a differential operator of order s, then the system of mappings $F_{-\infty} = \mathrm{id}_M$, $F_k = \Phi_\Delta^{(k-s)}$, $k \geq s$ (see the previous subsection), is a smooth mapping of $J^\infty(\pi)$ to $J^\infty(\pi')$.

We say that two smooth mappings $F = \{F_k\}, G = \{G_k\}\colon J^\infty(\pi) \to J^\infty(\pi')$ of degrees s and l respectively, $l \geq s$, are *equivalent*, if the diagrams

$$\begin{tikzcd} J^{k-s}(\pi') \arrow[rr, "\pi'_{k-s,k-l}"] & & J^{k-l}(\pi') \\ & J^k(\pi) \arrow[ul, "F_k"] \arrow[ur, "G_k"'] & \end{tikzcd}$$

are commutative for all admissible $k \geq 0$. When working with smooth mappings, one can always choose the representative of maximal degree in any class of equivalent mappings. In particular, it can be easily seen that mappings with negative degrees reduce to zero degree ones in such a way.

REMARK 1.3. The construction above can be literally generalized to the following situation. Consider the category $\mathcal{M}^\infty$, whose objects are chains

$$M_{-\infty} \xleftarrow{m} M_0 \xleftarrow{m_{1,0}} M_1 \leftarrow \cdots \leftarrow M_k \xleftarrow{m_{k+1,k}} M_{k+1} \leftarrow \cdots,$$

where $M_{-\infty}$ and all M_k, $k \geq 0$, are finite-dimensional smooth manifolds while m and $m_{k+1,k}$ are smooth mappings. Let us set

$$m_k \stackrel{\text{def}}{=} m \circ m_{1,0} \circ \cdots \circ m_{k,k-1}, \quad m_{k,l} \stackrel{\text{def}}{=} m_{l+1,l} \circ \cdots \circ m_{k,k-1}, \qquad k \geq l.$$

Define a *morphism* of two objects, $\{M_k\}$, $\{N_k\}$, as a system F of mappings $\{F_{-\infty}, F_k\}$ such that the diagram

$$\begin{tikzcd} M_k \arrow[r, "m_{k,l}"] \arrow[d, "F_k"'] & M_l \arrow[d, "F_l"] \\ N_{k-s} \arrow[r, "n_{k,l}"] & N_{l-s} \end{tikzcd}$$

is commutative for all admissible k and a fixed s (degree of F).

EXAMPLE 1.4. Let M and N be two smooth manifolds, $F\colon N \to M$ be a smooth mapping, and $\pi\colon E \to M$ a be vector bundle. Consider the pull-backs $F^*(\pi_k) \stackrel{\text{def}}{=} \pi_{F,k}\colon J_F^k(\pi) \to N$, where $J_F^k(\pi)$ denotes the corresponding total space. Thus $\{N, J_F^k(\pi)\}_{k\geq 0}$ is an object of $\mathcal{M}^\infty$.

To any section $\phi \in \Gamma(\pi_k)$, there corresponds the section $\phi_F \in \Gamma(\pi_{F,k})$ defined by $\phi_F(x) \stackrel{\text{def}}{=} (x, \phi F(x))$, $x \in N$ (for any $x \in N$, we set $\phi_F(x) \stackrel{\text{def}}{=} (x, \phi(F(x)))$. In particular, for $\phi = j_k(\varphi)$, $\varphi \in \Gamma(\pi)$ we obtain the section

$j_k(\varphi)_F$. Let $\xi\colon H \to N$ be another vector bundle and ψ be a section of the pull-back $\pi_{F,k}^*(\xi)$. Then the correspondence

$$\Delta = \Delta_\psi\colon \Gamma(\pi) \to \Gamma(\xi), \qquad \varphi \mapsto j_k(\varphi)_F^*(\psi),$$

is called a (*nonlinear*) *differential operator of order* $\leq k$ *over the mapping* F. As before, we can define prolongations $\Delta^{(l)}\colon \Gamma(\pi_{k+l}) \to \Gamma(\xi_l)$ and these prolongations would determine smooth mappings $\Phi_\Delta^{(l)}\colon J_F^{k+l}(\pi) \to J^l(\xi)$. The system $\{\Phi_\Delta^{(l)}\}_{l\geq 0}$ is a morphism of $\{J_F^k(\pi)\}$ to $\{J^k(\xi)\}$.

Note that if $F\colon N \to M$, $G\colon O \to N$ are two smooth maps and $\Delta, \square$ are two nonlinear operators over F and G respectively, then their composition is defined and is a nonlinear operator over $F \circ G$.

EXAMPLE 1.5. The category $\mathcal{M}$ of smooth manifolds is embedded into $\mathcal{M}^\infty$, if for any smooth manifold M one sets $M_\infty = \{M_k, m_{k,k-1}\}$ with $M_k = M$ and $m_{k,k-1} = \mathrm{id}_M$. For any smooth mapping $f\colon M \to N$ we also set $f_\infty = \{f_k\}$ with $f_k = f$. We say that F is a *smooth mapping* of $J^\infty(\pi)$ to a smooth manifold N, if $F = \{F_k\}$ is a morphism of $\{J^k(\pi), \pi_{k,k-1}\}$ to N_∞. In accordance to previous constructions, such a mapping is completely determined by some $f\colon J^k(\pi) \to N$.

Taking $\mathbb{R}$ for the manifold N in the previous example, we obtain a definition of a **smooth function** on $J^\infty(\pi)$. Thus, a smooth function on $J^\infty(\pi)$ is a function on $J^k(\pi)$ for some finite but an arbitrary k. The set $\mathcal{F}(\pi)$ of such functions is identified with $\bigcup_{k=0}^\infty \mathcal{F}_k(\pi)$ and forms a commutative filtered algebra. Using the well-known duality between smooth manifolds and algebras of smooth functions on these manifolds, we deal in what follows with the algebra $\mathcal{F}(\pi)$ rather than with the manifold $J^\infty(\pi)$ itself.

From this point of view, a **vector field** on $J^\infty(\pi)$ is a filtered derivation of $\mathcal{F}(\pi)$, i.e., an $\mathbb{R}$-linear map $X\colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ such that

$$X(fg) = fX(g) + gX(f), \qquad f, g \in \mathcal{F}(\pi), \quad X(\mathcal{F}_k(\pi)) \subset \mathcal{F}_{k+l}(\pi),$$

for all k and some $l = l(X)$. The latter is called the *filtration* of the field X. The set of all vector fields is a filtered Lie algebra over $\mathbb{R}$ with respect to commutator $[X, Y]$ and is denoted by $D(\pi) = \bigcup_{l\geq 0} D^{(l)}(\pi)$.

Differential forms of degree i on $J^\infty(\pi)$ are defined as elements of the filtered $\mathcal{F}(\pi)$-module $\Lambda^i(\pi) \stackrel{\text{def}}{=} \bigcup_{k\geq 0} \Lambda^i(\pi_k)$, where $\Lambda^i(\pi_k) \stackrel{\text{def}}{=} \Lambda^i(J^k(\pi))$ and the module $\Lambda^i(\pi_k)$ is considered to be embedded into $\Lambda^i(\pi_{k+1})$ by $\pi_{k+1,k}^*$. Defined in such a way, these forms possess all basic properties[4] of differential forms on finite-dimensional manifolds. Let us mention most important ones:

(i) The module $\Lambda^i(\pi)$ is the i-th external power of the module $\Lambda^1(\pi)$, $\Lambda^i(\pi) = \bigwedge^i \Lambda^1(\pi)$. Respectively, the operation of *wedge product* $\wedge\colon \Lambda^p(\pi) \otimes \Lambda^q(\pi) \to \Lambda^{p+q}(\pi)$ is defined and $\Lambda^*(\pi) = \sum_{i\geq 0} \Lambda^i(\pi)$ becomes a commutative graded algebra.

[4]In fact, as we shall see in Section 1 of Chapter 2, $\Lambda^i(\pi)$ is structurally much richer than forms on a finite-dimensional manifold.

(ii) The module $D(\pi)$ is dual to $\Lambda^1(\pi)$, i.e.,

$$D(\pi) = \hom^{\phi}_{\mathcal{F}(\pi)}(\Lambda^1(\pi), \mathcal{F}(\pi)), \tag{1.14}$$

where $\hom^{\phi}_{\mathcal{F}(\pi)}(\cdot,\cdot)$ denotes the module of all filtered homomorphisms over $\mathcal{F}(\pi)$. Moreover, equality (1.14) is established in the following way: there is a derivation $d\colon \mathcal{F}(\pi) \to \Lambda^1(\pi)$ such that for any vector field X there exists a uniquely defined filtered homomorphism f_X for which the diagram

$$\begin{tikzcd} \mathcal{F}(\pi) \arrow[rr, "d"] \arrow[dr, "X"'] & & \Lambda^1(\pi) \arrow[dl, "f_X"] \\ & \mathcal{F}(\pi) & \end{tikzcd}$$

is commutative.

(iii) The operator d is extended up to maps $d\colon \Lambda^i(\pi) \to \Lambda^{i+1}(\pi)$ in such a way that the sequence

$$0 \to \mathcal{F}(\pi) \xrightarrow{d} \Lambda^1(\pi) \to \cdots \to \Lambda^i(\pi) \xrightarrow{d} \Lambda^{i+1}(\pi) \to \cdots$$

becomes a complex, i.e., $d \circ d = 0$. This complex is called the *de Rham complex* on $J^\infty(\pi)$ while d is called the *de Rham differential*. The latter is a derivation of the superalgebra $\Lambda^*(\pi)$.

Using the identification (1.14), we can define the *inner product* (or *contraction*) of a field $X \in D(\pi)$ with a 1-form $\omega \in \Lambda^1(\pi)$:

$$\mathrm{i}_X\omega \stackrel{\text{def}}{=} f_X(\omega). \tag{1.15}$$

We shall also use the notation $X \lrcorner\, \omega$ for the contraction of X to ω. This operation extends onto $\Lambda^*(\pi)$, if we set

$$\mathrm{i}_X f = 0, \qquad \mathrm{i}_X(\omega \wedge \theta) = \mathrm{i}_X(\omega) \wedge \theta + (-1)^{\omega}\omega \wedge \mathrm{i}_X(\theta)$$

for all $f \in \mathcal{F}(\pi)$ and $\omega, \theta \in \Lambda^*(\pi)$ (here and below we always write $(-1)^\omega$ instead of $(-1)^{\deg\omega}$).

With the de Rham differential and interior product defined, we can introduce the *Lie derivative* of a form $\omega \in \Lambda^*(\pi)$ along a field X by setting

$$\mathrm{L}_X\omega \stackrel{\text{def}}{=} \mathrm{i}_X(d\omega) + d(\mathrm{i}_X\omega)$$

(the *infinitesimal Stokes formula*). We shall also denote the Lie derivative by $X(\omega)$. Other constructions related to differential calculus over $J^\infty(\pi)$ (and over infinite-dimensional objects of a more general nature) will be described in Chapter 4.

Linear differential operators over $J^\infty(\pi)$ generalize the notion of derivations and are defined as follows. Let P and Q be two filtered $\mathcal{F}(\pi)$-modules and $\Delta \in \hom^{\phi}_{\mathbb{R}}(P, Q)$. Then Δ is called a linear differential operator of order k acting from P to Q, if

$$(\delta_{f_0} \circ \delta_{f_1} \circ \dots \circ \delta_{f_k})\Delta = 0$$

for all $f_0, \dots, f_k \in \mathcal{F}(\pi)$, where $(\delta_f \Delta)p \stackrel{\text{def}}{=} f\Delta(p) - \Delta(fp)$. We write $k = \operatorname{ord}(\Delta)$.

Due to existence of filtrations in $\mathcal{F}(\pi), P$ and Q, one can define differential operators of *infinite order* acting from P to Q, [**51**]. Namely, let $P = \{P_l\}_l$, $Q = \{Q_l\}_l$, $P_l \subset P_{l+1}$, $Q_l \subset Q_{l+1}$, P_l, Q_l being $\mathcal{F}_l(\pi)$-modules. Let $\Delta \in \hom_{\mathbb{R}}^{\phi}(P, Q)$ and s be filtration of Δ, i.e., $\Delta(P_l) \subset Q_{l+s}$. We can always assume that $s \geq 0$. Suppose now that $\Delta_l \stackrel{\text{def}}{=} \Delta\,|_{P_l} : P_l \to Q_l$ is a linear differential operator of order o_l over $\mathcal{F}_l(\pi)$. Then we say that Δ is a linear differential operator of order growth o_l. In particular, if $o_l = \alpha l + \beta$, $\alpha, \beta \in \mathbb{R}$, we say that Δ is of constant growth α.

Distributions. Let $\theta \in J^\infty(\pi)$. The tangent plane to $J^\infty(\pi)$ at the point θ is the set of all tangent vectors to $J^\infty(\pi)$ at this point (see above). Denote such a plane by $T_\theta = T_\theta(J^\infty(\pi))$. Let $\theta = \{x, \theta_k\}$, $x \in M$, $\theta_k \in J^k(\pi)$ and $v = \{w, v_k\}$, $v' = \{w', v'_k\} \in T_\theta$. Then the linear combination $\lambda v + \mu v' = \{\lambda w + \mu w', \lambda v_k + \mu v'_k\}$ is again an element of T_θ and thus T_θ is a vector space. A correspondence $\mathcal{T} \colon \theta \mapsto \mathcal{T}_\theta \subset T_\theta$, where $\mathcal{T}_\theta$ is a linear subspace, is called a distribution on $J^\infty(\pi)$. Denote by $\mathcal{T}D(\pi) \subset D(\pi)$ the submodule of vector fields lying in $\mathcal{T}$, i.e., a field X belongs to $\mathcal{T}D(\pi)$ if and only if $X_\theta \in \mathcal{T}_\theta$ for all $\theta \in J^\infty(\pi)$. We say that the distribution $\mathcal{T}$ is *integrable*, if it satisfies formal Frobenius condition: for any vector fields $X, Y \in \mathcal{T}D(\pi)$ their commutator lies in $\mathcal{T}D(\pi)$ as well, or $[\mathcal{T}D(\pi), \mathcal{T}D(\pi)] \subset \mathcal{T}D(\pi)$.

This condition can expressed in a dual way as follows. Let us set

$$\mathcal{T}^1\Lambda(\pi) = \{\omega \in \Lambda^1(\pi) \mid \mathrm{i}_X\omega = 0, X \in \mathcal{T}D(\pi)\}$$

and consider the ideal $\mathcal{T}\Lambda^*(\pi)$ generated in $\Lambda^*(\pi)$ by $\mathcal{T}^1\Lambda(\pi)$. Then the distribution $\mathcal{T}$ is integrable if and only if the ideal $\mathcal{T}\Lambda^*(\pi)$ is differentially closed: $d(\mathcal{T}\Lambda^*(\pi)) \subset \mathcal{T}\Lambda^*(\pi)$.

Finally, we say that a submanifold $N \subset J^\infty(\pi)$ is an *integral manifold* of $\mathcal{T}$, if $T_\theta N \subset \mathcal{T}_\theta$ for any point $\theta \in N$. An integral manifold N is called *locally maximal* at a point $\theta \in N$, if there exist no other integral manifold N' such that $N \subset N'$.

2. Nonlinear PDE

In this section we introduce the notion of a nonlinear differential equation and discuss some important concepts related to this notion: solutions, symmetries, and prolongations.

2.1. Equations and solutions.

Let $\pi \colon E \to M$ be a vector bundle.

DEFINITION 1.9. A submanifold $\mathcal{E} \subset J^k(\pi)$ is called a (*nonlinear*) *differential equation* of order k in the bundle π. We say that $\mathcal{E}$ is a *linear* equation, if $\mathcal{E} \cap \pi_x^{-1}(x)$ is a linear subspace in $\pi_x^{-1}(x)$ for all $x \in M$.

We say that the equation $\mathcal{E}$ is *determined*, if $\operatorname{codim}\mathcal{E} = \dim\pi$, that it is *overdetermined*, if $\operatorname{codim}\mathcal{E} > \dim\pi$, and that it is *underdetermined*, if $\operatorname{codim}\mathcal{E} < \dim\pi$.

We shall always assume that $\mathcal{E}$ is projected surjectively onto E under $\pi_{k,0}$.

DEFINITION 1.10. A (local) section f of the bundle π is called a (local) *solution* of the equation $\mathcal{E}$, if its graph lies in $\mathcal{E}$: $j_k(f)(M) \subset \mathcal{E}$.

Let us show that these definitions are in agreement with the traditional ones. Choose in a neighborhood $\mathcal{U}$ of a point $\theta \in \mathcal{E}$ a special coordinate system $x_1, \dots, x_n, u^1, \dots, u^m, \dots, u^j_\sigma, \dots$, where $|\sigma| \le k$, $j = 1, \dots, m$. Then, in this coordinate system, $\mathcal{E}$ will be given by a system of equations

$$\begin{cases} F^1(x_1, \dots, x_n, u^1, \dots, u^m, \dots, u^1_\sigma, \dots, u^m_\sigma, \dots) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F^r(x_1, \dots, x_n, u^1, \dots, u^m, \dots, u^1_\sigma, \dots, u^m_\sigma, \dots) = 0, \end{cases} \tag{1.16}$$

where the functions $F^1, \dots, F^r$ are functionally independent. Now, let $f \in \Gamma_{\mathrm{loc}}(\pi)$ be a section locally expressed in the form of relations $u^1 = f^1(x_1, \dots, x_n), \dots, u^m = f^m(x_1, \dots, x_n)$. Then its k-jet is given by the equalities

$$u^j_\sigma = \frac{\partial^{|\sigma|} f^j}{\partial x_\sigma},$$

where $j = 1, \dots, m$, $0 \le |\sigma \le k$, and $j_k(f)(\bar{\mathcal{U}})$, $\bar{\mathcal{U}} = \pi_k(\mathcal{U}) \subset M$, lies in $\mathcal{E}$ if and only if the equations

$$\begin{cases} F^1(x_1, \dots, x_n, f^1, \dots, f^m, \dots, \dfrac{\partial^{|\sigma|} f^1}{\partial x_\sigma}, \dots, \dfrac{\partial^{|\sigma|} f^m}{\partial x_\sigma}, \dots) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F^r(x_1, \dots, x_n, f^1, \dots, f^m, \dots, \dfrac{\partial^{|\sigma|} f^1}{\partial x_\sigma}, \dots, \dfrac{\partial^{|\sigma|} f^m}{\partial x_\sigma}, \dots) = 0. \end{cases}$$

are satisfied. Thus we are in a complete correspondence with the analytical definition of a differential equation.

REMARK 1.4. There exists another way to represent differential equations. Namely, let $\pi' \colon \mathbb{R}^r \times \mathcal{U} \to \mathcal{U}$ be the trivial r-dimensional bundle. Then the set of functions $F^1, \dots, F^r$ can be understood as a section φ of the pull-back $(\pi_k|_{\mathcal{U}})^*(\pi')$, or as a nonlinear operator $\Delta = \Delta_\varphi$ defined in $\mathcal{U}$, while the equation $\mathcal{E}$ is characterized by the condition

$$\mathcal{E} \cap \mathcal{U} = \{\theta_k \in \mathcal{U} \mid \varphi(\theta_k) = 0\}. \tag{1.17}$$

More general, any equation $\mathcal{E} \subset J^k(\pi)$ can be represented in the form similar to (1.17). Namely, for any equation $\mathcal{E}$ there exists a fiber bundle $\pi' \colon E' \to M$ and a section $\varphi \in \mathcal{F}_k(\pi, \pi)$ such that $\mathcal{E}$ coincides with the set of zeroes for φ: $\mathcal{E} = \{\varphi = 0\}$. In this case we say that $\mathcal{E}$ is *associated to the operator* $\Delta = \Delta_\varphi \colon \Gamma(\pi) \to \Gamma(\pi')$ and use the notation $\mathcal{E} = \mathcal{E}_\Delta$.

EXAMPLE 1.6. Consider the bundles $\pi = \tau_p^*\colon \bigwedge^p T^*M \to M$, $\pi' = \tau_{p+1}^*\colon \bigwedge^{p+1} T^*M \to M$ and let $d\colon \Gamma(\pi) = \Lambda^p(M) \to \Gamma(\pi') = \Lambda^{p+1}(M)$ be the de Rham differential (see Example 1.2). Thus we obtain a first-order equation $\mathcal{E}_d$ in the bundle τ_p^*. Consider the case $p = 1$, $n \geq 2$ and choose local coordinates $x_1, \dots, x_n$ in M. Then any form $\omega \in \Lambda^1(M)$ is represented as $\omega = u^1 dx_1 + \dots + u^n dx_n$ and we have

$$\mathcal{E}_d = \{u^j_{1_i} = u^i_{1_j} \mid i < j\},$$

where 1_i denotes the multi-index $(0, \dots, 1, \dots, 0)$ with zeroes at all positions except for the i-th one. This equation is underdetermined when $n = 2$, determined for $n = 3$ and overdetermined for $n > 3$.

EXAMPLE 1.7 (see [**69**]). Consider an arbitrary vector bundle $\pi\colon E \to M$ and a differential form $\omega \in \Lambda^p(J^k(\pi))$, $p \leq \dim M$. The condition $j_k(\varphi)^*(\omega) = 0$, $\varphi \in \Gamma(\pi)$, determines a $(k+1)$-st order equation $\mathcal{E}_\omega$ in the bundle π. Consider the case $p = \dim M = 2$, $k = 1$ and choose a special coordinate system x, y, u, u_x, u_y in $J^k(\pi)$. Let $\varphi = \varphi(x, y)$ be a local section and

$$\begin{aligned}\omega = A\,du_x \wedge du_y &+ (B_1\,du_x + B_2\,du_y) \wedge du \\ &+ du_x \wedge (B_{11}\,dx + B_{12}\,dy) + du_y \wedge (B_{21}\,dx + B_{22}\,dy) \\ &+ du \wedge (C_1\,dx + C_2\,dy) + D\,dx \wedge dy,\end{aligned}$$

where A, B_i, B_{ij}, C_i, D are functions of x, y, u, u_x, u_y. Then we have

$$\begin{aligned}j_1(\varphi)^*\omega &= A^\varphi(\varphi_{xx}\,dx + \varphi_{xy}\,dy) \wedge (\varphi_{yx}\,dx + \varphi_{yy}\,dy) \\ &+ \Big(B_1^\varphi(\varphi_{xx}\,dx + \varphi_{xy}\,dy) + B_2^\varphi(\varphi_{yx}\,dx + \varphi_{yy}\,dy)\Big) \wedge (\varphi_x\,dx + \varphi_y\,dy) \\ &+ (\varphi_{xx}\,dx + \varphi_{xy}\,dy) \wedge (B_{11}^\varphi\,dx + B_{12}^\varphi\,dy) + (\varphi_{yx}\,dx + \varphi_{yy}\,dy) \wedge (B_{21}^\varphi\,dx + B_{22}^\varphi\,dy) \\ &+ (\varphi_x\,dx + \varphi_y\,dy) \wedge (C_1^\varphi\,dx + C_2^\varphi\,dy) + D^\varphi\,dx \wedge dy,\end{aligned}$$

where $F^\varphi \stackrel{\text{def}}{=} j_1(\varphi)^*F$ for any $F \in \mathcal{F}_1(\pi)$. Simplifying the last expression, we obtain

$$\begin{aligned}j_1(\varphi)^*\omega = \Big(&A^\varphi(\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2) + (\varphi_y B_1^\varphi + B_{12}^\varphi)\varphi_{xx} - (\varphi_x B_2^\varphi + B_{12}^\varphi)\varphi_{yy} \\ &+ (\varphi_y B_2^\varphi - \varphi_x B_1^\varphi + B_{22}^\varphi - B_{11}^\varphi)\varphi_{xy} + \varphi_x C_2^\varphi - \varphi_y C_1^\varphi + D^\varphi)\Big)\,dx \wedge dy.\end{aligned}$$

Hence, the equation $\mathcal{E}_\omega$ is of the form

$$a(u_{xx}u_{yy} - u_{xy}^2) + b_{11}u_{xx} + b_{12}u_{xy} + b_{22}u_{yy} + c = 0, \tag{1.18}$$

where $a = A$, $b_{11} = u_y B_1 + B_{12}$, $b_{12} = u_y B_2 - u_x B_1 + B_{22} - B_{11}$, $b_{22} = u_x B_2 + B_{12}$, $c = u_x C_2 - u_y C_1 + D$ are functions on $J^1(\pi)$. Equation (1.18) is the so-called two-dimensional *Monge–Ampere equation* and obviously any such an equation can be represented as $\mathcal{E}_\omega$ for some $\omega \in \Lambda^1(J^1(\pi))$.

Note that we have constructed a correspondence between p-forms on $J^k(\pi)$ and $(p+1)$-order operators. This correspondence will be described differently in Subsection 1.4 of Chapter 2

EXAMPLE 1.8. Consider again a fiber bundle $\pi\colon E \to M$ and a section $\nabla\colon E \to J^1(\pi)$ of the bundle $\pi_{1,0}\colon J^1(\pi) \to E$. Then the graph $\mathcal{E}_\nabla = \nabla(E) \subset J^1(\pi)$ is a first-order equation in the bundle π. Let $\theta_1 \in \mathcal{E}_\nabla$. Then, due to Proposition 1.2 on page 5, θ_1 is identified with the pair (θ_0, L_{θ_1}), where $\theta_0 = \pi_{1,0}(\theta_1) \in E$, while L_{θ_1} is the R-plane at θ_0 corresponding to θ_1. Hence, the section ∇ (or the equation $\mathcal{E}_\nabla$) may be understood as a distribution of horizontal[5] n-dimensional planes on E: $\mathcal{T}_\nabla\colon E \ni \theta \mapsto \theta_1 = L_{\nabla(\theta)}$. A solution of the equation $\mathcal{E}_\nabla$, by definition, is a section $\varphi \in \Gamma(\pi)$ such that $j_1(\varphi)(M) \subset \nabla(E)$. It means that at any point $\theta = \varphi(x) \in \varphi(M)$ the plane $\mathcal{T}_\nabla(\theta)$ is tangent to the graph of the section φ. Thus, solutions of $\mathcal{E}_\nabla$ coincide with integral manifolds of $\mathcal{T}_\nabla$.

In local coordinates $(x_1, \dots, x_n, u^1, \dots, u^m, \dots, u_i^j, \dots)$, where $u_i^j \overset{\text{def}}{=} u_{1_i}^j$, $i = 1, \dots, n$, $j = 1, \dots, m$, the equation $\mathcal{E}_\nabla$ is represented as

$$u_i^j = \nabla_i^j(x_1, \dots, x_n, u^1, \dots, u^m), \qquad i = 1, \dots, n,\ j = 1, \dots, m, \tag{1.19}$$

∇_i^j being smooth functions.

EXAMPLE 1.9. As we saw in the previous example, to solve the equation $\mathcal{E}_\nabla$ is the same as to find integral n-dimensional manifolds of the distribution $\mathcal{T}_\nabla$. Hence, the former to be solvable, the latter is to satisfy the Frobenius theorem conditions. Thus, for solvable $\mathcal{E}_\nabla$, we obtain conditions on the section $\nabla \in \Gamma(\pi_{1,0})$. Let us write down these conditions in local coordinates.

Using representation (1.19), note that $\mathcal{T}_\nabla$ is given by the 1-forms

$$\omega^j = du^j - \sum_{i=1^n} \nabla_i^j\, dx_i, \qquad j = 1, \dots, m.$$

Hence, the integrability conditions may be expressed as

$$d\omega^j = \sum_{i=1}^m \rho_i^j \wedge \omega_i, \qquad j = 1, \dots, m,$$

for some 1-forms ρ_i^i. After elementary computations, we obtain that the functions ∇_i^j must satisfy the following relations:

$$\frac{\partial \nabla_\alpha^j}{\partial x_\beta} + \sum_{\gamma=1}^m \nabla_\alpha^\gamma \frac{\partial \nabla_\beta^j}{\partial u^\gamma} = \frac{\partial \nabla_\beta^j}{\partial x_\alpha} + \sum_{\gamma=1}^m \nabla_\beta^\gamma \frac{\partial \nabla_\alpha^j}{\partial u^\gamma} \tag{1.20}$$

for all $j = 1, \dots, m$, $1 \le \alpha < \beta \le m$. Thus we got a naturally constructed first-order equation $\mathcal{I}(\pi) \subset J^1(\pi_{1,0})$ whose solutions are horizontal n-dimensional distributions in $E = J^1(\pi)$.

[5]An n-dimensional plane $L \subset T_{\theta_k}(J^k(\pi))$ is called *horizontal*, if it projects nondegenerately onto T_xM under $(\pi_k)_*$, $x = \pi_k(\theta_k)$.

REMARK 1.5. Let us consider the previous two examples from a bit different point of view. Namely, the horizontal distribution $\mathcal{T}_\nabla$ (or the section $\nabla\colon J^0(\pi) \to J^1(\pi)$, which is the same, as we saw above) may be understood as a *connection* in the bundle π. By the latter we understand the following.

Let X be a vector field on the manifold M. Then, for any point $x \in M$, the vector $X_x \in T_x M$ can be uniquely lifted up to a vector $\nabla_{X_x} \in T_\theta E$, $\pi(\theta) = x$, such that $X_x \in \mathcal{T}_\nabla(\theta)$. In such a way, we get the correspondence $D(M) \to D(E)$ which we shall denote by the same symbol ∇. This correspondence possesses the following properties:

(i) it is $C^\infty(M)$-linear, i.e., $\nabla(fX + gY) = f\nabla(X) + g\nabla(Y)$, $X, Y \in D(M)$, $f, g \in C^\infty(M)$;
(ii) for any $X \in D(M)$, the field $\nabla(X)$ is projected onto M in a well-defined way and $\pi_*\nabla(X) = X$.

Equation (1.20) is equivalent to *flatness* of the connection ∇, which means that

$$\nabla([X,Y]) - [\nabla(X), \nabla(Y)] = 0, \qquad X, Y \in M, \tag{1.21}$$

i.e., that ∇ is a homomorphism of the Lie algebra $D(M)$ of vector fields on M to the Lie algebra $D(E)$.

In Chapter 4 we shall deal with the concept of connection in a more extensive and general manner. In particular, it will allow us to construct equations (1.20) invariantly, without use of local coordinates.

EXAMPLE 1.10. Let $\pi\colon \mathbb{R}^m \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the trivial m-dimensional bundle. Then the system of equations

$$u^j_{1_{n+1}} = f^j(x_1, \dots, x_{n+1}, \dots, u^\alpha_{\sigma_1,\dots,\sigma_n,0}, \dots), \tag{1.22}$$

where $j, \alpha = 1, \dots, m$, is called *evolutionary*. In more conventional notations this system is written down as

$$\frac{\partial u^j}{\partial t} = f^j(x_1, \dots, x_n, t, \dots, \frac{\partial^{\sigma_1+\dots+\sigma_n} u^\alpha}{\partial x_1^{\sigma_1} \dots \partial x_n^{\sigma_n}}, \dots),$$

where the independent variable t corresponds to x_{n+1}.

2.2. The Cartan distributions. Now we know what a differential equation is, but cannot speak about *geometry* of these equation. The reason is that the notion of geometry implies the study of smooth manifolds (spaces) enriched with some additional structures. In particular, transformation groups preserving these structures are of great interest as it was stated in the *Erlangen Program* by Felix Klein [**45**].

Our nearest aim is to use this approach to PDE and the main question to be answered is

> *What are the structures making differential equations of smooth manifolds?*

At first glance, the answer is clear: solutions are those entities for the sake of which differential equations are studied. But this viewpoint can hardly considered to be constructive: to implement it, one needs to know the solutions of the equation at hand and this task, in general, is transcendental.

This means that we need to find a construction which, on one hand, contains all essential information about solutions and, on the other hand, can be efficiently studied by the tools of differential geometry.

DEFINITION 1.11. Let $\pi\colon E \to M$ be a vector bundle. Consider a point $\theta_k \in J^k(\pi)$ and the span $\mathcal{C}^k_{\theta_k} \subset T_{\theta_k}(J^k(\pi))$ of all R-planes (see Definition 1.4) at the point θ_k.

(i) The correspondence $\mathcal{C}^k = \mathcal{C}^k(\pi)\colon \theta_k \mapsto \mathcal{C}^k_{\theta_k}$ is called the *Cartan distribution* on $J^k(\pi)$.

(ii) Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation of order k. The correspondence $\mathcal{C}^k(\mathcal{E})\colon \mathcal{E} \ni \theta_k \mapsto \mathcal{C}^k_{\theta_k} \cap T_{\theta_k}\mathcal{E} \subset T_{\theta_k}\mathcal{E}$ is called the *Cartan distribution* on $\mathcal{E}$. We call elements of the Cartan distributions *Cartan planes.*

(iii) A point $\theta_k \in \mathcal{E}$ is called *regular*, if the Cartan plane $\mathcal{C}^k_{\theta_k}(\mathcal{E})$ is of maximal dimension. We say that $\mathcal{E}$ is a *regular equation*, if all its points are regular.

In what follows, we deal with regular equations or in neighborhoods of regular points[6].

We are now going to give an explicit description of Cartan distributions on $J^k(\pi)$ and to describe their integral manifolds. Let $\theta_k \in J^k(\pi)$ be represented in the form

$$\theta_k = [\varphi]^k_x, \qquad \varphi \in \Gamma(\pi), \quad x = \pi_k(\theta_k). \tag{1.23}$$

Then, by definition, the Cartan plain $\mathcal{C}^k_{\theta_k}$ is spanned by the vectors

$$j_k(\varphi)_{*,x}(v), \qquad v \in T_x M, \tag{1.24}$$

for all $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$ satisfying (1.23).

Let $x_1, \dots, x_n, \dots, u^j_\sigma, \dots$, $j = 1, \dots, m$, $|\sigma| \le k$, be a special coordinate system in a neighborhood of θ_k. Introduce the notation $\partial x_i \overset{\text{def}}{=} \partial/\partial x_i$, $\partial u_\sigma \overset{\text{def}}{=} \partial/\partial u_\sigma$. Then the vectors of the form (1.24) can be expressed as linear combinations of the vectors

$$\partial x_i + \sum_{|\sigma|\le k} \sum_{j=1}^m \frac{\partial^{|\sigma|+1}\varphi^j}{\partial x_\sigma \partial x_i} \partial u^j_\sigma, \tag{1.25}$$

where $i = 1, \dots, n$. Using this representation, we prove the following result:

PROPOSITION 1.6. *For any point $\theta_k \in J^k(\pi)$, $k \ge 1$, the Cartan plane $\mathcal{C}^k_{\theta_k}$ is of the form $\mathcal{C}^k_{\theta_k} = (\pi_{k,k-1})^{-1}_*(L_{\theta_k})$, where L_{θ_k} is the R-plane at the*

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[6] It is clear that for any regular point there exists a neighborhood of this point all points of which are regular.

point $\pi_{k,k-1}(\theta_k) \in J^{k-1}(\pi)$ determined by the point θ_k (see p. 5 for the definition of L_{θ_k}).

PROOF. Denote the vector (1.25) by $v_i^{k,\varphi}$. It is obvious that for any two sections φ, φ' satisfying (1.23) the difference $v_i^{k,\varphi} - v_i^{k,\varphi'}$ is a $\pi_{k,k-1}$-vertical vector and any such a vector can be obtained in this way. On the other hand, the vectors $v_i^{k-1,\varphi}$ do not depend on φ satisfying (1.23) and form a basis in the space L_{θ_k}. □

REMARK 1.6. From the result proved it follows that the Cartan distribution on $J^k(\pi)$ can be locally considered as generated by the vector fields

$$D_i^{[k]} = \partial x_i + \sum_{|\sigma|\le k-1}\sum_{j=1}^{m} u^j_{\sigma+1_i}\partial u^j_\sigma, \quad V^s_\tau = \partial u^s_\tau,\ |\tau| = k, s = 1,\dots,m. \tag{1.26}$$

From here, by direct computations, it follows that $[V^s_\tau, D_i^{[k]}] = V^s_{\tau-1_i}$, where

$$V^s_{(\tau_1,\dots,\tau_n)-1_i} = \begin{cases} V_{(\tau_1,\dots,\tau_i-1,\dots,\tau_n)}, & \text{if } \tau_i > 0,\\ 0, & \text{otherwise.}\end{cases}$$

But, as it follows from Proposition 1.6, vector fields V^j_σ for $|\sigma| \le k$ do not lie in $\mathcal{C}^k$.

Let us consider the following 1-forms in special coordinates on $J^{k+1}(\pi)$:

$$\omega^j_\sigma \overset{\text{def}}{=} du^j_\sigma - \sum_{i=1}^{n} u^j_{\sigma+1_i}\, dx_i, \tag{1.27}$$

where $j = 1,\dots,m$, $|\sigma| < k$. From the representation (1.26) we immediately obtain the following important property of the forms introduced:

PROPOSITION 1.7. *The system of forms (1.27) annihilates the Cartan distribution on $J^k(\pi)$, i.e., a vector field X lies in $\mathcal{C}^k$ if and only if $\mathrm{i}_X\omega^j_\sigma = 0$ for all $j = 1,\dots,m$, $|\sigma| < k$.*

DEFINITION 1.12. The forms (1.27) are called the *Cartan forms* on $J^k(\pi)$ associated to the special coordinate system x_i, u^j_σ.

Note that the $\mathcal{F}_k(\pi)$-submodule generated in $\Lambda^1(J^k(\pi))$ by the forms (1.27) is independent of the choice of coordinates.

DEFINITION 1.13. The $\mathcal{F}_k(\pi)$-submodule generated in $\Lambda^1(J^k(\pi))$ by the Cartan forms is called the *Cartan submodule*. We denote this submodule by $\mathcal{C}\Lambda^1(J^k(\pi))$.

Our last step is to describe maximal integral manifolds of the Cartan distribution on $J^k(\pi)$. To do this, we start with the "infinitesimal estimate".

Let $N \subset J^k(\pi)$ be an integral manifold of the Cartan distribution. Then from Proposition 1.7 it follows that the restriction of any Cartan form ω onto

N vanishes. Similarly, the differential $d\omega$ vanishes on N. Therefore, if vector fields X, Y are tangent to N, then $d\omega\,|_N\,(X, Y) = 0$.

DEFINITION 1.14. Let $\mathcal{C}^k_{\theta_k}$ be the Cartan plane at $\theta \in J^k(\pi)$.

(i) We say that two vectors $v, w \in \mathcal{C}^k_{\theta_k}$ are *in involution*, if the equality $d\omega\,|_{\theta_k}\,(v, w) = 0$ holds for any $\omega \in \mathcal{C}\Lambda^1(J^k(\pi))$.
(ii) A subspace $W \subset \mathcal{C}^k_{\theta_k}$ is said to be *involutive*, if any two vectors $v, w \in W$ are in involution.
(iii) An involutive subspace is called *maximal*, if it cannot be embedded into other involutive subspace.

Consider a point $\theta_k = [\varphi]^k_x \in J^k(\pi)$. Then from Proposition 1.7 it follows that the direct sum decomposition

$$\mathcal{C}^k_{\theta_k} = T^v_{\theta_k} \oplus T^\varphi_{\theta_k}$$

is valid, where $T^v_{\theta_k}$ denotes the tangent plane to the fiber of the projection $\pi_{k,k-1}$ passing through the point θ_k, while $T^\varphi_{\theta_k}$ is the tangent plane to the graph of $j_k(\varphi)$. Hence, the involutiveness is sufficient to be checked for the following pairs of vectors $v, w \in \mathcal{C}^k_{\theta_k}$:

(i) $v, w \in T^v_{\theta_k}$;
(ii) $v, w \in T^\varphi_{\theta_k}$;
(iii) $v \in T^v_{\theta_k}, w \in T^\varphi_{\theta_k}$.

Note now that the tangent space $T^v_{\theta_k}$ is identified with the tensor product $S^k(T^*_x) \otimes E_x$, $x = \pi_k(\theta_k) \in M$, where T^*_x is the fiber of the cotangent bundle to M at the point x, E_x is the fiber of the bundle π at the same point while S^k denotes the k-th symmetric power. Then any tangent vector $w \in T_x M$ determines the mapping $\delta_w \colon S^k(T^*_x) \otimes E_x \to S^{k-1}(T^*_x) \otimes E_x$ by

$$\delta_w(\rho_1 \odot \cdots \odot \rho_k) \otimes e = \sum_{i=1}^{k} \rho_1 \odot \cdots \odot \langle \rho_i, w \rangle \odot \cdots \odot \rho_k \otimes e,$$

where $\odot$ denotes multiplication in $S^k(T^*_x)$, $\rho_i \in T^*_x$, $e \in E_x$, while $\langle \cdot, \cdot \rangle$ is the natural pairing between T^*_x and T_x.

PROPOSITION 1.8. *Let* $v, w \in \mathcal{C}^k_{\theta_k}$. *Then*:

(i) *All pairs* $v, w \in T^v_{\theta_k}$ *are in involution.*
(ii) *All pairs* $v, w \in T^\varphi_{\theta_k}$ *are in involution too.*
(iii) *If* $v \in T^v_{\theta_k}$ *and* $w \in T^\varphi_{\theta_k}$, *then they are in involution if and only if* $\delta_{\pi_{k,*}(w)} v = 0$.

PROOF. Note first that the involutiveness conditions are sufficient to be checked for the Cartan forms (1.27) only. All three results follow from the representation (1.26) by straightforward computations. $\square$

Consider a point $\theta_k \in J^k(\pi)$. Let F_{θ_k} be the fiber of the bundle $\pi_{k,k-1}$ passing through the point θ_k and $H \subset T_xM$ be a subspace. Define the space[7]

$$\mathrm{Ann}(H) = \{v \in F_{\theta_k} \mid \delta_w v = 0,\ \forall w \in H\}.$$

Then, as it follows from Proposition 1.8, the following description of maximal involutive subspaces takes place:

COROLLARY 1.9. *Let $\theta_k = [\varphi]_x^k$, $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$. Then any maximal involutive subspace $V \subset \mathcal{C}_{\theta_k}^k(\pi)$ is of the form*

$$V = j_k(\varphi)_*(H) \oplus \mathrm{Ann}(H)$$

for some $H \subset T_xM$.

If V is a maximal involutive subspace, then the corresponding space H is obviously $\pi_{k,*}(V)$. We call dimension of H the *type* of the maximal involutive subspace V and denote it by $\mathrm{tp}(V)$.

PROPOSITION 1.10. *Let V be a maximal involutive subspace. Then*

$$\dim V = m\binom{n-r+k-1}{k} + r,$$

where $n = \dim M$, $m = \dim \pi$, $r = \mathrm{tp}(V)$.

PROOF. Choose local coordinates in M in such a way that the vectors $\partial x_1, \dots, \partial x_r$ form a basis in H. Then, in the corresponding special system in $J^k(\pi)$, coordinates along $\mathrm{Ann}(H)$ will consist of those functions u_σ^j, $|\sigma| = k$, for which $\sigma_1 = \dots = \sigma_r = 0$. $\square$

We can now describe maximal integral manifolds of the Cartan distribution on $J^k(\pi)$.

Let $N \subset J^k(\pi)$ be such a manifold $\theta_k \in N$. Then the tangent plane to N at the point θ_k is a maximal involutive plane. Assume that its type is equal to $r(\theta_k)$.

DEFINITION 1.15. The number

$$\mathrm{tp}(N) \stackrel{\text{def}}{=} \max_{\theta_k \in N} r(\theta_k).$$

is called the *type* of the maximal integral manifold N of the Cartan distribution.

Obviously, the set

$$g(N) \stackrel{\text{def}}{=} \{\theta_k \in N \mid r(\theta_k) = \mathrm{tp}(N)\}$$

is everywhere dense in N. We call the points $\theta_k \in g(N)$ *generic*. Let θ_k be such a point and $\mathcal{U}$ be its neighborhood in N consisting of generic points. Then:

[7]Using the linear structure, we identify the fiber F_{θ_k} of the bundle $\pi_{k,k-1}$ with its tangent space.

(i) $N' = \pi_{k,k-1}(N)$ is an integral manifold of the Cartan distribution on $J^{k-1}(\pi)$;
(ii) $\dim(N') = \mathrm{tp}(N)$;
(iii) $\pi_{k-1}|_{N'} : N' \to M$ is an immersion.

THEOREM 1.11. *Let $N \subset J^{k-1}(\pi)$ be an integral manifold of the Cartan distribution on $J^k(\pi)$ and $\mathcal{U} \subset N$ be an open domain consisting of generic points. Then*

$$\mathcal{U} = \{\theta_k \in J^k(\pi) \mid L_{\theta_k} \supset T_{\theta_{k-1}}\mathcal{U}'\},$$

where $\theta_{k-1} = \pi_{k,k-1}(\theta_k)$, $\mathcal{U}' = \pi_{k,k-1}(\mathcal{U})$.

PROOF. Let $\mathcal{V}' = \pi_{k-1}(\mathcal{U}') \subset M$. Denote its dimension (which equals the number $\mathrm{tp}(N)$) by r and choose local coordinates in M in such a way that the submanifold $\mathcal{V}'$ is determined by the equations $x_{r+1} = \cdots = x_n = 0$ in these coordinates. Then, since $\mathcal{U}' \subset J^{k-1}(\pi)$ is an integral manifold and $\pi_{k-1}|_{\mathcal{U}'} : \mathcal{U}' \to \mathcal{V}'$ is a diffeomorphism, in the corresponding special coordinates the manifold $\mathcal{U}'$ is given by the equations

$$u^j_\sigma = \begin{cases} \dfrac{\partial^{|\sigma|}\varphi^j}{\partial x_\sigma}, & \text{if } \sigma = (\sigma_1, \dots, \sigma_r, 0, \dots, 0), \\ 0, & \text{otherwise,} \end{cases}$$

for all $j = 1, \dots, m$, $|\sigma| \leq k-1$ and some smooth function $\varphi = \varphi(x_1, \dots, x_r)$. Hence, the tangent plane H to $\mathcal{U}'$ at θ_{k-1} is spanned by the vectors of the form (1.25) with $i = 1, \dots, r$. Consequently, a point θ_k, such that $L_{\theta_k} \supset H$, is determined by the coordinates

$$u^j_\sigma = \begin{cases} \dfrac{\partial^{|\sigma|}\varphi^j}{\partial x_\sigma}, & \text{if } \sigma = (\sigma_1, \dots, \sigma_r, 0, \dots, 0), \\ \text{arbitrary real numbers}, & \text{otherwise,} \end{cases}$$

where $j = 1, \dots, m$, $|\sigma| \leq k$. Hence, if θ_k, θ'_k are two such points, then the vector $\theta_k - \theta'_k$ lies in $\mathrm{Ann}(H)$, as it follows from the proof of Proposition 1.10. As it is easily seen, any integral manifold of the Cartan distribution projecting onto $\mathcal{U}'$ is contained in $\mathcal{U}$, which finishes the proof. □

REMARK 1.7. Note that maximal integral manifolds N of type $\dim M$ are exactly graphs of jets $j_k(\varphi)$, $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$. On the other hand, if $\mathrm{tp}(N) = 0$, then N coincides with a fiber of the projection $\pi_{k,k-1}\colon J^k(\pi) \to J^{k-1}(\pi)$.

2.3. Symmetries. The last remark shows that the Cartan distribution on $J^k(\pi)$ is in a sense sufficient to restore the structures specific to the jet manifolds. This motivates the following definition:

DEFINITION 1.16. Let $\mathcal{U}, \mathcal{U}' \subset J^k(\pi)$ be open domains.

(i) A diffeomorphism $F\colon \mathcal{U} \to \mathcal{U}'$ is called a *Lie transformation*, if it preserves the Cartan distribution, i.e.,

$$F_*(\mathcal{C}^k_{\theta_k}) = \mathcal{C}^k_{F(\theta_k)}$$

for any point $\theta_k \in \mathcal{U}$.

Let $\mathcal{E}, \mathcal{E}' \subset J^k(\pi)$ be differential equations.

(ii) A Lie transformation $F\colon \mathcal{U} \to \mathcal{U}$ is called a (*local*) *equivalence*, if $F(\mathcal{U} \cap \mathcal{E}) = \mathcal{U}' \cap \mathcal{E}'$.

(iii) A (local) equivalence is called a (*local*) *symmetry*, if $\mathcal{E} = \mathcal{E}'$ and $\mathcal{U} = \mathcal{U}'$. Such symmetries are also called *classical*[8].

Below we shall not distinguish between local and global versions of the concepts introduced.

REMARK 1.8. There is an alternative approach to the concept of a symmetry. Namely, we can introduce the Cartan distribution on $\mathcal{E}$ by setting

$$\mathcal{C}_\theta(\mathcal{E}) \stackrel{\text{def}}{=} \mathcal{C}_\theta \cap T_\theta \mathcal{E}, \qquad \theta \in \mathcal{E},$$

and define *interior symmetries* of $\mathcal{E}$ as a diffeomorphism $F\colon \mathcal{E} \to \mathcal{E}$ preserving $\mathcal{C}(\mathcal{E})$. In general, the group of these symmetries does not coincide with the above introduced. A detailed discussion of this matter can be found in [**60**].

EXAMPLE 1.11. Consider the case $J^0(\pi) = E$. Then, since any n-dimensional horizontal plane in $T_\theta E$ is tangent to some section of the bundle π, the Cartan plane $\mathcal{C}_\theta^0$ coincides with the whole space $T_\theta E$. Thus the Cartan distribution is trivial in this case and any diffeomorphism of E is a Lie transformation.

EXAMPLE 1.12. Since the Cartan distribution on $J^k(\pi)$ is locally determined by the Cartan forms (1.27), the condition of F to be a Lie transformation cam be reformulated as

$$F^*\omega_\sigma^j = \sum_{\alpha=1}^{m} \sum_{|\tau|<k} \lambda_{\sigma,\tau}^{j,\alpha} \omega_\tau^\alpha, \qquad j = 1, \dots, m, \quad |\sigma| < k, \tag{1.28}$$

where $\lambda_{\sigma,\tau}^{j,\alpha}$ are smooth functions on $J^k(\pi)$. Equations (1.28) are the base for computations in local coordinates.

In particular, if $\dim \pi = 1$ and $k = 1$, equations (1.28) reduce to the only condition $F^*\omega = \lambda\omega$, where $\omega = du - \sum_{i=1}^n u_{1_i}\, dx_i$. Hence, Lie transformations in this case are just *contact transformations* of the natural contact structure in $J^1(\pi)$.

EXAMPLE 1.13. Let $F\colon J^0(\pi) \to J^0(\pi)$ be a diffeomorphism (which can be considered as a general change of dependent and independent coordinates). Let us construct a Lie transformation $F^{(1)}$ of $J^1(\pi)$ such that the

[8] Contrary to higher, or generalized, symmetries which will be introduced in the next chapter.

diagram

$$\begin{tikzcd} J^1(\pi) \arrow[r, "F^{(1)}"] \arrow[d, "\pi_{1,0}"'] & J^1(\pi) \arrow[d, "\pi_{1,0}"] \\ J^0(\pi) \arrow[r, "F"] & J^0(\pi) \end{tikzcd}$$

is commutative, i.e., $\pi_{1,0} \circ F^{(1)} = F \circ \pi_{1,0}$. To do this, introduce local coordinates $x_1, \dots, x_n, u^1, \dots, u^m$ in $J^0(\pi)$ and consider the corresponding special coordinates in $J^1(\pi)$ denoting the functions $u^j_{1_i}$ by p^j_i. Express the transformation F in the form

$$x_i \mapsto X_i(x_1, \dots, x_n, u^1, \dots, u^m), \qquad u^j \mapsto U^j(x_1, \dots, x_n, u^1, \dots, u^m),$$

$i = 1, \dots, n$, $j = 1, \dots, m$, in these coordinates. Then, due to (1.28), to find

$$F^{(1)} \colon p^j_i \mapsto P^j_i(x_1, \dots, x_n, u^1, \dots, u^m, p^1_1, \dots, p^m_n),$$

one needs to solve the system

$$dU^j - \sum_{i=1}^n P^j_i \, dX_i = \sum_{\alpha=1}^m \lambda^{j,\alpha}(du^\alpha - \sum_{i=1}^n p^\alpha_i \, dx_i),$$

$j = 1, \dots, m$, with respect to the functions P^j_i for arbitrary smooth coefficients $\lambda^{j,\alpha}$. Using matrix notation $p = \|p^j_i\|, P = \|P^j_i\|$ and $\lambda = \|\lambda^{\alpha\beta}\|$, we see that

$$\lambda = \frac{\partial U}{\partial u} - P \circ \frac{\partial X}{\partial u}$$

and

$$P = \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial u} \circ p\right) \circ \left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial u} \circ p\right)^{-1}, \tag{1.29}$$

where

$$\frac{\partial X}{\partial x} = \left\|\frac{\partial X_\alpha}{\partial x_\beta}\right\|, \quad \frac{\partial X}{\partial u} = \left\|\frac{\partial X_\alpha}{\partial u^\beta}\right\|, \quad \frac{\partial U}{\partial x} = \left\|\frac{\partial U^\alpha}{\partial x_\beta}\right\|, \quad \frac{\partial U}{\partial u} = \left\|\frac{\partial U^\alpha}{\partial u^\beta}\right\|$$

denote Jacobi matrices. Note that the transformation $F^{(1)}$, as it follows from (1.29), is undefined at some points of $J^1(\pi)$, i.e., at the points where the matrix $\partial X/\partial x + \partial X/\partial u \circ p$ is not invertible.

EXAMPLE 1.14. Let $\pi \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\dim \pi = \dim M$ and consider the transformation $u^i \mapsto x_i$, $x_i \mapsto u^i$, $i = 1, \dots, n$. This transformation is called the *hodograph transformation*. From (1.29) it follows that the corresponding transformation of the functions p^j_i is defined by $P = p^{-1}$.

EXAMPLE 1.15. Let $\mathcal{E}_d$ be the equation determined by the de Rham differential (see Example 1.6), i.e., $\mathcal{E}_d = \{d\omega = 0\}$, $\omega \in \Lambda^i(M)$. Then for any diffeomorphism $F\colon M \to M$ one has $F^*(d\omega) = d(F^*\omega)$ which means that F determines a symmetry of $\mathcal{E}_d$. Symmetries of this type are called *gauge symmetries*.

The construction of Example 1.13 can be naturally generalized. Let $F\colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation. Note that from the definition it follows that for any maximal integral manifold N of the Cartan distribution on $J^k(\pi)$, the manifold $F(N)$ possesses the same property. In particular, graph of k-jets are taken to n-dimensional maximal integral manifolds. Let now θ_{k+1} be a point of $J^{k+1}(\pi)$ and let us represent θ_{k+1} as a pair $(\theta_k, L_{\theta_{k+1}})$, or, which is the same, as a class of graphs of k-jets tangent to each other at θ_k. Then, since diffeomorphisms preserve tangency, the image $F_*(L_{\theta_{k+1}})$ will almost always (cf. Example 1.13) be an R-plane at $F(\theta_k)$. Denote the corresponding point in $J^{k+1}(\pi)$ by $F^{(1)}(\theta_{k+1})$.

DEFINITION 1.17. Let $F\colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation. The above defined mapping $F^{(1)}\colon J^{k+1}(\pi) \to J^{k+1}(\pi)$ is called the 1-*lifting* of F.

The mapping $F^{(1)}$ is a Lie transformation at the domain of its definition, since almost everywhere it takes graphs of $(k+1)$-jets to graphs of the same kind. Hence, for any $l \geq 1$ we can define $F^{(l)} \overset{\text{def}}{=} (F^{(l-1)})^{(1)}$ and call this map the l-*lifting* of F.

THEOREM 1.12. *Let $\pi\colon E \to M$ be an m-dimensional vector bundle over an n-dimensional manifold M and $F\colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation. Then*:

(i) *If $m > 1$ and $k > 0$, the mapping F is of the form $F = G^{(k)}$ for some diffeomorphism $G\colon J^0(\pi) \to J^0(\pi)$;*
(ii) *If $m = 1$ and $k > 1$, the mapping F is of the form $F = G^{(k-1)}$ for some contact transformation $G\colon J^1(\pi) \to J^1(\pi)$.*

PROOF. Recall that fibers of the projection $\pi_{k,k-1}\colon J^k(\pi) \to J^{k-1}(\pi)$ for $k \geq 1$ are the only maximal integral manifolds of the Cartan distribution of type 0 (see Remark 1.7). Further, from Proposition 1.10 it follows that in the cases $m > 1$, $k > 0$ and $m = 1$, $k > 1$ they are integral manifolds of maximal dimension, provided $n > 1$. Therefore, the mapping F is $\pi_{k,\varepsilon}$-fiberwise, where $\varepsilon = 0$ for $m > 1$ and $\varepsilon = 1$ for $m = 1$.

Thus there exists a mapping $G\colon J^\varepsilon(\pi) \to J^\varepsilon(\pi)$ such that $\pi_{k,\varepsilon} \circ F = G \circ \pi_{k,\varepsilon}$ and G is a Lie transformation in an obvious way. Let us show that $F = G^{(k-\varepsilon)}$. To do this, note first that in fact, by the same reasons, the transformation F generates a series of Lie transformations $G_l\colon J^l(\pi) \to J^l(\pi)$, $l = \varepsilon, \dots, k$, satisfying $\pi_{l,l-1} \circ G_l = G_{l-1} \circ \pi_{l,l-1}$ and $G_k = F$, $G_\varepsilon = G$. Let us compare the mappings F and $G_{k-1}^{(1)}$.

From Proposition 1.6 and the definition of Lie transformations we obtain

$$F_*((\pi_{k,k-1})_*^{-1}(L_{\theta_k})) = F_*(\mathcal{C}_{\theta_k}^k) = \mathcal{C}_{F(\theta_k)} = (\pi_{k,k-1})_*^{-1}(L_{F(\theta_k)})$$

for any $\theta_k \in J^k(\pi)$. But $F_*((\pi_{k,k-1})_*^{-1}(L_{\theta_k})) = (\pi_{k,k-1})_*^{-1}(G_{k-1,*}(L_{\theta_k}))$ and consequently $G_{k-1,*}(L_{\theta_k}) = L_{F(\theta_k)}$. Hence, by the definition of 1-lifting we have $F = G_{k-1}^{(1)}$. Using this fact as a base of elementary induction, we obtain the result of the theorem for $\dim M > 1$.

Consider the case $n = 1$, $m = 1$ now. Since all maximal integral manifolds are one-dimensional in this case, it should treated in a special way. Denote by $\mathcal{V}$ the distribution consisting of vector fields tangent to the fibers of the projection $\pi_{k,k-1}$. Then

$$F_*\mathcal{V} = \mathcal{V} \tag{1.30}$$

for any Lie transformation F, which is equivalent to F being $\pi_{k,k-1}$-fiberwise.

Let us prove (1.30). To do it, consider an arbitrary distribution $\mathcal{P}$ on a manifold N and introduce the notation

$$\mathcal{P}D = \{X \in D(N) \mid X \text{ lies in } \mathcal{P}\} \tag{1.31}$$

and

$$D_{\mathcal{P}} = \{X \in D(N) \mid [X, Y] \in \mathcal{P},\ \forall Y \in \mathcal{P}D\}. \tag{1.32}$$

Then one can show (using coordinate representation, for example) that

$$D\mathcal{V} = D\mathcal{C}^k \cap D_{[D\mathcal{C}^k, D\mathcal{C}^k]}$$

for $k \geq 2$. But Lie transformations preserve the distributions at the right-hand side of the last equality and consequently preserve $D\mathcal{V}$. □

We pass now to infinitesimal analogues of Lie transformations:

Definition 1.18. Let $\pi\colon E \to M$ be a vector bundle and $\mathcal{E} \subset J^k(\pi)$ be a k-th order differential equation.

(i) A vector field X on $J^k(\pi)$ is called a *Lie field*, if the corresponding one-parameter group consists of Lie transformations.
(ii) A Lie field is called an *infinitesimal classical symmetry* of the equation $\mathcal{E}$, if it is tangent to $\mathcal{E}$.

It should be stressed that infinitesimal classical symmetries play an important role in applications of differential geometry to particular equations.

Since in the sequel we shall deal with infinitesimal symmetries only, we shall skip the adjective *infinitesimal* and call them just *symmetries*. By definition, one-parameter groups of transformations corresponding to symmetries preserve generalized solutions.

Remark 1.9. Similarly to the above considered situation, we may introduce the concepts both of *exterior* and *interior* infinitesimal symmetries (see Remark 1.8), but we do not treat the second ones below.

Let X be a Lie field on $J^k(\pi)$ and $F_t\colon J^k(\pi) \to J^k(\pi)$ be its one-parameter group. The we can construct l-liftings $F_t^{(l)}\colon J^{k+l}(\pi) \to J^{k+l}(\pi)$ and the corresponding Lie field $X^{(l)}$ on $J^{k+l}(\pi)$. This field is called the *l-lifting* of the field X. As we shall see a bit later, liftings of Lie fields, as opposed to those of Lie transformations, are defined globally and can be described explicitly.

An immediate consequence of the definition and of Theorem 1.12 is the following result:

THEOREM 1.13. *Let $\pi\colon E \to M$ be an m-dimensional vector bundle over an n-dimensional manifold M and X be a Lie field on $J^k(\pi)$. Then*:

(i) *If $m > 1$ and $k > 0$, the field X is of the form $X = Y^{(k)}$ for some vector field Y on $J^0(\pi)$;*
(ii) *If $m = 1$ and $k > 1$, the field X is of the form $X = Y^{(k-1)}$ for some contact vector field Y on $J^1(\pi)$.*

Coordinate expressions for Lie fields can be obtained as follows. Let $x_1, \dots, x_n, \dots, u_\sigma^j, \dots$ be a special coordinate system in $J^k(\pi)$ and ω_σ^j be the corresponding Cartan forms. Then X is a Lie field if and only if the following equations hold

$$\mathrm{L}_X \omega_\sigma^j = \sum_{\alpha=1}^{m} \sum_{|\tau|<k} \lambda_{\sigma,\tau}^{j,\alpha} \omega_\tau^\alpha, \qquad j = 1, \dots, m, \quad |\sigma| < k, \tag{1.33}$$

where $\lambda_{\sigma,\tau}^{j,\alpha}$ are arbitrary smooth functions. Let the vector field X be represented in the form

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \sum_{|\sigma|\le k} X_\sigma^j \frac{\partial}{\partial u_\sigma^j}.$$

Then from (1.33) it follows that the coefficients of the field X are related by the following recursion equalities

$$X_{\sigma+1_i}^j = D_i(X_\sigma^j) - \sum_{\alpha=1}^{n} u_{\sigma+1_\alpha}^j D_i(X_\alpha), \tag{1.34}$$

where

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \sum_{|\sigma|\ge 0} u_{\sigma+1_i}^j \frac{\partial}{\partial u_\sigma^j} \tag{1.35}$$

are the so-called *total derivatives*.

Recall now that a contact field X on $J^1(\pi)$, $\dim \pi = 1$, is completely determined by its *generating function* which is defined as $f \overset{\text{def}}{=} \mathrm{i}_X \omega$, where $\omega = du - \sum_i u_{1_i}\, dx_i$ is the Cartan (contact) form on $J^1(\pi)$. The contact field corresponding to a function $f \in \mathcal{F}_1(\pi)$ is denoted by X_f and is expressed as

$$X_f = -\sum_{i=1}^{n} \frac{\partial f}{\partial u_{1_i}} \frac{\partial}{\partial x_i} + \Big(f - \sum_{i=1}^{n} u_{1_i} \frac{\partial f}{\partial u_{1_i}}\Big) \frac{\partial}{\partial u} \\ + \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial x_i} + u_{1_i} \frac{\partial f}{\partial u}\Big) \frac{\partial}{\partial u_{1_i}} \tag{1.36}$$

in local coordinates.

Thus, starting with a field (1.36) in the case $\dim\pi = 1$ or with an arbitrary field on $J^0(\pi)$ for $\dim\pi > 1$ and using (1.34), we can obtain efficient expressions for Lie fields.

REMARK 1.10. Note that in the case $\dim\pi > 1$ we can introduce vector-valued generating functions by setting $f^j \overset{\text{def}}{=} \mathrm{i}_X\omega^j$, where $\omega^j = du^j - \sum_i u^j_{1_i}\,dx_i$ are the Cartan forms on $J^1(\pi)$. Such a function may be understood as an element of the module $\mathcal{F}_1(\pi,\pi)$. The local conditions that a section $f \in \mathcal{F}_1(\pi,\pi)$ corresponds to a Lie field is as follows:

$$\frac{\partial f^\alpha}{\partial u^\alpha_{1_i}} = \frac{\partial f^\beta}{\partial u^\beta_{1_i}}, \quad \frac{\partial f^\alpha}{\partial u^\beta_{1_i}} = 0, \qquad \alpha \neq \beta.$$

In Chapter 2 we shall generalize the theory and get rid of these conditions.

We call f the *generating section* (or *generating function*, depending on the dimension of π) of the Lie field X, if X is a lifting of the field X_f.

Let us finally write down the conditions of a Lie field to be a symmetry. Assume that an equation $\mathcal{E}$ is given by the relations $F^1 = 0,\dots,F^r = 0$, where $F^j \in \mathcal{F}_k(\pi)$. Then X is a symmetry of $\mathcal{E}$ if and only if

$$X(F^j) = \sum_{\alpha=1}^{r} \lambda^j_\alpha F^\alpha, \qquad j = 1,\dots,r,$$

where λ^j_α are smooth functions, or

$$X(F^j)\,|_{\mathcal{E}} = 0, \qquad j = 1,\dots,r. \tag{1.37}$$

These conditions can be rewritten in terms of generating sections and we shall do it in Chapter 2 in a more general situation.

Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation and X be its symmetry. Then for any solution φ of this equation, the one-parameter group $\{A_t\}$ corresponding to X transforms φ to some new solution φ_t almost everywhere. In special local coordinates, evolution of φ is governed by the following evolutionary equation:

$$\frac{\partial\varphi}{\partial t} = f(x_1,\dots,x_n,\varphi,\frac{\partial\varphi}{\partial x_1},\dots,\frac{\partial\varphi}{\partial x_n}), \tag{1.38}$$

if π is one-dimensional and f is the generating function of X, or by a system of evolutionary equations of the form

$$\frac{\partial\varphi^j}{\partial t} = f^j(x_1,\dots,x_n,\varphi^1,\dots,\varphi^m,\frac{\partial\varphi^1}{\partial x_1},\dots,\frac{\partial\varphi^m}{\partial x_n}), \tag{1.39}$$

where $j = 1,\dots,m = \dim\pi$ and f^j are the components of the generating section.

In particular, we say that a solution is *invariant* with respect to X, if it is transformed by $\{A_t\}$ to itself, which means that it has to satisfy the equation

$$f(x_1, \dots, x_n, \varphi, \frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n}) = 0 \tag{1.40}$$

or a similar system of equations when $\dim \pi > 1$. If $\mathfrak{g}$ is a subalgebra in the symmetry algebra of $\mathcal{E}$, we can also define $\mathfrak{g}$-*invariant solutions* as solutions invariant with respect to all elements of $\mathfrak{g}$.

2.4. Prolongations. The idea of prolongation originates from a simple observation that, a differential equation given, not all relations between dependent variables are explicitly encoded in this equation. To reconstruct these relations, it needs to analyze "differential consequences" of the initial equations.

EXAMPLE 1.16. Consider the system

$$u_{xxy} = v_y^2, \qquad u_{xyy} = v_x + u_y.$$

Then, differentiating the first equation with respect to y and the second one with respect to x, we obtain

$$u_{xxyy} = 2v_y v_{yy}, \qquad u_{xxyy} = v_{xx} + u_{xy}$$

and consequently

$$2v_y v_{yy} = v_{xx} + u_{xy}.$$

EXAMPLE 1.17. Let

$$v_x = u, \qquad v_t = \frac{1}{2}u^2 + u_x.$$

Then

$$u_t = uu_x + u_{xx}$$

by a similar procedure.

EXAMPLE 1.18. Consider equations (1.19) from Example 1.8 on p. 15. Then as consequences of these equations we obtain equations (1.20) which may be viewed at as *compatibility conditions* for equations (1.19). One can see that if the functions ∇_i^j satisfy (1.20), i.e., if the connection ∇ is flat, then these conditions are void; otherwise we obtain functional relations on the variables u_i^j.

Geometrically, the process of computation of differential consequences is expressed by the following definition:

DEFINITION 1.19. Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation of order k. Define the set

$$\mathcal{E}^1 = \{\theta_{k+1} \in J^{k+1}(\pi) \mid \pi_{k+1,k}(\theta_{k+1}) \in \mathcal{E}, L_{\theta_{k+1}} \subset T_{\pi_{k+1,k}(\theta_{k+1})}\mathcal{E}\}$$

and call it the *first prolongation* of the equation $\mathcal{E}$.

If the first prolongation $\mathcal{E}^1$ is a submanifold in $J^{k+1}(\pi)$, we define the second prolongation of $\mathcal{E}$ as $(\mathcal{E}^1)^1 \subset J^{k+2}(\pi)$, etc. Thus the *$l$-th prolongation* is a subset $\mathcal{E}^l \subset J^{k+l}(\pi)$.

Let us redefine the notion of l-th prolongation directly. Namely, take a point $\theta_k \in \mathcal{E}$ and consider a section $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$ such that the graph of $j_k(\varphi)$ is tangent to $\mathcal{E}$ with order l. Let $\pi_k(\theta_k) = x \in M$. Then $[\varphi]_x^{k+l}$ is a point of $J^{k+l}(\pi)$ and the set of all points obtained in such a way obviously coincides with $\mathcal{E}^l$, provided all intermediate prolongations $\mathcal{E}^1, \dots, \mathcal{E}^{l-1}$ be well defined in the sense of Definition 1.19.

Assume now that locally $\mathcal{E}$ is given by the equations

$$F^1 = 0, \dots, F^r = 0, \qquad F^j \in \mathcal{F}_k(\pi)$$

and $\theta_k \in \mathcal{E}$ is the origin of the chosen special coordinate system. Let $u^1 = \varphi^1(x_1, \dots, x_n), \dots, u^m = \varphi^m(x_1, \dots, x_n)$ be a local section of the bundle π. Then

$$\begin{aligned} j_k(\varphi)^* F^j &= F^j\Big(x_1, \dots, x_n, \dots, \frac{\partial^{|\sigma|}\varphi^\alpha}{\partial x_\sigma}, \dots\Big) \\ &= \sum_{i=1}^n \left(\frac{\partial F^j}{\partial x_i} + \sum_{\alpha,\sigma} \frac{\partial F^j}{\partial u_\sigma^\alpha} \frac{\partial^{|\sigma|+1}\varphi^\alpha}{\partial x_{\sigma+1_i}} \right)\Bigg|_{\theta_k} x_i + o(x), \end{aligned}$$

where the sums are taken over all admissible indices. From here it follows, that the graph of $j_k(\varphi)$ is tangent to $\mathcal{E}$ at the point under consideration if and only if

$$\sum_{i=1}^n \left(\frac{\partial F^j}{\partial x_i} + \sum_{\alpha,\sigma} \frac{\partial F^j}{\partial u_\sigma^\alpha} \frac{\partial^{|\sigma|+1}\varphi^\alpha}{\partial x_{\sigma+1_i}} \right)\Bigg|_{\theta_k} = 0.$$

Hence, the equations of the first prolongation are

$$\sum_{i=1}^n \left(\frac{\partial F^j}{\partial x_i} + \sum_{\alpha,\sigma} \frac{\partial F^j}{\partial u_\sigma^\alpha} u^\alpha_{\sigma+1_i} \right) = 0, \qquad i = 1, \dots, n.$$

From here and by comparison with the coordinate representation of prolongations for nonlinear differential operators (see Subsection 1.2), we obtain the following result:

Proposition 1.14. *Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation. Then*

(i) *If the equation $\mathcal{E}$ is determined by a differential operator $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$, then its l-th prolongation is given by the l-th prolongation $\Delta^{(l)}\colon \Gamma(\pi) \to \Gamma(\pi'_l)$ of the operator Δ.*

(ii) *If $\mathcal{E}$ is locally described by the system of equations*

$$F^1 = 0, \dots, F^r = 0, \qquad F^j \in \mathcal{F}_k(\pi),$$

then the system

$$D_\sigma F^j = 0, \qquad |\sigma| \le l, \quad j = 1, \dots, r, \tag{1.41}$$

where $D_\sigma \stackrel{\text{def}}{=} D_1^{\sigma_1} \circ \cdots \circ D_n^{\sigma_n}$, *corresponds to* $\mathcal{E}^l$. *Here* D_i *stands for the* i*-th total derivative* (*see* (1.35)).

From the definition it follows that for any $l \geq l' \geq 0$ one has the embeddings $\pi_{k+l,k+l'}(\mathcal{E}^l) \subset \mathcal{E}^{l'}$ and consequently one has the mappings $\pi_{k+l,k+l'}\colon \mathcal{E}^l \to \mathcal{E}^{l'}$.

DEFINITION 1.20. An equation $\mathcal{E} \subset J^k(\pi)$ is called *formally integrable*, if

(i) all prolongations $\mathcal{E}^l$ are smooth manifolds

and

(ii) all the mappings $\pi_{k+l+1,k+l}\colon \mathcal{E}^{l+1} \to \mathcal{E}^l$ are smooth fiber bundles.

In the sequel, we shall mostly deal with formally integrable equations.

The rest of this chapter is devoted to classical symmetries of some particular equations of mathematical physics.

3. Symmetries of the Burgers equation

As a first example, we shall discuss the computation of classical symmetries for the Burgers equation, which is described by

$$u_t = uu_x + u_{xx}. \tag{1.42}$$

The equation holds on $J^2(x,t;u) = J^2(\pi)$ for the trivial bundle $\pi\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ with x, t being coordinates in $\mathbb{R}^2$ (independent variables) and u a coordinate in the fiber (dependent variable). The total derivative operators are given by

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} \\
&\quad + u_{xt}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{xxt}\frac{\partial}{\partial u_{xt}} + u_{xtt}\frac{\partial}{\partial u_{tt}} + \cdots, \\
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} \\
&\quad + u_{tt}\frac{\partial}{\partial u_t} + u_{xxt}\frac{\partial}{\partial u_{xx}} + u_{xtt}\frac{\partial}{\partial u_{xt}} + u_{ttt}\frac{\partial}{\partial u_{tt}} + \cdots
\end{aligned} \tag{1.43}$$

We now introduce the vector field V of the form

$$V = V^x\frac{\partial}{\partial x} + V^t\frac{\partial}{\partial t} + V^u\frac{\partial}{\partial u} + \cdots + V^{u_{tt}}\frac{\partial}{\partial u_{tt}}, \tag{1.44}$$

where in (1.44) V^x, V^t, V^u are functions depending on x, t, u, while the components with respect to $\partial/\partial u_x$, $\partial/\partial u_t$, $\partial/\partial u_{xx}$, $\partial/\partial u_{xt}$, $\partial/\partial u_{tt}$, which are denoted by V^{u_x}, V^{u_t}, $V^{u_{xx}}$, $V^{u_{xt}}$, $V^{u_{tt}}$, are given by formula (1.34) and are of the form

$$\begin{aligned}
V^{u_x} &= D_x(V^u - u_xV^x - u_tV^t) + u_{xx}V^x + u_{xt}V^t, \\
V^{u_t} &= D_t(V^u - u_xV^x - u_tV^t) + u_{xt}V^x + u_{tt}V^t,
\end{aligned}$$

$$
\begin{aligned}
V^{u_{xx}} &= D_x^2(V^u - u_x V^x - u_t V^t) + u_{xxx}V^x + u_{xxt}V^t, \\
V^{u_{xt}} &= D_x D_t(V^u - u_x V^x - u_t V^t) + u_{xxt}V^x + u_{xtt}V^t, \\
V^{u_{tt}} &= D_t^2(V^u - u_x V^x - u_t V^t) + u_{xtt}V^x + u_{ttt}V^t. \qquad (1.45)
\end{aligned}
$$

The symmetry condition (1.37) on V, which is just the invariance condition of the hypersurface $\mathcal{E} \subset J^2(x,t;u)$ given by (1.42) under the vector field V, results in the equation

$$
V^{u_t} - u_x V^u - u V^{u_x} - V^{u_{xx}} = 0. \qquad (1.46)
$$

Calculation of the quantities V^{u_t}, V^{u_x}, $V^{u_{xx}}$ required in (1.46) yields

$$
\begin{aligned}
V^{u_x} &= \frac{\partial V^u}{\partial x} + u_x \frac{\partial V^u}{\partial u} - u_x \left(\frac{\partial V^x}{\partial x} + u_x \frac{\partial V^x}{\partial u} \right) - u_t \left(\frac{\partial V^t}{\partial x} + u_x \frac{\partial V^t}{\partial u} \right), \\
V^{u_t} &= \frac{\partial V^u}{\partial t} + u_t \frac{\partial V^u}{\partial u} - u_x \left(\frac{\partial V^x}{\partial t} + u_t \frac{\partial V^x}{\partial u} \right) - u_t \left(\frac{\partial V^t}{\partial t} + u_t \frac{\partial V^t}{\partial u} \right), \\
V^{u_{xx}} &= \frac{\partial^2 V^u}{\partial x^2} + 2u_x \frac{\partial^2 V^u}{\partial x \partial u} + u_x^2 \frac{\partial^2 V^u}{\partial u^2} + u_{xx} \frac{\partial V^u}{\partial u} \\
&\quad - 2u_{xx} \left(\frac{\partial V^x}{\partial x} + u_x \frac{\partial V^x}{\partial u} \right) - 2u_{xt} \left(\frac{\partial V^t}{\partial x} + u_x \frac{\partial V^t}{\partial u} \right) \\
&\quad - u_x \left(\frac{\partial^2 V^x}{\partial x^2} + 2u_x \frac{\partial^2 V^x}{\partial x \partial u} + u_x^2 \frac{\partial^2 V^x}{\partial u^2} + u_{xx} \frac{\partial V^x}{\partial u} \right) \\
&\quad - u_t \left(\frac{\partial^2 V^t}{\partial x^2} + 2u_x \frac{\partial^2 V^t}{\partial x \partial u} + u_x^2 \frac{\partial^2 V^t}{\partial u^2} + u_{xx} \frac{\partial V^t}{\partial u} \right). \qquad (1.47)
\end{aligned}
$$

Substitution of these expressions (1.47) together with

$$
\begin{aligned}
u_t &= u u_x + u_{xx}, \\
u_{xt} &= u_x^2 + u u_{xx} + u_{xxx}, \qquad (1.48)
\end{aligned}
$$

into (1.46) leads to a *polynomial* expression with respect to the variables u_{xxx}, u_{xx}, u_x, the coefficients of which should vanish.

The coefficient at u_{xxx}, which arises solely from the term u_{xt} in $V^{u_{xx}}$, leads to the first condition

$$
\frac{\partial V^t}{\partial x} + u_x \frac{\partial V^t}{\partial u} = 0, \qquad (1.49)
$$

from which we immediately obtain that $\partial V^t / \partial x = 0$, $\partial V^t / \partial u = 0$, or

$$
V^t(x,t,u) = F_0(t), \qquad (1.50)
$$

i.e., the function V^t is dependent just on the variable t.

REMARK 1.11. Although V^t is a function dependent just on one variable t, we prefer to write in the sequel partial derivatives instead of ordinary derivatives.

Now, using the obtained result for the function $V^t(x,t,u)$ we obtain from (1.46), (1.47), (1.48), (1.49) that the coefficients at the corresponding terms vanish:

$$
\begin{aligned}
u_{xx}u_x:&\quad 2\frac{\partial V^x}{\partial u}=0,\\
u_{xx}:&\quad -\frac{\partial V^t}{\partial t}+2\frac{\partial V^x}{\partial x}=0,\\
u_x^3:&\quad \frac{\partial^2 V^x}{\partial u^2}=0,\\
u_x^2:&\quad -\frac{\partial^2 V^u}{\partial u^2}+2\frac{\partial^2 V^x}{\partial x\partial u}=0,\\
u_x:&\quad -\frac{\partial V^x}{\partial t}-u\frac{\partial V^t}{\partial t}-V^u+u\frac{\partial V^x}{\partial x}-2\frac{\partial^2 V^u}{\partial x\partial u}+\frac{\partial^2 V^x}{\partial x^2}=0,\\
1:&\quad \frac{\partial V^u}{\partial t}-u\frac{\partial V^u}{\partial x}-\frac{\partial^2 V^u}{\partial x^2}=0.
\end{aligned}
\tag{1.51}
$$

From the first and the fourth equation in (1.51) we have

$$
V^x=F_1(x,t),\qquad V^u=F_2(x,t)+F_3(x,t)u. \tag{1.52}
$$

Substitution of this result into the second, fifth and sixth equation of (1.51) leads to

$$
\begin{aligned}
&\frac{\partial F_0(t)}{\partial t}-2\frac{\partial F_1(x,t)}{\partial x}=0,\\
&\frac{\partial F_1(x,t)}{\partial t}+u\frac{\partial F_0(t)}{\partial t}+F_2(x,t)+uF_3(x,t)\\
&\qquad -u\frac{\partial F_1(x,t)}{\partial x}+2\frac{\partial F_3(x,t)}{\partial x}-\frac{\partial^2 F_1(x,t)}{\partial x^2}=0,\\
&\frac{\partial F_2(x,t)}{\partial t}-u\frac{\partial F_2(x,t)}{\partial x}-\frac{\partial^2 F_2(x,t)}{\partial x^2}\\
&\qquad +u\left(\frac{\partial F_3(x,t)}{\partial t}-u\frac{\partial F_3(x,t)}{\partial x}-\frac{\partial^2 F_3(x,t)}{\partial x^2}\right)=0.
\end{aligned}
\tag{1.53}
$$

We now first solve the first equation in (1.53):

$$
F_1(x,t)=\frac{x}{2}\frac{\partial F_0(t)}{\partial t}+F_4(t), \tag{1.54}
$$

The second equation in (1.53) is an equation polynomial with respect to u, so we obtain from this the following relations:

$$
\begin{aligned}
&\frac{\partial F_0(t)}{\partial t}+2F_3(x,t)=0,\\
&4\frac{\partial F_3(x,t)}{\partial x}+2\frac{\partial F_4(t)}{\partial t}+x\frac{\partial^2 F_0(t)}{\partial t^2}+2F_2(x,t)=0,
\end{aligned}
\tag{1.55}
$$

while from the third equation in (1.53) we obtain

$$
\frac{\partial F_3(x,t)}{\partial x}=0,
$$

$$\begin{aligned}
&\frac{\partial F_3(x,t)}{\partial t} - \frac{\partial^2 F_3(x,t)}{\partial x^2} - \frac{\partial F_2(x,t)}{\partial x} = 0, \\
&\frac{\partial F_2(x,t)}{\partial t} - \frac{\partial^2 F_2(x,t)}{\partial x} = 0.
\end{aligned} \tag{1.56}$$

From (1.55) we can obtain the form of $F_3(x,t)$ and $F_2(x,t)$, i.e.,

$$\begin{aligned}
F_3(x,t) &= -\frac{1}{2}\frac{\partial F_0(t)}{\partial t}, \\
F_2(x,t) &= -\frac{\partial F_4(t)}{\partial t} - \frac{x}{2}\frac{\partial^2 F_0(t)}{\partial t^2}.
\end{aligned} \tag{1.57}$$

The first and second equation in (1.56) now fulfill automatically, while the third equation is a polynomial with respect to x; hence we have

$$\frac{\partial^2 F_4(t)}{\partial t^2} + \frac{x}{2}\frac{\partial^3 F_0(t)}{\partial t^3} = 0, \tag{1.58}$$

from which we finally arrive at

$$F_0(t) = c_1 + c_2 t + c_3 t^2, \quad F_4(t) = c_4 + c_5 t. \tag{1.59}$$

Combining the obtained results we finally have:

$$\begin{aligned}
V^x(x,t,u) &= c_4 + \frac{1}{2}c_2 x + c_5 t + c_3 x t, \\
V^t(x,t,u) &= c_1 + c_2 t + c_3 t^2, \\
V^u(x,t,u) &= -c_5 - c_3 x - \frac{1}{2}c_2 u - c_3 t u,
\end{aligned}$$

which are the components of the vector field V, whereas $c_1, \dots, c_5$ are arbitrary constants.

From (1.59) we have that the Lie algebra of classical symmetries of the Burgers equation is generated by five vector fields

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial t}, \\
V_2 &= \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}, \\
V_3 &= xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (x + tu)\frac{\partial}{\partial u}, \\
V_4 &= \frac{\partial}{\partial x}, \\
V_5 &= t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.
\end{aligned} \tag{1.60}$$

The commutator table for the generators (1.60) is presented on Fig. 1.1.

Note that the generating functions $\varphi_i = V_i \lrcorner (du - u_x\,dx - u_t\,dt)$ corresponding to symmetries (1.60) are

$$\begin{aligned}
\varphi_1 &= -u_t, \\
\varphi_2 &= -\frac{1}{2}(u + x u_x + 2t u_t),
\end{aligned}$$

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	V_1	$2V_2$	0	V_4
V_2		0	V_3	$-\frac{1}{2}V_4$	$\frac{1}{2}V_5$
V_3			0	$-V_5$	0
V_4				0	0
V_5					0

FIGURE 1.1. Commutator table for classical symmetries of the Burgers equation

$$\begin{aligned}
\varphi_3 &= -(x + tu + xtu_x + t^2u_t),\\
\varphi_4 &= -u_x,\\
\varphi_5 &= -(tu_x + 1).
\end{aligned} \tag{1.61}$$

The computations carried through in this application indicate the way one has to take to solve overdetermined systems of partial differential equations for the components of a vector field arising from the symmetry condition (1.37). We also refer to Chapter 8 for description of computer-based computations of symmetries.

4. Symmetries of the nonlinear diffusion equation

The $(3+1)$-nonlinear diffusion equation is given by

$$\Delta(u^{p+1}) + ku^q = u_t, \tag{1.62}$$

where $u = u(x,y,z,t)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, $p, k, q \in \mathbb{Q}$, and $p \neq -1$.

We shall state the results for the Lie algebras of symmetries for all distinct values of p, k, q.

First of all we derived that there are no contact symmetries, i.e., the coefficients of any symmetry V,

$$V = V^x\frac{\partial}{\partial x} + V^y\frac{\partial}{\partial y} + V^z\frac{\partial}{\partial z} + V^t\frac{\partial}{\partial t} + V^u\frac{\partial}{\partial u},$$

V^x, V^y, V^z, V^t, V^u depend on x, y, z, t, u only.

REMARK 1.12. Such symmetries are called *point symmetries* contrary to general contact symmetries whose coefficients at $\partial/\partial x_i$ and $\partial/\partial u$ may depend on coordinates in $J^1(\pi)$ (see Theorem 1.12 (ii)).

Secondly, for any value of p, k, q, equation (1.62) admits the following seven symmetries:

$$\begin{aligned}
&V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial y},\ V_3 = \frac{\partial}{\partial z},\ V_4 = \frac{\partial}{\partial t},\\
&V_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ V_6 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ V_7 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.
\end{aligned} \tag{1.63}$$

We now summarize the final results, while the complete Lie algebras are given for all the cases that should be distinguished.

4.1. Case 1: $p = 0,\ k = 0$. The complete Lie algebra of symmetries of the equation

$$\Delta(u) = u_t \tag{1.64}$$

is spanned by the vector fields $V_1, \ldots, V_7$ given in (1.63) and

$$\begin{aligned}
V_8 &= u\frac{\partial}{\partial u},\\
V_9 &= 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u},\\
V_{10} &= 2t\frac{\partial}{\partial y} - yu\frac{\partial}{\partial u},\\
V_{11} &= 2t\frac{\partial}{\partial z} - zu\frac{\partial}{\partial u},\\
V_{12} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t},\\
V_{13} &= xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + zt\frac{\partial}{\partial z} + t^2\frac{\partial}{\partial t} + \frac{1}{4}u(-x^2 - y^2 - z^2 - 6t)\frac{\partial}{\partial u}
\end{aligned} \tag{1.65}$$

together with the continuous part $F(x, y, z, t)\partial/\partial u$, where $F(x, y, z, t)$ is an arbitrary function which has to satisfy (1.64). In fact, all linear equations possess symmetries of this type.

4.2. Case 2: $p = 0,\ k \neq 0,\ q = 1$. The complete Lie algebra of symmetries of the equation

$$\Delta(u) + ku = u_t \tag{1.66}$$

is spanned by the fields $V_1, \ldots, V_7$ given in (1.63) and

$$\begin{aligned}
V_8 &= u\frac{\partial}{\partial u},\\
V_9 &= 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u},\\
V_{10} &= 2t\frac{\partial}{\partial y} - yu\frac{\partial}{\partial u},\\
V_{11} &= 2t\frac{\partial}{\partial z} - zu\frac{\partial}{\partial u},\\
V_{12} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t} + 2kut\frac{\partial}{\partial u},\\
V_{13} &= xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + zt\frac{\partial}{\partial z} + t^2\frac{\partial}{\partial t} + \frac{1}{4}u(4kt^2 - x^2 - y^2 - z^2 - 6t)\frac{\partial}{\partial u}.
\end{aligned} \tag{1.67}$$

Since (1.66) is a linear equation, it also possesses symmetries of the form $F(x, y, z, t)\partial/\partial u$, where

$$\Delta(F) + kF = F_t. \tag{1.68}$$

4.3. Case 3: $p = 0,\ k \neq 0,\ q \neq 1$. The complete Lie algebra of symmetries of the equation

$$\Delta(u) + ku^q = u_t \tag{1.69}$$

is spanned by $V_1, \dots, V_7$ given in (1.63) and the field

$$V_8 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t} - \frac{2}{q-1}u\frac{\partial}{\partial u}. \tag{1.70}$$

4.4. Case 4: $p = -4/5,\ k = 0$. The complete Lie algebra of symmetries of

$$\Delta(u^{1/5}) = u_t \tag{1.71}$$

is spanned by $V_1, \dots, V_7$ given in (1.63) together with the fields

$$\begin{aligned}
V_8 &= 4t\frac{\partial}{\partial t} + 5u\frac{\partial}{\partial u},\\
V_9 &= 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} - 5u\frac{\partial}{\partial u},\\
V_{10} &= (x^2 - y^2 - z^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} - 5xu\frac{\partial}{\partial u},\\
V_{11} &= 2xy\frac{\partial}{\partial x} + (-x^2 + y^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} - 5yu\frac{\partial}{\partial u},\\
V_{12} &= 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (-x^2 - y^2 + z^2)\frac{\partial}{\partial z} - 5zu\frac{\partial}{\partial u}.
\end{aligned} \tag{1.72}$$

4.5. Case 5: $p \neq -4/5,\ p \neq 0,\ k = 0$. The complete Lie algebra of symmetries of the equation

$$\Delta(u^{p+1}) = u_t \tag{1.73}$$

is spanned by $V_1, \dots, V_7$ given in (1.63) and two additional vector fields

$$\begin{aligned}
V_8 &= -pt\frac{\partial}{\partial t} + u\frac{\partial}{\partial u},\\
V_9 &= px\frac{\partial}{\partial x} + py\frac{\partial}{\partial y} + pz\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u}.
\end{aligned} \tag{1.74}$$

4.6. Case 6: $p = -4/5,\ k \neq 0,\ q = 1$. The complete Lie algebra of symmetries of the equation

$$\Delta(u^{1/5}) + ku = u_t \tag{1.75}$$

is spanned by $V_1, \dots, V_7$ given in (1.63) and

$$V_8 = e^{\frac{4kt}{5}}\frac{\partial}{\partial t} + kue^{\frac{4kt}{5}}\frac{\partial}{\partial u},$$

$$\begin{aligned}
V_9 &= 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} - 5u\frac{\partial}{\partial u},\\
V_{10} &= (x^2 - y^2 - z^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} - 5xu\frac{\partial}{\partial u},\\
V_{11} &= 2xy\frac{\partial}{\partial x} + (-x^2 + y^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} - 5yu\frac{\partial}{\partial u},\\
V_{12} &= 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (-x^2 - y^2 + z^2)\frac{\partial}{\partial z} - 5zu\frac{\partial}{\partial u}.
\end{aligned} \tag{1.76}$$

4.7. Case 7: $p \neq 0$, $p \neq -4/5$, $k \neq 0$, $q = 1$. The complete Lie algebra of symmetries of the equation

$$\Delta(u^{p+1}) + ku = u_t \tag{1.77}$$

is spanned by $V_1, \ldots, V_7$ given in (1.63) and by

$$\begin{aligned}
V_8 &= e^{-pkt}\left(\frac{\partial}{\partial t} + 4ku\frac{\partial}{\partial u}\right),\\
V_9 &= px\frac{\partial}{\partial x} + py\frac{\partial}{\partial y} + pz\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u}.
\end{aligned} \tag{1.78}$$

4.8. Case 8: $p \neq 0$, $p \neq -4/5$, $q = p + 1$. The complete Lie algebra of symmetries of the equation

$$\Delta(u^{p+1}) + ku^{p+1} = u_t \tag{1.79}$$

is spanned by $V_1, \ldots, V_7$ given in (1.63) and by the field

$$V_8 = pt\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{1.80}$$

4.9. Case 9: $p \neq 0$, $p \neq -4/5$, $q \neq 1$, $q \neq p + 1$. The complete Lie algebra of symmetries of the equation

$$\Delta(u^{p+1}) + ku^q = u_t \tag{1.81}$$

is spanned by $V_1, \ldots, V_7$ given in (1.63) and by the field

$$V_8 = (-p + q - 1)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) + 2(q - 1)t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}. \tag{1.82}$$

The results in these nine cases are a generalization of the results of other authors [**13**]. We leave to the reader to describe the corresponding Lie algebra structures in the cases above.

5. The nonlinear Dirac equations

In this section, we consider the nonlinear Dirac equations and compute their classical symmetries [**33**]. Symmetry classification of these equations leads to four different cases: linear Dirac equations with vanishing and non-vanishing rest mass, nonlinear Dirac equation with vanishing rest mass, and general nonlinear Dirac equation (with nonvanishing rest mass). We continue to study the last case in the next chapter (Subsection 2.2) and compute there conservation laws associated to some symmetries.

We shall only give here a short idea of the solution procedure, since all computations follow to standard lines. The Dirac equations are of the form [**11**]:

$$\sum_{k=1}^{3} \hbar \frac{\partial(\gamma_k \psi)}{\partial x_k} - i\hbar \frac{\partial(\gamma_4 \psi)}{\partial x_4} + m_0 c \psi + n_0 \psi(\overline{\psi}\psi) = 0, \tag{1.83}$$

where

$$\begin{aligned} x_4 &= ct, \\ \psi &= (\psi_1, \psi_2, \psi_3, \psi_4)^T, \\ \overline{\psi} &= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*), \end{aligned} \tag{1.84}$$

T stands for transposition, $*$ is complex conjugate and γ_1, γ_2, γ_3, γ_4 are 4×4-matrices defined by

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.85}$$

After introduction of the parameter

$$\lambda = \frac{\hbar}{m_0 c}, \tag{1.86}$$

we obtain

$$\lambda \sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\gamma_k \psi) - \lambda i \frac{\partial}{\partial x_4}(\gamma_4 \psi) + \psi + \lambda^3 \epsilon \psi(\overline{\psi}\psi) = 0. \tag{1.87}$$

In computation of the symmetry algebra of (1.87) we have to distinguish the following cases:

1. $\epsilon = 0$, $\lambda^{-1} = 0$: Dirac equations with vanishing rest mass,
2. $\epsilon = 0$, $\lambda^{-1} \neq 0$: Dirac equations with nonvanishing rest mass,
3. $\epsilon \neq 0$, $\lambda^{-1} = 0$: nonlinear Dirac equations with vanishing rest mass,
4. $\epsilon \neq 0$, $\lambda^{-1} \neq 0$: nonlinear Dirac equations.

These cases are equivalent to the respective choices of m_0 and n_0 in (1.83): e.g., $\epsilon = 0$, $\lambda^{-1} = 0$ is the same as $m_0 = n_0 = 0$, etc.

We put $\psi_j = u^j + iv^j$, $j = 1, \dots, 4$, and obtain a system of eight coupled partial differential equations

$$\lambda v_1^4 - \lambda u_2^4 + \lambda v_3^3 + \lambda v_4^1 + (1 + \lambda^3 \epsilon K) u^1 = 0,$$

$$
\begin{aligned}
\lambda v_1^3 + \lambda u_2^3 - \lambda v_3^4 + \lambda v_4^2 + (1+\lambda^3\epsilon K)u^2 &= 0,\\
-\lambda v_1^2 + \lambda u_2^2 - \lambda v_3^1 - \lambda v_4^3 + (1+\lambda^3\epsilon K)u^3 &= 0,\\
-\lambda v_1^1 - \lambda u_2^1 + \lambda v_3^2 - \lambda v_4^4 + (1+\lambda^3\epsilon K)u^4 &= 0,\\
-\lambda u_1^4 - \lambda v_2^4 - \lambda u_3^3 - \lambda u_4^1 + (1+\lambda^3\epsilon K)v^1 &= 0,\\
-\lambda u_1^3 + \lambda v_2^3 + \lambda u_3^4 - \lambda u_4^2 + (1+\lambda^3\epsilon K)v^2 &= 0,\\
\lambda u_1^2 + \lambda v_2^2 + \lambda u_3^1 + \lambda u_4^3 + (1+\lambda^3\epsilon K)v^3 &= 0,\\
\lambda u_1^1 - \lambda v_2^1 - \lambda u_3^2 + \lambda u_4^4 + (1+\lambda^3\epsilon K)v^4 &= 0,
\end{aligned}
\tag{1.88}
$$

where

$$
u_k^j = \frac{\partial u^j}{\partial x_k}, \quad v_k^j = \frac{\partial v^j}{\partial x_k}, \qquad j,k = 1,\dots,4,
$$

and

$$
K = (u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2 + (v^1)^2 + (v^2)^2 - (v^3)^2 - (v^4)^2. \tag{1.89}
$$

Thus (1.87) is a determined system $\mathcal{E} \subset J^1(\pi)$ in the trivial bundle $\pi\colon \mathbb{R}^8 \times \mathbb{R}^4 \to \mathbb{R}^4$.

Using relations (1.34) and symmetry conditions (1.37), we construct the overdetermined system of partial differential equations for the coefficients of the vector field V

$$
V = F^{x_1}\frac{\partial}{\partial x_1} + \cdots + F^{x_4}\frac{\partial}{\partial x_4} + F^{u^1}\frac{\partial}{\partial u^1} + \cdots + F^{v^4}\frac{\partial}{\partial v^4}. \tag{1.90}
$$

From the resulting overdetermined system of partial differential equations we derive in a straightforward way the following intermediate result:

$$
\begin{aligned}
&1: \qquad F^{x_1},\dots,F^{x_4} \ \text{ are independent of } u^1,\dots,v^4,\\
&2: \qquad F^{x_1},\dots,F^{x_4} \ \text{ are polynomials of degree 3 in } x_1,\dots,x_4,\\
&3: \qquad F^{u^1},\dots,F^{v^4} \ \text{ are linear with respect to } u^1,\dots,v^4.
\end{aligned}
\tag{1.91}
$$

Combination of this intermediate result (1.91) with the remaining system of partial differential equations leads to the following description of symmetry algebras in the four specific cases.

5.1. Case 1: $\epsilon = 0$, $\lambda^{-1} = 0$. The complete Lie algebra of classical symmetries for the Dirac equations with vanishing rest mass is spanned by 23 generators. In addition, there is a continuous part generated by functions $F^{u^1},\dots,F^{v^4}$ dependent on $x_1,\dots,x_4$ and satisfying the Dirac equations (1.88) due to the linearity of these equations. The Lie algebra contains the fifteen infinitesimal generators of the conformal group $X_1,\dots,X_{15}$ and eight vertical vector fields $X_{16},\dots,X_{23}$:

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial x_1},\ X_2 = \frac{\partial}{\partial x_2},\ X_3 = \frac{\partial}{\partial x_3},\ X_4 = \frac{\partial}{\partial x_4},\\
&X_5 = 2x_2\frac{\partial}{\partial x_1} - 2x_1\frac{\partial}{\partial x_2} - v^1\frac{\partial}{\partial u^1} + v^2\frac{\partial}{\partial u^2} - v^3\frac{\partial}{\partial u^3} + v^4\frac{\partial}{\partial u^4}
\end{aligned}
$$

$$
\begin{aligned}
&+ u^1\frac{\partial}{\partial v^1} - u^2\frac{\partial}{\partial v^2} + u^3\frac{\partial}{\partial v^3} - u^4\frac{\partial}{\partial v^4},\\
X_6 &= 2x_3\frac{\partial}{\partial x_1} - 2x_1\frac{\partial}{\partial x_3} - u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2} - u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4}\\
&- v^2\frac{\partial}{\partial v^1} + v^1\frac{\partial}{\partial v^2} - v^4\frac{\partial}{\partial v^3} + v^3\frac{\partial}{\partial v^4},\\
X_7 &= -2x_3\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3} + v^2\frac{\partial}{\partial u^1} + v^1\frac{\partial}{\partial u^2} + v^4\frac{\partial}{\partial u^3} + v^3\frac{\partial}{\partial u^4}\\
&- u^2\frac{\partial}{\partial v^1} - u^1\frac{\partial}{\partial v^2} - u^4\frac{\partial}{\partial v^3} - u^3\frac{\partial}{\partial v^4},\\
X_8 &= 2x_4\frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial x_4} + u^4\frac{\partial}{\partial u^1} + u^3\frac{\partial}{\partial u^2} + u^2\frac{\partial}{\partial u^3} + u^1\frac{\partial}{\partial u^4}\\
&+ v^4\frac{\partial}{\partial v^1} + v^3\frac{\partial}{\partial v^2} + v^2\frac{\partial}{\partial v^3} + v^1\frac{\partial}{\partial v^4},\\
X_9 &= 2x_4\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_4} + v^4\frac{\partial}{\partial u^1} - v^3\frac{\partial}{\partial u^2} + v^2\frac{\partial}{\partial u^3} - v^1\frac{\partial}{\partial u^4}\\
&- u^4\frac{\partial}{\partial v^1} + u^3\frac{\partial}{\partial v^2} - u^2\frac{\partial}{\partial v^3} + u^1\frac{\partial}{\partial v^4},\\
X_{10} &= 2x_4\frac{\partial}{\partial x_3} + 2x_3\frac{\partial}{\partial x_4} + u^3\frac{\partial}{\partial u^1} - u^4\frac{\partial}{\partial u^2} + u^1\frac{\partial}{\partial u^3} - u^2\frac{\partial}{\partial u^4}\\
&+ v^3\frac{\partial}{\partial v^1} - v^4\frac{\partial}{\partial v^2} + v^1\frac{\partial}{\partial v^3} - v^2\frac{\partial}{\partial v^4},\\
X_{11} &= x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4},\\
X_{12} &= (x_1^2 - x_2^2 - x_3^2 + x_4^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2} + 2x_1x_3\frac{\partial}{\partial x_3} + 2x_1x_4\frac{\partial}{\partial x_4}\\
&- (3x_1u^1 - x_2v^1 - x_3u^2 - x_4u^4)\frac{\partial}{\partial u^1}\\
&- (3x_1u^2 + x_2v^2 + x_3u^1 - x_4u^3)\frac{\partial}{\partial u^2}\\
&- (3x_1u^3 - x_2v^3 - x_3u^4 - x_4u^2)\frac{\partial}{\partial u^3}\\
&- (3x_1u^4 + x_2v^4 + x_3u^3 - x_4u^1)\frac{\partial}{\partial u^4}\\
&- (3x_1v^1 + x_2u^1 - x_3v^2 - x_4v^4)\frac{\partial}{\partial v^1}\\
&- (3x_1v^2 - x_2u^2 + x_3v^1 - x_4v^3)\frac{\partial}{\partial v^2}\\
&- (3x_1v^3 + x_2u^3 - x_3v^4 - x_4v^2)\frac{\partial}{\partial v^3}\\
&- (3x_1v^4 - x_2u^4 + x_3v^3 - x_4v^1)\frac{\partial}{\partial v^4},
\end{aligned}
$$

$$
\begin{aligned}
X_{13} = {} & 2x_1x_2\frac{\partial}{\partial x_1} - (x_1^2 - x_2^2 + x_3^2 - x_4^2)\frac{\partial}{\partial x_2} + 2x_2x_3\frac{\partial}{\partial x_3} + 2x_2x_4\frac{\partial}{\partial x_4} \\
& - (3x_2u^1 + x_1v^1 - x_3v^2 - x_4v^4)\frac{\partial}{\partial u^1} \\
& - (3x_2u^2 - x_1v^2 - x_3v^1 + x_4v^3)\frac{\partial}{\partial u^2} \\
& - (3x_2u^3 + x_1v^3 - x_3v^4 - x_4v^2)\frac{\partial}{\partial u^3} \\
& - (3x_2u^4 - x_1v^4 - x_3v^3 + x_4v^1)\frac{\partial}{\partial u^4} \\
& - (3x_2v^1 - x_1u^1 + x_3u^2 + x_4u^4)\frac{\partial}{\partial v^1} \\
& - (3x_2v^2 + x_1u^2 + x_3u^1 - x_4u^3)\frac{\partial}{\partial v^2} \\
& - (3x_2v^3 - x_1u^3 + x_3u^4 + x_4u^2)\frac{\partial}{\partial v^3} \\
& - (3x_2v^4 + x_1u^4 + x_3u^3 - x_4u^1)\frac{\partial}{\partial v^4}, \\
X_{14} = {} & 2x_1x_3\frac{\partial}{\partial x_1} + 2x_2x_3\frac{\partial}{\partial x_2} - (x_1^2 + x_2^2 - x_3^2 - x_4^2)\frac{\partial}{\partial x_3} + 2x_3x_4\frac{\partial}{\partial x_4} \\
& - (3x_3u^1 + x_2v^2 + x_1u^2 - x_4u^3)\frac{\partial}{\partial u^1} \\
& - (3x_3u^2 + x_2v^1 - x_1u^1 + x_4u^4)\frac{\partial}{\partial u^2} \\
& - (3x_3u^3 + x_2v^4 + x_1u^4 - x_4u^1)\frac{\partial}{\partial u^3} \\
& - (3x_3u^4 + x_2v^3 - x_1u^3 + x_4u^2)\frac{\partial}{\partial u^4} \\
& - (3x_3v^1 - x_2u^2 + x_1v^2 - x_4v^3)\frac{\partial}{\partial v^1} \\
& - (3x_3v^2 - x_2u^1 - x_1v^1 + x_4v^4)\frac{\partial}{\partial v^2} \\
& - (3x_3v^3 - x_2u^4 + x_1v^4 - x_4v^1)\frac{\partial}{\partial v^3} \\
& - (3x_3v^4 - x_2u^3 - x_1v^3 + x_4v^2)\frac{\partial}{\partial v^4}, \\
X_{15} = {} & 2x_1x_4\frac{\partial}{\partial x_1} + 2x_2x_4\frac{\partial}{\partial x_2} + 2x_3x_4\frac{\partial}{\partial x_3} + (x_1^2 + x_2^2 + x_3^2 + x_4^2)\frac{\partial}{\partial x_4} \\
& - (3x_4u^1 - x_2v^4 - x_3u^3 - x_1u^4)\frac{\partial}{\partial u^1} \\
& - (3x_4u^2 + x_2v^3 + x_3u^4 - x_1u^3)\frac{\partial}{\partial u^2} \\
& - (3x_4u^3 - x_2v^2 - x_3u^1 - x_1u^2)\frac{\partial}{\partial u^3}
\end{aligned}
$$

$$
\begin{aligned}
&- (3x_4u^4 + x_2v^1 + x_3u^2 - x_1u^1)\frac{\partial}{\partial u^4} \\
&- (3x_4v^1 + x_2u^4 - x_3v^3 - x_1v^4)\frac{\partial}{\partial v^1} \\
&- (3x_4v^2 - x_2u^3 + x_3v^4 - x_1v^3)\frac{\partial}{\partial v^2} \\
&- (3x_4v^3 + x_2u^2 - x_3v^1 - x_1v^2)\frac{\partial}{\partial v^3} \\
&- (3x_4v^4 - x_2u^1 + x_3v^2 - x_1v^1)\frac{\partial}{\partial v^4}, \\
X_{16} = {} & u^1\frac{\partial}{\partial u^1} + u^2\frac{\partial}{\partial u^2} + u^3\frac{\partial}{\partial u^3} + u^4\frac{\partial}{\partial u^4} + v^1\frac{\partial}{\partial v^1} \\
&+ v^2\frac{\partial}{\partial v^2} + v^3\frac{\partial}{\partial v^3} + v^4\frac{\partial}{\partial v^4}, \\
X_{17} = {} & u^2\frac{\partial}{\partial u^1} - u^1\frac{\partial}{\partial u^2} - u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4} - v^2\frac{\partial}{\partial v^1} \\
&+ v^1\frac{\partial}{\partial v^2} + v^4\frac{\partial}{\partial v^3} - v^3\frac{\partial}{\partial v^4}, \\
X_{18} = {} & u^3\frac{\partial}{\partial u^1} + u^4\frac{\partial}{\partial u^2} + u^1\frac{\partial}{\partial u^3} + u^2\frac{\partial}{\partial u^4} + v^3\frac{\partial}{\partial v^1} \\
&+ v^4\frac{\partial}{\partial v^2} + v^1\frac{\partial}{\partial v^3} + v^2\frac{\partial}{\partial v^4}, \\
X_{19} = {} & u^4\frac{\partial}{\partial u^1} - u^3\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^3} + u^1\frac{\partial}{\partial u^4} - v^4\frac{\partial}{\partial v^1} \\
&+ v^3\frac{\partial}{\partial v^2} + v^2\frac{\partial}{\partial v^3} - v^1\frac{\partial}{\partial v^4}, \\
X_{20} = {} & v^1\frac{\partial}{\partial u^1} + v^2\frac{\partial}{\partial u^2} + v^3\frac{\partial}{\partial u^3} + v^4\frac{\partial}{\partial u^4} - u^1\frac{\partial}{\partial v^1} \\
&- u^2\frac{\partial}{\partial v^2} - u^3\frac{\partial}{\partial v^3} - u^4\frac{\partial}{\partial v^4}, \\
X_{21} = {} & v^2\frac{\partial}{\partial u^1} - v^1\frac{\partial}{\partial u^2} - v^4\frac{\partial}{\partial u^3} + v^3\frac{\partial}{\partial u^4} + u^2\frac{\partial}{\partial v^1} \\
&- u^1\frac{\partial}{\partial v^2} - u^4\frac{\partial}{\partial v^3} + u^3\frac{\partial}{\partial v^4}, \\
X_{22} = {} & v^3\frac{\partial}{\partial u^1} + v^4\frac{\partial}{\partial u^2} + v^1\frac{\partial}{\partial u^3} + v^2\frac{\partial}{\partial u^4} - u^3\frac{\partial}{\partial v^1} \\
&- u^4\frac{\partial}{\partial v^2} - u^1\frac{\partial}{\partial v^3} - u^2\frac{\partial}{\partial v^4}, \\
X_{23} = {} & v^4\frac{\partial}{\partial u^1} - v^3\frac{\partial}{\partial u^2} - v^2\frac{\partial}{\partial u^3} + v^1\frac{\partial}{\partial u^4} + u^4\frac{\partial}{\partial v^1} \\
&- u^3\frac{\partial}{\partial v^2} - u^2\frac{\partial}{\partial v^3} + u^1\frac{\partial}{\partial v^4}.
\end{aligned} \tag{1.92}
$$

The result is in full agreement with that of Ibragimov [**5**].

5.2. Case 2: $\epsilon = 0$, $\lambda^{-1} \neq 0$. The complete Lie algebra of symmetries for the Dirac equations with nonvanishing rest mass is spanned by fourteen generators, including ten infinitesimal generators of the Poincaré group $X_1, \ldots, X_{10}$ and the generators X_{19}, X_{20}, X_{23}, X_{16}. There is also a continuous part generated by the functions $F^{u^1}, \ldots, F^{v^4}$ dependent on $x_1, \ldots, x_4$, which satisfy Dirac equations (1.83) with nonvanishing rest mass.

5.3. Case 3: $\epsilon \neq 0$, $\lambda^{-1} = 0$. The complete Lie algebra in this situation is spanned by fourteen generators. These generators are $X_1, \ldots, X_{10}$, X_{19}, X_{20}, X_{23}, and $X_{11} - X_{16}/2$.

5.4. Case 4: $\epsilon \neq 0$, $\lambda^{-1} \neq 0$. The complete Lie algebra of symmetries for the nonlinear Dirac equations with nonvanishing rest mass is spanned by thirteen generators. The generators in this case are the ten infinitesimal generators of the Poincaré group, $X_1, \ldots, X_{10}$, and X_{19}, X_{20}, X_{23}. This result generalizes the result by Steeb [**94**] where X_{20} was found as additional symmetry to the generators of the Poincaré group.

6. Symmetries of the self-dual $SU(2)$ Yang–Mills equations

We study here classical symmetries of the self-dual $SU(2)$ Yang–Mills equations. Two cases are considered: the general one and of the so-called static gauge fields. In the first case we obtain two instanton solutions (the Belavin–Polyakov–Schwartz–Tyupkin [**6**] and 't Hooft instantons [**84**]) as invariant solutions for a special choice of symmetry subalgebras. In a similar way, for the second case we derive a monopole solution [**83**].

We start with a concise description of the $SU(2)$-gauge theory referring the reader to the survey paper by M. K. Prasad [**83**] for a more extensive exposition.

6.1. Self-dual $SU(2)$ Yang–Mills equations. Let M be a 4-dimensional Euclidean space with the coordinates $x_1, \ldots, x_4$. Due to nondegenerate metric in M, we make no distinction between contravariant and covariant indices, $x_\mu = x^\mu$. The basic object in the gauge theory is the Yang–Mills *gauge potential*. The gauge potential is a set of fields $A^a_\mu \in C^\infty(M)$, $a = 1, \ldots, 3$, $\mu = 1, \ldots, 4$. It is convenient to introduce a matrix-valued vector field $A_\mu(x)$, by setting

$$A_\mu = gT^a A^a_\mu, \qquad T^a = \frac{\sigma^a}{2i}, \ a = 1, \ldots, 3, \ \mu = 1, \ldots, 4, \tag{1.93}$$

where σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.94}$$

g being a constant, called the *gauge coupling constant*. Throughout this section we shall use the Einstein summation convention when an index occurs twice. From the matrix gauge potential $A_\mu \, dx_\mu$ one constructs the matrix-valued field strength $F_{\mu\nu}(x)$ by

$$F_{\mu\nu} = \frac{\partial}{\partial x_\mu}A_\nu - \frac{\partial}{\partial x_\nu}A_\mu + [A_\mu, A_\nu], \qquad \mu, \nu = 1, \ldots, 4, \tag{1.95}$$

where $[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu$. If one defines the covariant derivative

$$D_\mu = \frac{\partial}{\partial x_\mu} + A_\mu, \tag{1.96}$$

then (1.95) is rewritten as

$$F_{\mu\nu} = [D_\mu, D_\nu]. \tag{1.97}$$

In explicit component form, one has

$$F_{\mu\nu} = gT^a F^a_{\mu\nu}, \tag{1.98}$$

where

$$F^a_{\mu\nu} = \frac{\partial}{\partial x_\mu}A^a_\nu - \frac{\partial}{\partial x_\nu}A^a_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu \tag{1.99}$$

and

$$\epsilon_{abc} = \begin{cases} +1 & \text{if } abc \text{ is an even permutation of (1,2,3)}, \\ -1 & \text{if } abc \text{ is an odd permutation of (1,2,3)}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.100}$$

We shall use the expression *static gauge field* to refer to gauge potentials that are independent of x_4 (x_4 to be considered as time), i.e.,

$$\frac{\partial}{\partial x_4}A_\mu(x) = 0, \qquad \mu = 1, \ldots, 4. \tag{1.101}$$

For gauge potentials that depend on all four coordinates $x_1, \ldots, x_4$, the action functional is defined by

$$S = \frac{1}{4}\int F^a_{\mu\nu}F^a_{\mu\nu}\, d^4x, \tag{1.102}$$

the integral taken over $\mathbb{R}^4$, while for static gauge fields we define the energy functional by

$$E = \frac{1}{4}\int F^a_{\mu\nu}F^a_{\mu\nu}\, d^3x, \tag{1.103}$$

whereas in (1.103) the integral is taken over $\mathbb{R}^3$.

The extremals of the action S (or of the energy E for static gauge fields) are found by standard calculus of variations techniques leading to the Euler–Lagrange equations

$$\frac{\partial}{\partial x_\mu}F_{\mu\nu} + [A_\mu, F_{\mu\nu}] \equiv [D_\mu, F_{\mu\nu}] = 0, \tag{1.104}$$

or in components

$$\frac{\partial}{\partial x_\mu}F^a_{\mu\nu} + g\epsilon_{abc}A^b_\mu F^c_{\mu\nu} = 0. \tag{1.105}$$

Equations (1.105) is a system of second order nonlinear partial differential equations for the twelve unknown functions A^a_μ, $a = 1, \ldots, 3$, $\mu = 1, \ldots, 4$, that seems hard to solve.

Then one introduces the *dual gauge field strength* $^*F_{\mu\nu}$ as

$$^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}, \tag{1.106}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor on M defined by

$$\epsilon_{\mu\nu\lambda\rho} = \begin{cases} +1 & \text{if } \mu\nu\lambda\rho \text{ is an even permutation of } (1,2,3,4), \\ -1 & \text{if } \mu\nu\lambda\rho \text{ is an odd permutation of } (1,2,3,4), \\ 0 & \text{otherwise.} \end{cases} \tag{1.107}$$

Since the fields D_μ (1.96) satisfy the Jacobi identity

$$[D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] = 0, \tag{1.108}$$

multiplication of (1.108) by $\epsilon_{\mu\nu\lambda\rho}$ and summation result in

$$[D_\mu, {}^*F_{\mu\nu}] = 0. \tag{1.109}$$

If we compare (1.104) with (1.109), we see that any gauge field which is *self-dual*, i.e., for which

$$^*F_{\mu\nu} = F_{\mu\nu}, \tag{1.110}$$

automatically satisfies (1.101). Consequently, the only equations to solve are (1.110) with $^*F_{\mu\nu}$ given by (1.106). This is a system of first order nonlinear partial differential equations.

Instanton solutions for general Yang–Mills equations and monopole solutions for static gauge fields satisfy (1.110) under the condition that S (1.102) or E (1.103) are finite.

Written in components, (1.110) takes the form

$$F_{12} = F_{34}, \quad F_{13} = -F_{24}, \quad F_{14} = F_{23}. \tag{1.111}$$

So in components, the self-dual Yang–Mills equations are described as a system of nine nonlinear partial differential equations,

$$\begin{aligned}
-A^1_{4,1} + A^1_{3,2} - A^1_{2,3} + A^1_{1,4} - g(A^2_1A^3_4 - A^2_2A^3_3 + A^2_3A^3_2 - A^2_4A^3_1) &= 0,\\
-A^2_{4,1} + A^2_{3,2} - A^2_{2,3} + A^2_{1,4} + g(A^1_1A^3_4 - A^1_2A^3_3 + A^1_3A^3_2 - A^1_4A^3_1) &= 0,\\
-A^3_{4,1} + A^3_{3,2} - A^3_{2,3} + A^3_{1,4} - g(A^1_1A^2_4 - A^1_2A^2_3 + A^1_3A^2_3 - A^1_4A^2_1) &= 0,\\
A^1_{3,1} + A^1_{4,2} - A^1_{1,3} - A^1_{2,4} + g(A^2_1A^3_3 + A^2_2A^3_4 - A^2_3A^3_1 - A^2_4A^3_2) &= 0,\\
A^2_{3,1} + A^2_{4,2} - A^2_{1,3} - A^2_{2,4} - g(A^1_1A^3_3 - A^1_2A^3_4 - A^1_3A^3_1 - A^1_4A^3_2) &= 0,\\
A^3_{3,1} + A^3_{4,2} - A^3_{1,3} - A^3_{2,4} + g(A^1_1A^2_3 + A^1_2A^2_4 - A^1_3A^2_1 - A^1_4A^2_2) &= 0,\\
A^1_{2,1} - A^1_{1,2} - A^1_{4,3} + A^1_{3,4} + g(A^2_1A^3_2 - A^2_2A^3_1 - A^2_3A^3_4 + A^2_4A^3_3) &= 0,\\
A^2_{2,1} - A^2_{1,2} - A^2_{4,3} + A^2_{3,4} - g(A^1_1A^3_2 - A^1_2A^3_1 - A^1_3A^3_4 + A^1_4A^3_3) &= 0,\\
A^3_{2,1} - A^3_{1,2} - A^3_{4,3} + A^3_{3,4} + g(A^1_1A^2_2 - A^1_2A^2_1 - A^1_3A^2_4 + A^1_4A^2_3) &= 0,
\end{aligned} \tag{1.112}$$

whereas in (1.112)

$$A^a_{\mu,\nu} = \frac{\partial}{\partial x_\nu} A^a_\mu, \qquad a = 1, \dots, 3,\ \mu, \nu = 1, \dots, 4. \tag{1.113}$$

Thus, we obtain a system $\mathcal{E} \subset J^1(\pi)$ for $\pi\colon \mathbb{R}^{12} \times \mathbb{R}^4 \to \mathbb{R}^4$.

6.2. Classical symmetries of self-dual Yang–Mills equations. In order to construct the Lie algebra of classical symmetries of (1.112), we start at a vector field V given by

$$V = V^{x_1} \frac{\partial}{\partial x_1} + \dots + V^{x_4} \frac{\partial}{\partial x_4} + V^{A^1_1} \frac{\partial}{\partial A^1_1} + \dots + V^{A^3_4} \frac{\partial}{\partial A^3_4}. \tag{1.114}$$

The condition for V to be a symmetry of equations (1.112) now leads to an overdetermined system of partial differential equations for the components $V^{x_1}, \dots, V^{x_4}$, $V^{A^1_1}, \dots, V^{A^3_4}$, which are functions dependent of the variables $x_1, \dots, x_4$, $A^1_1, \dots, A^3_4$.

The general solution of this overdetermined system of partial differential equations constitutes a Lie algebra of symmetries, generated by the vector fields

$$\begin{aligned}
V_1^{f^1} &= f^1_{x_1} \frac{\partial}{\partial A^1_1} + f^1_{x_2} \frac{\partial}{\partial A^1_2} + f^1_{x_3} \frac{\partial}{\partial A^1_3} + f^1_{x_4} \frac{\partial}{\partial A^1_4} \\
&\quad + f^1 g A^3_1 \frac{\partial}{\partial A^2_1} + f^1 g A^3_2 \frac{\partial}{\partial A^2_2} + f^1 g A^3_3 \frac{\partial}{\partial A^2_3} + f^1 g A^3_4 \frac{\partial}{\partial A^2_4} \\
&\quad - f^1 g A^2_1 \frac{\partial}{\partial A^3_1} - f^1 g A^2_2 \frac{\partial}{\partial A^3_2} - f^1 g A^2_3 \frac{\partial}{\partial A^3_3} - f^1 g A^2_4 \frac{\partial}{\partial A^3_4}, \\
V_2^{f^2} &= -f^2 g A^2_1 \frac{\partial}{\partial A^1_1} - f^2 g A^2_2 \frac{\partial}{\partial A^1_2} - f^2 g A^2_3 \frac{\partial}{\partial A^1_3} - f^2 g A^2_4 \frac{\partial}{\partial A^1_4} \\
&\quad + f^2 g A^1_1 \frac{\partial}{\partial A^2_1} + f^2 g A^1_2 \frac{\partial}{\partial A^2_2} + f^2 g A^1_3 \frac{\partial}{\partial A^2_3} + f^2 g A^1_4 \frac{\partial}{\partial A^2_4} \\
&\quad - f^2_{x_1} \frac{\partial}{\partial A^3_1} - f^2_{x_2} \frac{\partial}{\partial A^3_2} - f^2_{x_3} \frac{\partial}{\partial A^3_3} - f^2_{x_4} \frac{\partial}{\partial A^3_4}, \\
V_3^{f^3} &= f^3 g A^3_1 \frac{\partial}{\partial A^1_1} + f^3 g A^3_2 \frac{\partial}{\partial A^1_2} + f^3 g A^3_3 \frac{\partial}{\partial A^1_3} + f^3 g A^3_4 \frac{\partial}{\partial A^1_4} \\
&\quad - f^3_{x_1} \frac{\partial}{\partial A^2_1} - f^3_{x_2} \frac{\partial}{\partial A^2_2} - f^3_{x_3} \frac{\partial}{\partial A^2_3} - f^3_{x_4} \frac{\partial}{\partial A^2_4} \\
&\quad - f^3 g A^1_1 \frac{\partial}{\partial A^3_1} - f^3 g A^1_2 \frac{\partial}{\partial A^3_2} - f^3 g A^1_3 \frac{\partial}{\partial A^3_3} - f^3 g A^1_4 \frac{\partial}{\partial A^3_4}, \\
V_4 &= \frac{\partial}{\partial x_1},\ V_5 = \frac{\partial}{\partial x_2},\ V_6 = \frac{\partial}{\partial x_3},\ V_7 = \frac{\partial}{\partial x_4}, \\
V_8 &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + A^1_2 \frac{\partial}{\partial A^1_1} - A^1_1 \frac{\partial}{\partial A^1_2} \\
&\quad + A^2_2 \frac{\partial}{\partial A^2_1} - A^2_1 \frac{\partial}{\partial A^2_2} + A^3_2 \frac{\partial}{\partial A^3_1} - A^3_1 \frac{\partial}{\partial A^3_2},
\end{aligned}$$

$$
\begin{aligned}
V_9 = &-x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3} - A_3^1\frac{\partial}{\partial A_1^1} + A_1^1\frac{\partial}{\partial A_3^1}\\
&- A_3^2\frac{\partial}{\partial A_1^2} + A_1^2\frac{\partial}{\partial A_3^2} - A_3^3\frac{\partial}{\partial A_1^3} + A_1^3\frac{\partial}{\partial A_3^3},\\
V_{10} = &-x_4\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_4} - A_4^1\frac{\partial}{\partial A_1^1} + A_1^1\frac{\partial}{\partial A_4^1}\\
&- A_4^2\frac{\partial}{\partial A_1^2} + A_1^2\frac{\partial}{\partial A_4^2} - A_4^3\frac{\partial}{\partial A_1^3} + A_1^3\frac{\partial}{\partial A_4^3},\\
V_{11} = &-x_3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} - A_3^1\frac{\partial}{\partial A_2^1} + A_2^1\frac{\partial}{\partial A_3^1}\\
&- A_3^2\frac{\partial}{\partial A_2^2} + A_2^2\frac{\partial}{\partial A_3^2} - A_3^3\frac{\partial}{\partial A_2^3} + A_2^3\frac{\partial}{\partial A_3^3},\\
V_{12} = &\,x_4\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_4} + A_4^1\frac{\partial}{\partial A_2^1} - A_2^1\frac{\partial}{\partial A_4^1}\\
&+ A_4^2\frac{\partial}{\partial A_2^2} - A_2^2\frac{\partial}{\partial A_4^2} + A_4^3\frac{\partial}{\partial A_2^3} - A_2^3\frac{\partial}{\partial A_4^3},\\
V_{13} = &-x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4} - A_4^1\frac{\partial}{\partial A_3^1} + A_3^1\frac{\partial}{\partial A_4^1}\\
&- A_4^2\frac{\partial}{\partial A_3^2} + A_3^2\frac{\partial}{\partial A_4^2} - A_4^3\frac{\partial}{\partial A_3^3} + A_3^3\frac{\partial}{\partial A_4^3},\\
V_{14} = &\,x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}\\
&- A_1^1\frac{\partial}{\partial A_1^1} - A_2^1\frac{\partial}{\partial A_2^1} - A_3^1\frac{\partial}{\partial A_3^1} - A_4^1\frac{\partial}{\partial A_4^1}\\
&- A_1^2\frac{\partial}{\partial A_1^2} - A_2^2\frac{\partial}{\partial A_2^2} - A_3^2\frac{\partial}{\partial A_3^2} - A_4^2\frac{\partial}{\partial A_4^2}\\
&- A_1^3\frac{\partial}{\partial A_1^3} - A_2^3\frac{\partial}{\partial A_2^3} - A_3^3\frac{\partial}{\partial A_3^3} - A_4^3\frac{\partial}{\partial A_4^3},\\
V_{15} = &\,(-x_1^2 + x_2^2 + x_3^2 + x_4^2)\frac{\partial}{\partial x_1} - 2x_1x_2\frac{\partial}{\partial x_2} - 2x_1x_3\frac{\partial}{\partial x_3} - 2x_1x_4\frac{\partial}{\partial x_4}\\
&+ 2(x_1A_1^1 + x_2A_2^1 + x_3A_3^1 + x_4A_4^1)\frac{\partial}{\partial A_1^1} + 2(x_1A_2^1 - x_2A_1^1)\frac{\partial}{\partial A_2^1}\\
&+ 2(x_1A_3^1 - x_3A_1^1)\frac{\partial}{\partial A_3^1} + 2(x_1A_4^1 - x_4A_1^1)\frac{\partial}{\partial A_4^1}\\
&+ 2(x_1A_1^2 + x_2A_2^2 + x_3A_3^2 + x_4A_4^2)\frac{\partial}{\partial A_1^2} + 2(x_1A_2^2 - x_2A_1^2)\frac{\partial}{\partial A_2^2}\\
&+ 2(x_1A_3^2 - x_3A_1^2)\frac{\partial}{\partial A_3^2} + 2(x_1A_4^2 - x_4A_1^2)\frac{\partial}{\partial A_4^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(x_1A^3_1 + x_2A^3_2 + x_3A^3_3 + x_4A^3_4)\frac{\partial}{\partial A^3_1} + 2(x_1A^3_2 - x_2A^3_1)\frac{\partial}{\partial A^3_2}\\
&+ 2(x_1A^3_3 - x_3A^3_1)\frac{\partial}{\partial A^3_3} + 2(x_1A^3_4 - x_4A^3_1)\frac{\partial}{\partial A^3_4},\\
V_{16} &= -2x_2x_1\frac{\partial}{\partial x_1} + (x_1^2 - x_2^2 + x_3^2 + x_4^2)\frac{\partial}{\partial x_2} - 2x_2x_3\frac{\partial}{\partial x_3} - 2x_2x_4\frac{\partial}{\partial x_4}\\
&+ 2(-x_1A^1_2 + x_2A^1_1)\frac{\partial}{\partial A^1_1} + 2(x_1A^1_1 + x_2A^1_2 + x_3A^1_3 + x_4A^1_4)\frac{\partial}{\partial A^1_2}\\
&+ 2(x_2A^1_3 - x_3A^1_2)\frac{\partial}{\partial A^1_3} + 2(x_2A^1_4 - x_4A^1_2)\frac{\partial}{\partial A^1_4}\\
&+ 2(-x_1A^2_2 + x_2A^2_1)\frac{\partial}{\partial A^2_1} + 2(x_1A^2_1 + x_2A^2_2 + x_3A^2_3 + x_4A^2_4)\frac{\partial}{\partial A^2_2}\\
&+ 2(x_2A^2_3 - x_3A^2_2)\frac{\partial}{\partial A^2_3} + 2(x_2A^2_4 - x_4A^2_2)\frac{\partial}{\partial A^2_4}\\
&+ 2(-x_1A^3_2 + x_2A^3_1)\frac{\partial}{\partial A^3_1} + 2(x_1A^3_1 + x_2A^3_2 + x_3A^3_3 + x_4A^3_4)\frac{\partial}{\partial A^3_2}\\
&+ 2(x_2A^3_3 - x_3A^3_2)\frac{\partial}{\partial A^3_3} + 2(x_2A^3_4 - x_4A^3_2)\frac{\partial}{\partial A^3_4},\\
V_{17} &= -2x_3x_1\frac{\partial}{\partial x_1} - 2x_3x_2\frac{\partial}{\partial x_2} + (x_1^2 + x_2^2 - x_3^2 + x_4^2)\frac{\partial}{\partial x_3} - 2x_3x_4\frac{\partial}{\partial x_4}\\
&+ 2(-x_1A^1_3 + x_3A^1_1)\frac{\partial}{\partial A^1_1} + 2(-x_2A^1_3 + x_3A^1_2)\frac{\partial}{\partial A^1_2}\\
&+ 2(x_1A^1_1 + x_2A^1_2 + x_3A^1_3 + x_4A^1_4)\frac{\partial}{\partial A^1_3} + 2(x_3A^1_4 - x_4A^1_3)\frac{\partial}{\partial A^1_4}\\
&+ 2(-x_1A^2_3 + x_3A^2_1)\frac{\partial}{\partial A^2_1} + 2(-x_2A^2_3 + x_3A^2_2)\frac{\partial}{\partial A^2_2}\\
&+ 2(x_1A^2_1 + x_2A^2_2 + x_3A^2_3 + x_4A^2_4)\frac{\partial}{\partial A^2_3} + 2(x_3A^2_4 - x_4A^2_3)\frac{\partial}{\partial A^2_4}\\
&+ 2(-x_1A^3_3 + x_3A^3_1)\frac{\partial}{\partial A^3_1} + 2(-x_2A^3_3 + x_3A^3_2)\frac{\partial}{\partial A^3_2}\\
&+ 2(x_1A^3_1 + x_2A^3_2 + x_3A^3_3 + x_4A^3_4)\frac{\partial}{\partial A^3_3} + 2(x_3A^3_4 - x_4A^3_3)\frac{\partial}{\partial A^3_4},\\
V_{18} &= -2x_4x_1\frac{\partial}{\partial x_1} - 2x_4x_2\frac{\partial}{\partial x_2} - 2x_4x_3\frac{\partial}{\partial x_3} + (x_1^2 + x_2^2 + x_3^2 - x_4^2)\frac{\partial}{\partial x_4}\\
&+ 2(-x_1A^1_4 + x_4A^1_1)\frac{\partial}{\partial A^1_1} + 2(-x_2A^1_4 + x_4A^1_2)\frac{\partial}{\partial A^1_2}\\
&+ 2(-x_3A^1_4 + x_4A^1_3)\frac{\partial}{\partial A^1_3} + 2(x_1A^1_1 + x_2A^1_2 + x_3A^1_3 + x_4A^1_4)\frac{\partial}{\partial A^1_4}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(-x_1A_4^2 + x_4A_1^2)\frac{\partial}{\partial A_1^2} + 2(-x_2A_4^2 + x_4A_2^2)\frac{\partial}{\partial A_2^2} \\
&+ 2(-x_3A_4^2 + x_4A_3^2)\frac{\partial}{\partial A_3^2} + 2(x_1A_1^2 + x_2A_2^2 + x_3A_3^2 + x_4A_4^2)\frac{\partial}{\partial A_4^2} \\
&+ 2(-x_1A_4^3 + x_4A_1^3)\frac{\partial}{\partial A_1^3} + 2(-x_2A_4^3 + x_4A_2^3)\frac{\partial}{\partial A_2^3} \\
&+ 2(-x_3A_4^3 + x_4A_3^3)\frac{\partial}{\partial A_3^3} + 2(x_1A_1^3 + x_2A_2^3 + x_3A_3^3 + x_4A_4^3)\frac{\partial}{\partial A_4^3}.
\end{aligned}
\tag{1.115}
$$

The functions F^1, F^2, F^3 in the symmetries V_1, V_2, V_3 are arbitrary, depending on the variables x_1, x_2, x_3, x_4. The vector fields V_1, V_2, V_3 are just the generators of the *gauge transformations.*

The vector fields V_4, V_5, V_6, V_7 are generators of translations while the fields $V_8, \dots, V_{13}$ refer to infinitesimal rotations in $\mathbb{R}^4$, $X_4, \dots, X_{18}$ being the infinitesimal generators of the conformal group.

6.3. Instanton solutions. In order to construct invariant solutions associated to symmetries of the self-dual Yang–Mills equations (1.112), we start from the vector fields X_1, X_2, X_3 defined by

$$
\begin{aligned}
X_1 &= V_8 + V_1^{f_1^1} + V_2^{f_1^2} + V_3^{f_1^3}, \\
X_2 &= V_9 + V_1^{f_2^1} + V_2^{f_2^2} + V_3^{f_2^3}, \\
X_3 &= V_{10} + V_1^{f_3^1} + V_2^{f_3^2} + V_3^{f_3^3},
\end{aligned}
\tag{1.116}
$$

i.e., we take a combination of a rotation and a special choice for the

gauge transformations choosing particular values f_i^j of arbitrary functions f^j. We also construct commutators of the vector fields X_1, X_2, X_3,

$$
[X_1, X_2], \quad [X_1, X_3], \quad [X_2, X_3]
\tag{1.117}
$$

and make the following choice for the gauge transformations

$$
\begin{aligned}
&f_1^1 = 0, && f_1^2 = -1, && f_1^3 = 0, \\
&f_2^1 = 0, && f_2^2 = 0, && f_2^3 = -1, \\
&f_3^1 = -1, && f_3^2 = 0, && f_3^3 = 0.
\end{aligned}
\tag{1.118}
$$

In order to derive invariant solutions (see equations (1.40) on p. 28), we impose the additional conditions. Namely, we compute generating functions

$$
(\varphi_i)_\mu^j = Y_i \lrcorner\, \omega_{A_\mu^j}, \qquad j = 1, \dots, 3, \quad \mu = 1, \dots, 4,
\tag{1.119}
$$

whereas in (1.119) $\omega_{A_\mu^j}$ is the contact form associated to A_μ^j, i.e.,

$$
\omega_{A_\mu^j} = dA_\mu^j - A_{\mu,\nu}^j\, dx_\nu,
$$

while Y_i refers to the fields X_1, X_2, X_3, $[X_1, X_2]$, $[X_1, X_3]$, $[X_2, X_3]$. Then we impose additional equations

$$\left(\varphi_i\right)^j_\mu(x_1, \dots, x_4, \dots, A^j_\mu, \dots, A^j_{\mu\nu}, \dots) = 0 \tag{1.120}$$

and solve them together with the initial system. From conditions (1.119) we arrive at a system of $6 \times 12 = 72$ equations.

The resulting system can be solved in a straightforward way, leading to the following intermediate presentation

$$\begin{array}{llll} A^1_1 = x_4 F(r), & A^1_2 = x_3 F(r), & A^1_3 = -x_2 F(r), & A^1_4 = -x_1 F(r), \\ A^2_1 = -x_3 F(r), & A^2_2 = x_4 F(r), & A^2_3 = x_1 F(r), & A^2_4 = -x_2 F(r), \\ A^3_1 = x_2 F(r), & A^3_2 = -x_1 F(r), & A^3_3 = x_4 F(r), & A^3_4 = -x_3 F(r), \end{array} \tag{1.121}$$

where

$$r = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}. \tag{1.122}$$

When obtaining the monopole solution (see below), we shall discuss in some more detail how to solve a system of partial differential equations like (1.120). Substitution of (1.122) in (1.95) yields an *ordinary* differential equation for the function $F(r)$, i.e.,

$$\frac{dF(r)}{dr} + grF(r)^2 = 0, \tag{1.123}$$

the solution of which is given by

$$F(r) = \frac{2g^{-1}}{r^2 + C}, \tag{1.124}$$

C being a constant. The result (1.124) is just the *Belavin–Polyakov–Schwartz–Tyupkin instanton* solution!

More general, if we choose

$$f^2_1 = \pm 1, \quad f^3_2 = \pm 1, \quad f^1_3 = \pm 1, \tag{1.125}$$

and

$$f^2_1 f^3_2 f^1_3 = -1, \tag{1.126}$$

or equivalently

$$f^1_3 = -f^2_1 f^3_2, \tag{1.127}$$

we arrive at

$$\begin{array}{llll} A^1_1 = x_4 F(r), & A^1_2 = x_3 F(r), & A^1_3 = -x_2 F(r), & A^1_4 = -x_1 F(r), \\ A^2_1 = x_3 F(r) f^2_1, & A^2_2 = -x_4 F(r) f^2_1, & A^2_3 = -x_1 F(r) f^2_1, & A^2_4 = -x_2 F(r) f^2_1, \\ A^3_1 = -x_2 F(r) f^2_3, & A^3_2 = x_1 F(r) f^2_3, & A^3_3 = -x_4 F(r) f^2_3, & A^3_4 = -x_3 F(r) f^2_3, \end{array} \tag{1.128}$$

while (1.128) with $F(r)$ has to satisfy (1.112), which results in

$$\frac{\partial F(r)}{\partial r} + grf_1^2 f_3^2 F(r)^2 = 0. \tag{1.129}$$

Choosing f_1^2, f_2^3, f_3^1 as in (1.125) but with

$$f_1^2 f_2^3 f_3^1 = +1, \tag{1.130}$$

then the result is

$$\begin{aligned}
&A_1^1 = x_4 F(r), && A_2^1 = -\, x_3 F(r), && A_3^1 = x_2 F(r), && A_4^1 = -\, x_1 F(r),\\
&A_1^2 = -x_3 F(r) f_1^2, && A_2^2 = -\, x_4 F(r) f_1^2, && A_3^2 = x_1 F(r) f_1^2, && A_4^2 = x_2 F(r) f_1^2,\\
&A_1^3 = x_2 F(r) f_3^2, && A_2^3 = -\, x_1 F(r) f_3^2, && A_3^3 = -x_4 F(r) f_3^2, && A_4^3 = x_3 F(r) f_3^2,
\end{aligned} \tag{1.131}$$

while $F(r)$ has to satisfy

$$r\frac{dF(r)}{dr} + 4F(r) + gr^2 f_1^2 f_2^3 F(r)^2 = 0. \tag{1.132}$$

The solution of (1.132)

$$F(r) = -\frac{2}{g}(f_1^2 f_2^3)^{-1}\frac{a^2}{(r^2+a^2)r^2} \tag{1.133}$$

together with (1.131) is just the *'t Hooft instanton* solution with instanton number $k = 1$. This solution can be obtained from (1.124) by a gauge transformation.

6.4. Classical symmetries for static gauge fields. The equations for the static $SU(2)$ gauge field are described by (1.109) and (1.101). The symmetries for the static gauge field are obtained from those for the time-dependent case or straightforwardly in the following way . The respective computations then results in the following Lie algebra of symmetries for the static self-dual $SU(2)$ Yang–Mills equations

$$\begin{aligned}
V_1^{C^1} ={}& C^1_{x_1}\frac{\partial}{\partial A_1^1} + C^1_{x_2}\frac{\partial}{\partial A_2^1} + C^1_{x_3}\frac{\partial}{\partial A_3^1}\\
&+ C^1 g A_1^3 \frac{\partial}{\partial A_1^2} + C^1 g A_2^3 \frac{\partial}{\partial A_2^2} + C^1 g A_3^3 \frac{\partial}{\partial A_3^2} + C^1 g A_4^3 \frac{\partial}{\partial A_4^2}\\
&- C^1 g A_1^2 \frac{\partial}{\partial A_1^3} - C^1 g A_2^2 \frac{\partial}{\partial A_2^3} - C^1 g A_3^2 \frac{\partial}{\partial A_3^3} - C^1 g A_4^2 \frac{\partial}{\partial A_4^3},\\
V_2^{C^2} ={}& -C^2 g A_1^2 \frac{\partial}{\partial A_1^1} - C^2 g A_2^2 \frac{\partial}{\partial A_2^1} - C^2 g A_3^2 \frac{\partial}{\partial A_3^1} - C^2 g A_4^2 \frac{\partial}{\partial A_4^1}\\
&+ C^2 g A_1^1 \frac{\partial}{\partial A_1^2} + C^2 g A_2^1 \frac{\partial}{\partial A_2^2} + C^2 g A_3^1 \frac{\partial}{\partial A_3^2} + C^2 g A_4^1 \frac{\partial}{\partial A_4^2}\\
&- C^2_{x_1}\frac{\partial}{\partial A_1^3} - C^2_{x_2}\frac{\partial}{\partial A_2^3} - C^2_{x_3}\frac{\partial}{\partial A_3^3},
\end{aligned}$$

$$
\begin{aligned}
V_3^{C^3} &= C^3 g A_1^3 \frac{\partial}{\partial A_1^1} + C^3 g A_2^3 \frac{\partial}{\partial A_2^1} + C^3 g A_3^3 \frac{\partial}{\partial A_3^1} + C^3 g A_4^3 \frac{\partial}{\partial A_4^1} \\
&- C^3_{x_1} \frac{\partial}{\partial A_1^2} - C^3_{x_2} \frac{\partial}{\partial A_2^2} - C^3_{x_3} \frac{\partial}{\partial A_3^2} \\
&- C^3 g A_1^1 \frac{\partial}{\partial A_1^3} - C^3 g A_2^1 \frac{\partial}{\partial A_2^3} - C^3 g A_3^1 \frac{\partial}{\partial A_3^3} - C^3 g A_4^1 \frac{\partial}{\partial A_4^3}, \\
V_4 &= \frac{\partial}{\partial x_1}, \ V_5 = \frac{\partial}{\partial x_2}, \ V_6 = \frac{\partial}{\partial x_3}, \\
V_7 &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + A_2^1 \frac{\partial}{\partial A_1^1} - A_1^1 \frac{\partial}{\partial A_2^1} \\
&+ A_2^2 \frac{\partial}{\partial A_1^2} - A_1^2 \frac{\partial}{\partial A_2^2} + A_2^3 \frac{\partial}{\partial A_1^3} - A_1^3 \frac{\partial}{\partial A_2^3}, \\
V_8 &= -x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3} + A_1^1 \frac{\partial}{\partial A_3^1} - A_3^1 \frac{\partial}{\partial A_1^1} \\
&- A_3^2 \frac{\partial}{\partial A_1^2} + A_1^2 \frac{\partial}{\partial A_3^2} - A_3^3 \frac{\partial}{\partial A_1^3} + A_1^3 \frac{\partial}{\partial A_3^3}, \\
V_9 &= -x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} - A_3^1 \frac{\partial}{\partial A_2^1} + A_2^1 \frac{\partial}{\partial A_3^1} \\
&- A_3^2 \frac{\partial}{\partial A_2^2} + A_2^2 \frac{\partial}{\partial A_3^2} - A_3^3 \frac{\partial}{\partial A_2^3} + A_2^3 \frac{\partial}{\partial A_3^3}, \\
V_{10} &= x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \\
&- A_1^1 \frac{\partial}{\partial A_1^1} - A_2^1 \frac{\partial}{\partial A_2^1} - A_3^1 \frac{\partial}{\partial A_3^1} - A_4^1 \frac{\partial}{\partial A_4^1} \\
&- A_1^2 \frac{\partial}{\partial A_1^2} - A_2^2 \frac{\partial}{\partial A_2^2} - A_3^2 \frac{\partial}{\partial A_3^2} \\
&- A_4^2 \frac{\partial}{\partial A_4^2} - A_1^3 \frac{\partial}{\partial A_1^3} - A_2^3 \frac{\partial}{\partial A_2^3} - A_3^3 \frac{\partial}{\partial A_3^3} - A_4^3 \frac{\partial}{\partial A_4^3}.
\end{aligned} \tag{1.134}
$$

In (1.134) C^1, C^2, C^3 are arbitrary functions of $x_1, \ldots, x_3$, while V_1, V_2, V_3 themselves are just the generators of the gauge transformations. The fields V_7, V_8, V_9 generate rotations, while V_{10} is the generator of the scale change of variables.

6.5. Monopole solution. In order to construct invariant solutions to the static $SU(2)$ gauge field, we proceed in a way analogously to the one for the time-dependent field setting. We define the vector fields Y_1, Y_2, Y_3 by

$$
\begin{aligned}
Y_1 &= V_7 - V_2^1, \\
Y_2 &= V_8 - V_3^1, \\
Y_3 &= V_9 - V_1^1,
\end{aligned} \tag{1.135}
$$

i.e., put C^1, C^2, and C^3 equal to g^{-1}. It results in 36 equations for the functions A^a_μ:

$$
\begin{aligned}
1: &\quad A^2_1 + A^1_2 - x_2 A^1_{1,1} + x_1 A^1_{1,2} = 0, nonumber && (1.136)\\
2: &\quad -A^1_1 + A^2_2 - x_2 A^2_{1,1} + x_1 A^2_{1,2} = 0,\\
3: &\quad A^3_2 - x_2 A^3_{1,1} + x_1 A^3_{1,2} = 0,\\
4: &\quad -A^1_1 + A^2_2 - x_2 A^1_{2,1} + x_1 A^1_{2,2} = 0,\\
5: &\quad -A^2_1 - A^1_2 - x_2 A^2_{2,1} + x_1 A^2_{2,2} = 0,\\
6: &\quad -A^3_1 - x_2 A^3_{2,1} + x_1 A^3_{2,2} = 0,\\
7: &\quad A^2_3 - x_2 A^1_{3,1} + x_1 A^1_{3,2} = 0,\\
8: &\quad -A^1_3 - x_2 A^2_{3,1} + x_1 A^2_{3,2} = 0,\\
9: &\quad -x_2 A^3_{3,1} + x_1 A^3_{3,2} = 0,\\
10: &\quad A^2_4 - x_2 A^1_{4,1} + x_1 A^1_{4,2} = 0,\\
11: &\quad -A^1_4 - x_2 A^2_{4,1} + x_1 A^2_{4,2} = 0,\\
12: &\quad -x_2 A^3_{4,1} + x_1 A^3_{4,2} = 0, && (1.137)\\
13: &\quad -A^3_1 - A^1_3 - x_1 A^1_{1,3} + x_3 A^1_{1,1} = 0,\\
14: &\quad -A^2_3 - x_1 A^2_{1,3} + x_3 A^2_{1,1} = 0,\\
15: &\quad A^1_1 - A^3_3 - x_1 A^3_{1,3} + x_3 A^3_{1,1} = 0,\\
16: &\quad -A^3_2 - x_1 A^1_{2,3} + x_3 A^1_{2,1} = 0,\\
17: &\quad -x_1 A^2_{2,3} + x_3 A^2_{2,1} = 0,\\
18: &\quad A^1_2 - A^3_3 - x_1 A^3_{2,3} + x_3 A^3_{2,1} = 0,\\
19: &\quad A^1_1 - x_1 A^1_{3,3} + x_3 A^1_{3,1} = 0,\\
20: &\quad A^2_1 - x_1 A^2_{3,3} + x_3 A^2_{3,1} = 0,\\
21: &\quad A^3_1 + A^1_3 - x_1 A^3_{3,3} + x_3 A^3_{3,1} = 0,\\
22: &\quad -A^3_4 - x_1 A^1_{4,3} + x_3 A^1_{4,1} = 0, && (1.138)\\
23: &\quad -x_1 A^2_{4,3} + x_3 A^2_{4,1} = 0,\\
24: &\quad A^1_4 - x_1 A^3_{4,3} + x_3 A^3_{4,1} = 0, && (1.139)\\
25: &\quad -x_2 A^1_{1,3} + x_3 A^1_{1,2} = 0,\\
26: &\quad -A^3_1 - x_2 A^2_{1,3} + x_3 A^2_{1,2} = 0,\\
27: &\quad A^2_1 - x_2 A^3_{1,3} + x_3 A^3_{1,2} = 0,\\
28: &\quad -A^1_3 - x_2 A^1_{2,3} + x_3 A^1_{2,2} = 0,\\
29: &\quad -A^3_2 - A^2_3 - x_2 A^2_{2,3} + x_3 A^2_{2,2} = 0,\\
30: &\quad A^2_2 - A^3_3 - x_2 A^3_{2,3} + x_3 A^3_{2,2} = 0,
\end{aligned}
$$

$$
\begin{aligned}
31: &\quad A^1_2 - x_2A^1_{3,3} + x_3A^1_{3,2} = 0,\\
32: &\quad A^2_2 - A^3_3 - x_2A^2_{3,3} + x_3A^2_{3,2} = 0.\\
33: &\quad A^3_2 + A^2_3 - x_2A^3_{3,3} + x_3A^3_{3,2} = 0.\\
34: &\quad -x_2A^1_{4,3} + x_3A^1_{4,2} = 0.\\
35: &\quad -A^3_4 - x_2A^2_{4,3} + x_3A^2_{4,2} = 0.\\
36: &\quad A^2_4 - x_2A^3_{4,3} + x_3A^3_{4,2} = 0.
\end{aligned} \tag{1.140}
$$

We shall now indicate in more detail how to solve (1.140).

Note, that due to (1.137)

$$A^3_4 = F^1(r_{1,2}, x_3), \tag{1.141}$$

where

$$r_{1,2} = (x_1^2 + x_2^2)^{\frac{1}{2}}, \tag{1.142}$$

and due to (1.139)

$$A^1_4 = x_1\left(\frac{\partial F^1(r_{1,2}, x_3)}{\partial x_3} - \frac{x_3}{r_{1,2}}\frac{\partial F^1(r_{1,2}, x_3)}{\partial r_{1,2}}\right). \tag{1.143}$$

Now let

$$\frac{\partial F^1(r_{1,2}, x_3)}{\partial x_3} - \frac{x_3}{r_{1,2}}\frac{\partial F^1(r_{1,2}, x_3)}{\partial r_{1,2}} \stackrel{\text{def}}{=} H(r_{1,2}, x_3). \tag{1.144}$$

Substitution of (1.141) and (1.144) into (1.138) results in

$$F^1(r_{1,2}, x_3) = x_3H(r_{1,2}, x_3) + \frac{x_1^2}{r_{1,2}}\frac{\partial H(r_{1,2}, x_3)}{\partial r_{1,2}} - x_1^2\frac{\partial H(r_{1,2}, x_3)}{\partial x_3}, \tag{1.145}$$

or

$$F^1(r_{1,2}, x_3) - x_3H(r_{1,2}, x_3) = x_1^2\left(\frac{1}{r_{1,2}}\frac{\partial H(r_{1,2}, x_3)}{\partial r_{1,2}} - \frac{\partial H(r_{1,2}, x_3)}{\partial x_3}\right). \tag{1.146}$$

Differentiation of (1.146) with respect to x_1, x_2 yields

$$
\begin{gathered}
\frac{1}{r_{1,2}}\frac{\partial H(r_{1,2}, x_3)}{\partial r_{1,2}} - \frac{\partial H(r_{1,2}, x_3)}{\partial x_3} = 0,\\
F^1(r_{1,2}, x_3) = x_3H(r_{1,2}, x_3).
\end{gathered} \tag{1.147}
$$

From the second equation in (1.147) and equation (1.144) we obtain

$$H(r_{1,2}, x_3) = l(r), \tag{1.148}$$

where

$$r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \tag{1.149}$$

and finally, due to (1.147) and (1.123), one has

$$A^2_4 = x_1l(r), \quad A^2_4 = x_2l(r), \quad A^3_4 = x_3l(r). \tag{1.150}$$

Handling the remaining system in a similar way, a straightforward but tedious computation leads to the general solution of (1.123), i.e.,

$$
\begin{aligned}
A_1^1 &= \frac{1}{2}x_1^2 f(r) + k(r), & A_2^1 &= \frac{1}{2}x_1 x_2 f(r) - x_3 u(r),\\
A_3^1 &= \frac{1}{2}x_1 x_3 f(r) + x_2 u(r), & A_1^2 &= \frac{1}{2}x_1 x_2 l(r) + x_3 u(r),\\
A_2^2 &= \frac{1}{2}x_2^2 f(r) + k(r), & A_3^2 &= \frac{1}{2}x_2 x_3 f(r) - x_1 u(r),\\
A_1^3 &= \frac{1}{2}x_1 x_3 f(r) - x_2 u(r), & A_2^3 &= \frac{1}{2}x_2 x_3 f(r) + x_1 u(r),\\
A_3^3 &= \frac{1}{2}x_3^2 f(r) + k(r), & A_4^2 &= x_1 l(r),\\
A_4^2 &= x_2 l(r), & A_4^3 &= x_3 l(r),
\end{aligned}
\tag{1.151}
$$

where u, l, k, f are functions of r.

Substitution of (1.151) into (1.95) and (1.95) yields a system of three ordinary differential equations for the functions u, l, k, f:

$$
\begin{aligned}
&l' + u' - gru^2 - grul + \frac{1}{2}grfk = 0,\\
&r^2u' + 2ru - rl - gr^3ul + grk^2 + \frac{1}{2}gr^3fk = 0,\\
&k' - \frac{1}{2}rf - grku - grlk - \frac{1}{2}gr^3fu = 0.
\end{aligned}
\tag{1.152}
$$

If we choose

$$
f(r) = k(r) = 0, \quad l(r) = \frac{h(r)}{r}, \quad u(r) = -\frac{a(r)}{r}, \tag{1.153}
$$

we are led by (1.151), (1.153) to the *monopole solution* obtained by Prasad and Sommerfeld [**84**] by imposing the *ansatz* (1.151), (1.153).

CHAPTER 2

Higher symmetries and conservation laws

In this chapter, we specify general constructions described for infinite jets to *infinitely prolonged differential equations.* We describe basic structures existing on these objects, give an outline of differential calculus over them and introduce the notions of a *higher symmetry* and of a *conservation law.*

We also compute higher symmetries and conservation laws for some equations of mathematical physics.

1. Basic structures

Now we introduce the main object of our interest:

DEFINITION 2.1. The inverse limit $\operatorname{proj\,lim}_{l\to\infty} \mathcal{E}^l$ with respect to projections $\pi_{l+1,l}$ is called the *infinite prolongation* of the equation $\mathcal{E}$ and is denoted by $\mathcal{E}^\infty \subset J^\infty(\pi)$.

In the sequel, we shall mostly deal with formally integrable equations $\mathcal{E} \subset J^k(\pi)$ (see Definition 1.20 on p. 30), which means that all $\mathcal{E}^l$ are smooth manifolds while the mappings $\pi_{k+l+1,k+l}\colon \mathcal{E}^{l+1} \to \mathcal{E}^l$ are smooth locally trivial fiber bundles.

Infinite prolongations are objects of the category $\mathcal{M}^\infty$ (see Example 1.5 on p. 10). Hence, general approach exposed in Subsection 1.3 of Chapter 1 can be applied to them just in the same manner as it was done for manifolds of infinite jets. In this section, we give a brief outline of calculus over $\mathcal{E}^\infty$ and describe essential structures specific for infinite prolongations of differential equations.

1.1. Calculus. Let $\pi\colon E \to M$ be a vector bundle and $\mathcal{E} \subset J^k(\pi)$ be a k-th order differential equation. Then we have the embeddings $\varepsilon_l\colon \mathcal{E}^l \subset J^{k+l}(\pi)$ for all $l \geq 0$. We define a *smooth function* on $\mathcal{E}^l$ as the restriction $f\,|_{\mathcal{E}^l}$ of a smooth function $f \in \mathcal{F}_{k+l}(\pi)$. The set $\mathcal{F}_l(\mathcal{E})$ of all functions on $\mathcal{E}^l$ forms an $\mathbb{R}$-algebra in a natural way and $\varepsilon_l^*\colon \mathcal{F}_{k+l}(\pi) \to \mathcal{F}_l(\mathcal{E})$ is a homomorphism. In the case of formally integrable equations, the algebra $\mathcal{F}_l(\mathcal{E})$ coincides with $C^\infty(\mathcal{E}^l)$. Let $I_l \overset{\text{def}}{=} \ker \varepsilon_l^*$.

Due to commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{F}_{k+l}(\pi) & \xrightarrow{\varepsilon_l^*} & \mathcal{F}_l(\mathcal{E}) \\
\Big\downarrow{\scriptstyle \pi^*_{k+l+1,k+l}} & & \Big\downarrow{\scriptstyle \pi^*_{k+l+1,k+l}} \\
\mathcal{F}_{k+l+1}(\pi) & \xrightarrow{\varepsilon_{l+1}^*} & \mathcal{F}_{l+1}(\mathcal{E})
\end{array}$$

one has $I_l(\mathcal{E}) \subset I_{l+1}(\mathcal{E})$. Then $I(\mathcal{E}) = \bigcup_{l\geq 0} I_l(\mathcal{E})$ is an ideal in $\mathcal{F}(\pi)$ which is called the *ideal of the equation* $\mathcal{E}$. The *function algebra* on $\mathcal{E}^\infty$ is the quotient $\mathcal{F}(\mathcal{E}) = \mathcal{F}(\pi)/I(\mathcal{E})$ and coincides with $\operatorname{inj\,lim}_{l\to\infty} \mathcal{F}_l(\mathcal{E})$ with respect to the system of homomorphisms $\pi^*_{k+l+1,k+l}$. For all $l \geq 0$, we have the homomorphisms $\varepsilon_l^* \colon \mathcal{F}_l(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$. When $\mathcal{E}$ is formally integrable, they are monomorphic, but in any case the algebra $\mathcal{F}(\mathcal{E})$ is filtered by the images of ε_l^*.

Now, to construct differential calculus on $\mathcal{E}^\infty$, one needs the general algebraic scheme exposed in Chapter 4 and applied to the filtered algebra $\mathcal{F}(\mathcal{E})$. However, in the case of formally integrable equations, due to the fact that all $\mathcal{E}^l$ are smooth manifolds, this scheme may be simplified and combined with a purely geometrical approach (cf. similar constructions of Subsection 1.3 of Chapter 1).

Namely, *differential forms* in this case are defined as elements of the module $\Lambda^*(\mathcal{E}) \stackrel{\text{def}}{=} \bigcup_{l\geq 0} \Lambda^*(\mathcal{E}^l)$, where $\Lambda^*(\mathcal{E}^l)$ is considered to be embedded into $\Lambda^*(\mathcal{E}^{l+1})$ by $\pi^*_{k+l+1,k+l}$. A *vector field* on $\mathcal{E}^\infty$ is a derivation $X \colon \mathcal{F}(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$ agreed with filtration, i.e., such that $X(\mathcal{F}_l(\mathcal{E})) \subset \mathcal{F}_{l+\alpha}(\mathcal{E})$ for some integer $\alpha = \alpha(X) \in \mathbb{Z}$. Just like in the case $J^\infty(\pi)$, we define the *de Rham complex* over $\mathcal{E}^\infty$ and obtain "usual" relations between standard operations (contractions, de Rham differential and Lie derivatives).

In special coordinates the infinite prolongation of the equation $\mathcal{E}$ is determined by the system similar to (1.41) on p. 29 with the only difference that $|\sigma|$ is unlimited now. Thus, the ideal $I(\mathcal{E})$ is generated by the functions $D_\sigma F^j$, $|\sigma| \geq 0$, $j = 1, \dots, m$. From these remarks we obtain the following important fact.

EXAMPLE 2.1. Let $\mathcal{E}$ be a formally integrable equation. Then from the above said it follows that the ideal $I(\mathcal{E})$ is stable with respect to the action of the total derivatives D_i, $i = 1, \dots, n = \dim M$. Consequently, the action $D_i^{\mathcal{E}} \stackrel{\text{def}}{=} D_i\big|_{\mathcal{F}(\mathcal{E})} : \mathcal{F}(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$ is well defined and $D_i^{\mathcal{E}}$ are filtered derivations. We can reformulate it in other words by saying that the vector fields D_i are tangent to any infinite prolongation and thus determine vector fields on $\mathcal{E}^\infty$. We shall often skip the superscript $\mathcal{E}$ in the notation of the above defined restrictions.

The fact established in the last example plays a crucial role in the theory of infinite prolongations. We continue to discuss it in the next section.

To finish this one, let us make a remark concerning local coordinates. Let $\mathcal{E}$ be locally represented with equations (1.41). Assume that the latter is resolved in the form

$$u^{\alpha_j}_{\sigma^j} = f^j(x_1, \dots, x_n, \dots, u^\beta_\sigma, \dots), \quad j = 1, \dots, r,$$

in such a way that

(i) the set of functions $u^{\alpha_1}_{\sigma^1}, \dots, u^{\alpha_r}_{\sigma^r}$ has at the left-hand side the empty intersection with the set of functions u^β_σ at the left-hand side and

(ii) $u^{\alpha_i}_{\sigma^i+\tau} = u^{\alpha_j}_{\sigma^j+\tau'}$ for no τ, τ' unless $i = j$.

In this case, all coordinate functions in the system under consideration may be partitioned into two parts: those of the form $u^{\alpha_j}_{\sigma^j+\tau}$, $|\tau| \geq 0, j = 1, \dots, r$, and all others. We call the latter ones *internal coordinates* on $\mathcal{E}^\infty$. Note that all constructions of differential calculus over $\mathcal{E}^\infty$ can be expressed in terms of internal coordinates.

EXAMPLE 2.2. Consider a system of evolution equations of the form (1.22) (see p. 16). Then the functions $x_1, \dots, x_n, t, \dots, u^j_{\sigma_1,\dots,\sigma_n,0}$, $\sigma_i \geq 0$, $j = 1, \dots, m$, where $t = x_{n+1}$, may be taken for internal coordinates on $\mathcal{E}^\infty$. The total derivatives restricted onto $\mathcal{E}^\infty$ are expressed as

$$\begin{aligned} D_i &= \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \sum_{|\sigma|\geq 0} u^j_{\sigma+1_i} \frac{\partial}{\partial u_\sigma}, \ i = 1, \dots, n, \\ D_t &= \frac{\partial}{\partial t} + \sum_{j=1}^{n} \sum_{|\sigma|\geq 0} D_\sigma(f^j) \frac{\partial}{\partial u_\sigma} \end{aligned} \tag{2.1}$$

in these coordinates, while the Cartan forms are written down as

$$\omega^j_\sigma = du^j_\sigma - \sum_{i=1}^{n} u^j_{\sigma+1_i}\, dx_i - D_\sigma(f^j)\, dt, \tag{2.2}$$

where all multi-indices σ are of the form $\sigma = (\sigma_1, \dots, \sigma_n, 0)$.

1.2. Cartan distribution. Let $\pi\colon E \to M$ be a vector bundle and $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation.

DEFINITION 2.2. Let $\theta \in J^\infty(\pi)$. Then

(i) The *Cartan plane* $\mathcal{C}_\theta = \mathcal{C}_\theta(\pi) \subset T_\theta J^\infty(\pi)$ at θ is the linear envelope of tangent planes to all manifolds $j_\infty(\varphi)(M)$, $\varphi \in \Gamma(\pi)$, passing through the point θ.

(ii) If $\theta \in \mathcal{E}^\infty$, the intersection $\mathcal{C}_\theta(\mathcal{E}) \overset{\text{def}}{=} \mathcal{C}_\theta(\pi) \cap T_\theta\mathcal{E}^\infty$ is called *Cartan plane* of $\mathcal{E}^\infty$ at θ.

The correspondence $\theta \mapsto \mathcal{C}_\theta(\pi)$, $\theta \in J^\infty(\pi)$ (respectively, $\theta \mapsto \mathcal{C}_\theta(\mathcal{E}^\infty)$, $\theta \in \mathcal{E}^\infty$) is called the *Cartan distribution* on $J^\infty(\pi)$ (respectively, on $\mathcal{E}^\infty$).

The following result shows the crucial difference between the Cartan distributions on finite and infinite jets (or between those on finite and infinite prolongations).

PROPOSITION 2.1. *For any vector bundle $\pi\colon E \to M$ and a formally integrable equation $\mathcal{E} \subset J^k(\pi)$ one has:*

(i) *The Cartan plane $\mathcal{C}_\theta(\pi)$ is n-dimensional at any point $\theta \in J^\infty(\pi)$.*
(ii) *Any point $\theta \in \mathcal{E}^\infty$ is generic, i.e., $\mathcal{C}_\theta(\pi) \subset T_\theta\mathcal{E}^\infty$ and thus $\mathcal{C}_\theta(\mathcal{E}^\infty) = \mathcal{C}_\theta(J^\infty)$.*
(iii) *Both distributions, $\mathcal{C}(J^\infty)$ and $\mathcal{C}(\mathcal{E}^\infty)$, are integrable.*

PROOF. Let $\theta \in J^\infty(\pi)$ and $\pi_\infty(\theta) = x \in M$. Then the point θ completely determines all partial derivatives of any section $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$ such that its graph passes through θ. Consequently, all such graphs have a common tangent plane at this point which coincides with $\mathcal{C}_\theta(\pi)$. This proves the first statement.

To prove the second one, recall Example 2.1: locally, any vector field D_i is tangent to $\mathcal{E}^\infty$. But as it follows from (1.27) on p. 18, one has $\mathrm{i}_{D_i}\omega_\sigma^j = 0$ for any D_i and any Cartan form ω_σ^j. Hence, linear independent vector fields $D_1, \dots, D_n$ locally lie both in $\mathcal{C}(\pi)$ and in $\mathcal{C}(\mathcal{E}^\infty)$ which gives the result.

Finally, as it follows from the above said, the module

$$\mathcal{C}D(\pi) \stackrel{\text{def}}{=} \{X \in D(\pi) \mid X \text{ lies in } \mathcal{C}(\pi)\} \tag{2.3}$$

is locally generated by the fields $D_1, \dots, D_n$. But it is easily seen that $[D_\alpha, D_\beta] = 0$ for all $\alpha, \beta = 1, \dots, n$ and consequently $[\mathcal{C}D(\pi), \mathcal{C}D(\pi)] \subset \mathcal{C}D(\pi)$. The same reasoning is valid for

$$\mathcal{C}D(\mathcal{E}) \stackrel{\text{def}}{=} \{X \in D(\mathcal{E}^\infty) \mid X \text{ lies in } \mathcal{C}(\mathcal{E}^\infty)\}. \tag{2.4}$$

This finishes the proof of the proposition. $\square$

We shall describe now maximal integral manifolds of the Cartan distributions on $J^\infty(\pi)$ and $\mathcal{E}^\infty$.

PROPOSITION 2.2. *Maximal integral manifolds of the Cartan distribution $\mathcal{C}(\pi)$ are graph of $j_\infty(\varphi)$, $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$.*

PROOF. Note first that graphs of infinite jets are integral manifolds of the Cartan distribution of maximal dimension (equaling to n) and that any integral manifold projects onto $J^k(\pi)$ and M without singularities.

Let now $N \subset J^\infty(\pi)$ be an integral manifold and $N^k \stackrel{\text{def}}{=} \pi_{\infty,k}N \subset J^k(\pi)$, $N' \stackrel{\text{def}}{=} \pi_\infty N \subset M$. Hence, there exists a diffeomorphism $\varphi'\colon N' \to N^0$ such that $\pi \circ \varphi' = \mathrm{id}_{N'}$. Then by the Whitney theorem on extension for smooth functions [**71**], there exists a local section $\varphi\colon M \to E$ satisfying $\varphi|_{N'} = \varphi'$ and $j_k(\varphi)(M) \supset N^k$ for all $k > 0$. Consequently, $j_\infty(\varphi)(M) \supset N$. $\square$

COROLLARY 2.3. *Maximal integral manifolds of the Cartan distribution on $\mathcal{E}^\infty$ coincide locally with graphs of infinite jets of solutions.*

We use the results obtained here in the next subsection.

1.3. Cartan connection. Consider a point $\theta \in J^\infty(\pi)$ and let $x = \pi_\infty(\theta) \in M$. Let v be a tangent vector to M at the point x. Then, since the Cartan plane $\mathcal{C}_\theta$ isomorphically projects onto $T_x M$, there exists a unique tangent vector $\mathcal{C}v \in T_\theta J^\infty(\pi)$ such that $\pi_{\infty,*}(\mathcal{C}v) = v$. Hence, for any vector field $X \in D(M)$ we can define a vector field $\mathcal{C}X \in D(\pi)$ by setting $(\mathcal{C}X)_\theta \stackrel{\text{def}}{=} \mathcal{C}(X_{\pi_\infty(\theta)})$. Then, by construction, the field $\mathcal{C}X$ is projected by $\pi_{\infty,*}$ to X while the correspondence $\mathcal{C}\colon D(M) \to D(\pi)$ is $C^\infty(M)$-linear. In other words, this correspondence is a connection in the bundle $\pi_\infty\colon J^\infty(\pi) \to M$.

DEFINITION 2.3. The connection $\mathcal{C}\colon D(M) \to D(\pi)$ defined above is called the *Cartan connection* in $J^\infty(\pi)$.

Let now $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation. Then, due to the fact that $\mathcal{C}_\theta(\mathcal{E}^\infty) = \mathcal{C}_\theta(\pi)$ at any point $\theta \in \mathcal{E}^\infty$, we see that the fields $\mathcal{C}X$ are tangent to $\mathcal{E}^\infty$ for all vector fields $X \in D(M)$. Thus we obtain the Cartan connection in the bundle $\pi_\infty\colon \mathcal{E}^\infty \to M$ which is denoted by the same symbol $\mathcal{C}$.

Let $x_1, \dots, x_n, \dots, u^j_\sigma, \dots$ be special coordinates in $J^\infty(\pi)$ and $X = X_1 \partial/\partial x_1 + \dots + X_n \partial/\partial x_n$ be a vector field on M represented in this coordinate system. Then the field $\mathcal{C}X$ is to be of the form $\mathcal{C}X = X + X^v$, where $X^v = \sum_{j,\sigma} X^j_\sigma \partial/\partial u^j_\sigma$ is a π_∞-vertical field. The defining conditions $\mathrm{i}_{\mathcal{C}X}\omega^j_\sigma = 0$, where ω^j_σ are the Cartan forms on $J^\infty(\pi)$, imply

$$\mathcal{C}X = \sum_{i=1}^n X_i \left(\frac{\partial}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma+1_i} \frac{\partial}{\partial u^j_\sigma} \right) = \sum_{i=1}^n X_i D_i. \tag{2.5}$$

In particular, we see that $\mathcal{C}(\partial/\partial x_i) = D_i$, i.e., total derivatives are just liftings to $J^\infty(\pi)$ of the corresponding partial derivatives by the Cartan connection.

To obtain a similar expression for the Cartan connection on $\mathcal{E}^\infty$, it needs only to obtain coordinate representation for total derivatives in internal coordinates. For example, in the case of equations (1.22) (see p. 16) we have

$$\mathcal{C}\left(\sum_{i=1}^n X_i \frac{\partial}{\partial x_i} + T\frac{\partial}{\partial t} \right) = \sum_{i=1}^n X_i D_i + T D_t,$$

where $D_1, \dots, D_n, D_t$ are given by formulas (2.1) and $X_i, T \in C^\infty(M)$ are the coefficients of the field $X \in D(M)$.

Consider the following construction now. Let V be a vector field on $\mathcal{E}^\infty$ and $\theta \in \mathcal{E}^\infty$ be a point. Then the vector V_θ can be projected parallel to the Cartan plane $\mathcal{C}_\theta$ onto the fiber of the projection $\pi_\infty\colon \mathcal{E}^\infty \to M$ passing through θ. Thus we get a vertical vector field V^v. Hence, for any $f \in \mathcal{F}(\mathcal{E})$ a differential one-form $U_\mathcal{E}(f) \in \Lambda^1(\mathcal{E})$ is defined by

$$\mathrm{i}_V(U_\mathcal{E}(f)) \stackrel{\text{def}}{=} V^v(f), \quad V \in D(\mathcal{E}). \tag{2.6}$$

The correspondence $f \mapsto U_{\mathcal{E}}(f)$ is a derivation of the algebra $\mathcal{F}(\mathcal{E})$ with the values in the $\mathcal{F}(\mathcal{E})$-module $\Lambda^1(\mathcal{E})$, i.e.,

$$U_{\mathcal{E}}(fg) = fU_{\mathcal{E}}(g) + gU_{\mathcal{E}}(f)$$

for all $f, g \in \mathcal{F}(\mathcal{E})$. This correspondence contains all essential data about the equation $\mathcal{E}$.

DEFINITION 2.4. The derivation $U_{\mathcal{E}} \colon \mathcal{F}(\mathcal{E}) \to \Lambda^1(\mathcal{E})$ is called the *structural element* of the equation $\mathcal{E}$.

For the "empty" equation, i.e., in the case $\mathcal{E}^\infty = J^\infty(\pi)$, the structural element U_π is locally represented in the form

$$U_\pi = \sum_{j,\sigma} \omega_\sigma^j \otimes \frac{\partial}{\partial u_\sigma^j}, \tag{2.7}$$

where ω_σ^j are the Cartan forms on $J^\infty(\pi)$. To obtain the expression in the general case, one needs to rewrite (2.7) in local coordinates. For example, in the case of evolution equations we get the same expression with $\sigma = (\sigma_1, \dots, \sigma_n, 0)$ and the forms ω_σ^j given by (2.2). Contrary to the Cartan forms, the structural element is independent of local coordinates.

We shall now give a "more algebraic" version of the Cartan connection definition.

PROPOSITION 2.4. *For any vector field $X \in D(M)$, the equality*

$$j_\infty(\varphi)^*(\mathcal{C}X(f)) = X(j_\infty(\varphi)^*(f)) \tag{2.8}$$

takes place, where $f \in \mathcal{F}(\pi)$ and $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$. Equality (2.8) *uniquely determines the Cartan connection in $J^\infty(\pi)$.*

PROOF. Both statements follow from the fact that in special coordinates the right-hand side of (2.8) is of the form

$$\sum_{j,\sigma} \left.\frac{\partial f}{\partial u_\sigma^j}\right|_{j_\infty(\varphi)(M)} X\left(\frac{\partial^{|\sigma|}\varphi^j}{\partial x_\sigma}\right).$$

$\square$

COROLLARY 2.5. *The Cartan connection in $\mathcal{E}^\infty$ is flat, i.e.,*

$$\mathcal{C}[X, Y] = [\mathcal{C}X, \mathcal{C}Y]$$

for any $X, Y \in D(M)$.

PROOF. Consider the case $\mathcal{E}^\infty = J^\infty(\pi)$. Then from Proposition 2.4 we have

$$\begin{aligned} j_\infty(\varphi)^*(\mathcal{C}[X, Y](f)) &= [X, Y](j_\infty(\varphi)^*(f)) \\ &= X(Y(j_\infty(\varphi)^*(f))) - Y(X(j_\infty(\varphi)^*(f))) \end{aligned}$$

for any $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$, $f \in \mathcal{F}(\pi)$. On the other hand,

$$\begin{aligned} j_\infty(\varphi)^*([\mathcal{C}X,\mathcal{C}Y](f)) &= j_\infty(\varphi)^*(\mathcal{C}X(\mathcal{C}Y(f)) - \mathcal{C}Y(\mathcal{C}X(f))) \\ &= X(j_\infty(\varphi)^*(Y(f))) - Y(j_\infty(\varphi)^*(\mathcal{C}X(f))) \\ &= X(Y(j_\infty(\varphi)^*(f))) - Y(X(j_\infty(\varphi)^*(f))) \end{aligned}$$

To prove the statement for an arbitrary formally integrable equation $\mathcal{E}$, it suffices to note that the Cartan connection in $\mathcal{E}^\infty$ is obtained by restricting the fields $\mathcal{C}X$ onto infinite prolongation of $\mathcal{E}$. ☐

The construction of Proposition 2.4 can be generalized.

1.4. $\mathcal{C}$-differential operators. Let $\pi\colon E \to M$ be a vector bundle and $\xi_1\colon E_1 \to M$, $\xi_2\colon E_2 \to M$ be another two vector bundles.

DEFINITION 2.5. Let $\Delta\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ be a linear differential operator. The *lifting* $\mathcal{C}\Delta\colon \mathcal{F}(\pi,\xi_1) \to \mathcal{F}(\pi,\xi_2)$ of the operator Δ is defined by

$$j_\infty(\varphi)^*(\mathcal{C}\Delta(f)) = \Delta(j_\infty(\varphi)^*(f)), \tag{2.9}$$

where $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$ and $f \in \mathcal{F}(\pi,\xi_1)$ are arbitrary sections.

Immediately from the definition, we obtain the following properties of operators $\mathcal{C}\Delta$:

PROPOSITION 2.6. *Let $\pi\colon E \to M$, $\xi_i\colon E_i \to M$, $i = 1,2,3$, be vector bundles. Then*

(i) *For any $C^\infty(M)$-linear differential operator $\Delta\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$, the operator $\mathcal{C}\Delta$ is an $\mathcal{F}(\pi)$-linear differential operator of the same order.*

(ii) *For any $\Delta,\Box\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ and $f,g \in \mathcal{F}(\pi)$, one has*
$$\mathcal{C}(f\Delta + g\Box) = f\mathcal{C}\Delta + g\mathcal{C}\Box.$$

(iii) *For $\Delta_1\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$, $\Delta_2\colon \Gamma(\xi_2) \to \Gamma(\xi_3)$, one has*
$$\mathcal{C}(\Delta_2 \circ \Delta_1) = \mathcal{C}\Delta_2 \circ \mathcal{C}\Delta_1.$$

From this proposition and from Proposition 2.4 it follows that if Δ is a scalar differential operator $C^\infty(M) \to C^\infty(M)$ locally represented as $\Delta = \sum_\sigma a_\sigma \partial^{|\sigma|}/\partial x_\sigma$, $a_\sigma \in C^\infty(M)$, then

$$\mathcal{C}\Delta = \sum_\sigma a_\sigma D_\sigma \tag{2.10}$$

in the corresponding special coordinates. If $\Delta = \|\Delta_{ij}\|$ is a matrix operator, then $\mathcal{C}\Delta = \|\mathcal{C}\Delta_{ij}\|$.

From Proposition 2.6 it follows that $\mathcal{C}\Delta$ may be understood as a differential operator acting from sections of the bundle π to linear differential operators from $\Gamma(\xi_1)$ to $\Gamma(\xi_2)$. This observation is generalized as follows.

DEFINITION 2.6. An $\mathcal{F}(\pi)$-linear differential operator $\Delta\colon \mathcal{F}(\pi,\xi_1) \to \mathcal{F}(\pi,\xi_2)$ is called a *$\mathcal{C}$-differential operator*, if it admits restriction onto graphs of infinite jets, i.e., if for any section $\varphi \in \Gamma(\pi)$ there exists an operator $\Delta_\varphi\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ such that

$$j_\infty(\varphi)^*(\Delta(f)) = \Delta_\varphi(j_\infty(\varphi)^*(f)) \tag{2.11}$$

for all $f \in \mathcal{F}(\pi, \xi_1)$.

Thus, $\mathcal{C}$-differential operators are nonlinear differential operators taking their values in $C^\infty(M)$-modules of linear differential operators. The following proposition gives a complete description of such operators.

PROPOSITION 2.7. *Let π, ξ_1, ξ_2 be vector bundles over M. Then any $\mathcal{C}$-differential operator $\Delta\colon \mathcal{F}(\pi, \xi_1) \to \mathcal{F}(\pi, \xi_2)$ can be presented in the form*

$$\Delta = \sum_\alpha a_\alpha \mathcal{C}\Delta_\alpha, \quad a_\alpha \in \mathcal{F}(\pi),$$

where Δ_α are linear differential operators acting from $\Gamma(\xi_1)$ to $\Gamma(\xi_2)$.

PROOF. Recall first that we consider the filtered theory; in particular, there exists an integer l such that $\Delta(\mathcal{F}_k(\pi, \xi_1)) \subset \mathcal{F}_{k+l}(\pi, \xi_2)$ for all k. Consequently, since $\Gamma(\xi_1)$ is embedded into $\mathcal{F}_0(\pi, \xi_1)$, we have $\Delta(\Gamma(\xi_1)) \subset \mathcal{F}_l(\pi, \xi_2)$ and the restriction $\bar{\Delta} = \Delta\big|_{\Gamma(\xi_1)}$ is a $C^\infty(M)$-differential operator taking its values in $\mathcal{F}_l(\pi, \xi_2)$. Then one can easily see that the equality $\Delta_\varphi = j_\infty(\varphi)^* \circ \bar{\Delta}$ holds, where $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$ and Δ_φ is the operator from (2.11). It means that any $\mathcal{C}$-differential Δ operator is completely determined by its restriction $\bar{\Delta}$.

On the other hand, the operator $\bar{\Delta}$ is represented in the form $\bar{\Delta} = \sum_\alpha a_\alpha \Delta_\alpha$, $a_\alpha \in \mathcal{F}_l(\pi)$ and $\Delta_\alpha\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ being $C^\infty(M)$-linear differential operators. Let us define $\mathcal{C}\bar{\Delta} \stackrel{\text{def}}{=} \sum_\alpha a_\alpha \mathcal{C}\Delta_\alpha$. Then the difference $\Delta - \mathcal{C}\bar{\Delta}$ is a $\mathcal{C}$-differential operator such that its restriction onto $\Gamma(\xi_1)$ vanishes. Therefore $\Delta = \mathcal{C}\bar{\Delta}$. □

REMARK 2.1. From the result obtained it follows that $\mathcal{C}$-differential operators are operators "in total derivatives". By this reason, they are called *total differential operators* sometimes.

COROLLARY 2.8. *$\mathcal{C}$-differential operators admit restrictions onto infinite prolongations: if $\Delta\colon \mathcal{F}(\pi, \xi_1) \to \mathcal{F}(\pi, \xi_2)$ is a $\mathcal{C}$-differential operator and $\mathcal{E} \subset J^k(\pi)$ is a k-th order equation, then there exists a linear differential operator $\Delta_{\mathcal{E}}\colon \mathcal{F}(\mathcal{E}, \xi_1) \to \mathcal{F}(\mathcal{E}, \xi_2)$ such that*

$$\varepsilon^* \circ \Delta = \Delta_{\mathcal{E}} \circ \varepsilon^*,$$

where $\varepsilon\colon \mathcal{E}^\infty \hookrightarrow J^\infty(\pi)$ is the natural embedding.

PROOF. The result immediately follows from Example 2.1 and Proposition 2.7. □

We shall now consider an example which will play a very important role in the sequel.

EXAMPLE 2.3. Let $\xi_1 = \tau_i^*$, $\xi_2 = \tau_{i+1}^*$, where $\tau_p^*\colon \bigwedge^p T^*M \to M$ (see Example 1.2 on p. 6), and $\Delta = d\colon \Lambda^i(M) \to \Lambda^{i+1}(M)$ be the de Rham differential. Then we obtain the first-order operator $d_h \stackrel{\text{def}}{=} \mathcal{C}d\colon \Lambda_h^i(\pi) \to \Lambda_h^{i+1}(\pi)$, where $\Lambda_h^p(\pi)$ denotes the module $\mathcal{F}(\pi, \tau_p^*)$. Due Corollary 2.8, the operators $d\colon \Lambda_h^i(\mathcal{E}) \to \Lambda_h^{i+1}(\mathcal{E})$ are also defined, where $\Lambda_h^p(\mathcal{E}) = \mathcal{F}(\mathcal{E}, \tau_p^*)$.

DEFINITION 2.7. Let $\mathcal{E} \subset J^k(\pi)$ be an equation.

(i) Elements of the module $\Lambda_h^i(\mathcal{E})$ are called *horizontal i-forms* on $\mathcal{E}^\infty$.

(ii) The operator $d_h \colon \Lambda_h^i(\mathcal{E}) \to \Lambda_h^{i+1}(\mathcal{E})$ is called the *horizontal de Rham differential* on $\mathcal{E}^\infty$.

(iii) The sequence

$$0 \to \mathcal{F}(\mathcal{E}) \xrightarrow{d} \Lambda_h^1(\mathcal{E}) \to \cdots \to \Lambda_h^i(\mathcal{E}) \xrightarrow{d} \Lambda_h^{i+1}(\mathcal{E}) \to \cdots$$

is called the *horizontal de Rham sequence* of the equation $\mathcal{E}$.

From Proposition 2.6 (iii) it follows that $d \circ d = 0$ and hence the de Rham sequence is a complex. It cohomologies are called the *horizontal de Rham cohomologies* of $\mathcal{E}$ and are denoted by $H_h^*(\mathcal{E}) = \sum_{i \geq 0} H_h^i(\mathcal{E})$.

In local coordinates, horizontal forms of degree p on $\mathcal{E}^\infty$ are represented as $\omega = \sum_{i_1 < \cdots < i_p} a_{i_1 \dots i_p} \, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, where $a_{i_1 \dots i_p} \in \mathcal{F}(E)$, while the horizontal de Rham differential acts as

$$d_h(\omega) = \sum_{i=1}^{n} \sum_{i_1 < \cdots < i_p} D_i(a_{i_1 \dots i_p}) \, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}. \tag{2.12}$$

In particular, we see that both $\Lambda_h^i(\mathcal{E})$ and $H_h^i(\mathcal{E})$ vanish for $i > \dim M$.

REMARK 2.2. In fact, the above introduced cohomologies are horizontal cohomologies with *trivial coefficients*. The case of more general coefficients will be considered in Chapter 4 (see also [**98, 52**]). Below we make the first step to deal with a nontrivial case.

Consider the algebra $\Lambda^*(\mathcal{E})$ of all differential forms on $\mathcal{E}^\infty$ and let us note that one has the embedding $\Lambda_h^*(\mathcal{E}) \hookrightarrow \Lambda^*(\mathcal{E})$. Let us extend the horizontal de Rham differential onto this algebra as follows:

(i) $d_h(d\omega) = -d(d_h(\omega))$,

(ii) $d_h(\omega \wedge \theta) = d_h(\omega) \wedge \theta + (-1)^\omega \omega \wedge d_h(\theta)$.

Obviously, conditions (i), (ii) define the differential $d_h \colon \Lambda^i(\mathcal{E}) \to \Lambda^{i+1}(\mathcal{E})$ and its restriction onto $\Lambda_h^*(\mathcal{E})$ coincides with the horizontal de Rham differential.

Let us also set $d_{\mathcal{C}} \overset{\text{def}}{=} d - d_h \colon \Lambda_h^*(\mathcal{E}) \to \Lambda_h^*(\mathcal{E})$. Then, by definition,

$$d = d_h + d_{\mathcal{C}}, \quad d_h \circ d_h = d_{\mathcal{C}} \circ d_{\mathcal{C}} = 0, \quad d_{\mathcal{C}} \circ d_h + d_h \circ d_{\mathcal{C}} = 0.$$

In other words, the pair $(d_h, d_{\mathcal{C}})$ forms a bicomplex in $\Lambda^*(\mathcal{E})$ with the total differential d. Hence, the corresponding spectral sequence converges to the de Rham cohomology of $\mathcal{E}^\infty$.

REMARK 2.3. We shall redefine this bicomplex in a more general algebraic situation in Chapter 4. On the other hand, it should be noted that the above mentioned spectral sequence (in the case, when d_h is taken for the first differential and $d_{\mathcal{C}}$ for the second one) is a particular case of the Vinogradov $\mathcal{C}$-spectral sequence (or the so-called variational bicomplex) which is essential to the theory of conservation laws and Lagrangian formalism with constraints; cf. Subsection 2.2 below.

To conclude this section, let us write down the coordinate representation for the differential $d_{\mathcal{C}}$ and the extended d_h. First note that by definition and due to (2.12), one has

$$d_{\mathcal{C}}(u^j_\sigma) = d(u^j_\sigma) - d_h(u^j_\sigma) = du^j_\sigma - \sum_{i=1}^n u^j_{\sigma+1_i}\,dx_i,$$

i.e., $d_{\mathcal{C}}$ takes coordinate functions u^j_σ to the corresponding Cartan forms. Since obviously $d_{\mathcal{C}}(x_i) = 0$ for any coordinate function on the base, we obtain

$$d_{\mathcal{C}}(f) = \sum_{j,\sigma} \frac{\partial f}{\partial u^j_\sigma}\omega^j_\sigma, \quad f \in \mathcal{F}(\pi). \tag{2.13}$$

The same representation, written in internal coordinates, is valid on $\mathcal{E}^\infty$. Therefore, the image of $d_{\mathcal{C}}$ spans the Cartan submodule $\mathcal{C}\Lambda^1(\mathcal{E})$ in $\Lambda^1(\mathcal{E})$. By this reason, we call $d_{\mathcal{C}}$ the *Cartan differential* on $\mathcal{E}^\infty$. From the equality $d = d_h + d_{\mathcal{C}}$ it follows that the direct sum decomposition

$$\Lambda^1(\mathcal{E}) = \Lambda^1_h(\mathcal{E}) \oplus \mathcal{C}\Lambda^1(\mathcal{E})$$

takes place which extends to the decomposition

$$\Lambda^i(\mathcal{E}) = \bigoplus_{p+q=i} \Lambda^q_h(\mathcal{E}) \otimes \mathcal{C}^p\Lambda(\mathcal{E}). \tag{2.14}$$

Here the notation

$$\mathcal{C}^p\Lambda(\mathcal{E}) \stackrel{\text{def}}{=} \underbrace{\mathcal{C}\Lambda^1(\mathcal{E}) \wedge \dots \wedge \mathcal{C}\Lambda^1(\mathcal{E})}_{p \text{ times}}$$

is used. Consequently, to finish computations, it suffices to compute $d_h(\omega^j_\sigma)$. But we have

$$d_h(\omega^j_\sigma) = d_h d_{\mathcal{C}}(u^j_\sigma) = -d_{\mathcal{C}} d_h(u^j_\sigma)$$

and thus

$$d_h(\omega^j_\sigma) = -\sum_{i=1}^n \omega^j_{\sigma+1_i} \wedge dx_i. \tag{2.15}$$

Note that from the results obtained it follows, that

$$d_h(\Lambda^q_h(\mathcal{E}) \otimes \mathcal{C}^p\Lambda(\mathcal{E})) \subset \Lambda^{q+1}_h(\mathcal{E}) \otimes \mathcal{C}^p\Lambda(\mathcal{E}),$$
$$d_{\mathcal{C}}(\Lambda^q_h(\mathcal{E}) \otimes \mathcal{C}^p\Lambda(\mathcal{E})) \subset \Lambda^q_h(\mathcal{E}) \otimes \mathcal{C}^{p+1}\Lambda(\mathcal{E}).$$

REMARK 2.4. Note that the sequence $d_h \colon \Lambda^q_h(\mathcal{E}) \otimes \mathcal{C}^*(\mathcal{E}) \to \Lambda^{q+1}_h(\mathcal{E}) \otimes \mathcal{C}^*(\mathcal{E})$ can be considered as the *horizontal de Rham complex with coefficients in Cartan forms*

REMARK 2.5. From (2.14) it follows that to any form $\omega \in \Lambda^*(\mathcal{E})$ we can put into correspondence its "purely horizontal" component $\omega_h \in \Lambda_h^*(\mathcal{E})$. Moreover, if the form ω "lives" on $J^k(\pi)$, then, due to the equality $du_\sigma^j = \sum_{i=1}^n u_{\sigma+1_i}\, dx_i + \omega_\sigma^j$, the form ω_h belongs to $\Lambda^*(J^{k+1}(\pi))$. This correspondence coincides with the one used in Example 1.7 on p. 14 to construct Monge–Ampere equations.

2. Higher symmetries and conservation laws

In this section, we briefly expose the theory of higher (or Lie–Bäcklund) symmetries and conservation laws for nonlinear partial differential equations (for more details and examples see [**60, 12**]).

2.1. Symmetries. Let $\pi\colon E \to M$ be a vector bundle and $\mathcal{E} \subset J^k(\pi)$ be a differential equation. We shall still assume $\mathcal{E}$ to be formally integrable, though is it not restrictive in this context.

Consider a symmetry F of the equation $\mathcal{E}$ and let θ_{k+1} be a point of $\mathcal{E}^1$ such that $\pi_{k+1,k}(\theta_{k+1}) = \theta_k \in \mathcal{E}$. Then the R-plane $L_{\theta_{k+1}}$ is taken to an R-plane $F_*(L_{\theta_{k+1}})$ by F, since F is a Lie transformation, and $F_*(L_{\theta_{k+1}}) \subset T_{F(\theta_k)}$, since F is a symmetry. Consequently, the lifting $F^{(1)}\colon J^{k+1}(\pi) \to J^{k+1}(\pi)$ is a symmetry of $\mathcal{E}^1$. By the same reasons, $F^{(l)}$ is a symmetry of the l-th prolongation of $\mathcal{E}$. From here it also follows that for any infinitesimal symmetry X of the equation $\mathcal{E}$, its l-th lifting is a symmetry of $\mathcal{E}^l$ as well. In fact, the following result is valid:

PROPOSITION 2.9. *Symmetries of a formally integrable equation $\mathcal{E} \subset J^k(\pi)$ coincide with symmetries of any prolongation of this equation. The same is valid for infinitesimal symmetries.*

PROOF. We have shown already that to any (infinitesimal) symmetry of $\mathcal{E}$ there corresponds an (infinitesimal) symmetry of $\mathcal{E}^l$. Consider now an (infinitesimal) symmetry of $\mathcal{E}^l$. Then, due to Theorems 1.12 and 1.13 (see pp. 24 and 26), it is $\pi_{k+l,k}$-fiberwise and therefore generates an (infinitesimal) symmetry of $\mathcal{E}$. $\square$

The result proved means that a symmetry of $\mathcal{E}$ generates a symmetry of $\mathcal{E}^\infty$ which preserves every prolongation up to finite order. A natural step to generalize the concept of symmetry is to consider "all symmetries" of $\mathcal{E}^\infty$. Let us clarify such a generalization.

First of all note that only infinitesimal point of view may be efficient in the setting under consideration. Otherwise we would have to deal with diffeomorphisms of infinite-dimensional manifolds with all natural difficulties arising as a consequence. Keeping this in mind, we proceed with the following definition. Recall the notation

$$\mathcal{C}D(\pi) \stackrel{\text{def}}{=} \{X \in D(\pi) \mid X \text{ lies in } \mathcal{C}(\pi)\},$$

cf. (1.31) on p. 25.

Definition 2.8. Let π be a vector bundle. A vector field $X \in D(\pi)$ is called a *symmetry* of the Cartan distribution $\mathcal{C}(\pi)$ on $J^\infty(\pi)$, if $[X, \mathcal{C}D(\pi)] \subset \mathcal{C}D(\pi)$.

Thus, the set of symmetries coincides with $D_{\mathcal{C}}(\pi)$ (see (1.32) on p. 25) and forms a Lie algebra over $\mathbb{R}$ and a module over $\mathcal{F}(\pi)$. Note that since the Cartan distribution on $J^\infty(\pi)$ is integrable, one has $\mathcal{C}D(\pi) \subset D_{\mathcal{C}}(\pi)$ and, moreover, $\mathcal{C}D(\pi)$ is an ideal in the Lie algebra $D_{\mathcal{C}}(\pi)$.

Note also that symmetries belonging to $\mathcal{C}D(\pi)$ are of a special type: they are tangent to any integral manifold of the Cartan distribution. By this reason, we call such symmetries *trivial*. Respectively, the elements of the quotient Lie algebra

$$\operatorname{sym}(\pi) \stackrel{\text{def}}{=} D_{\mathcal{C}}(\pi)/\mathcal{C}D(\pi)$$

are called *nontrivial symmetries* of the Cartan distribution on $J^\infty(\pi)$.

Let now $\mathcal{E}^\infty$ be the infinite prolongation of an equation $\mathcal{E} \subset J^k(\pi)$. Then, since $\mathcal{C}D(\pi)$ is spanned by the fields of the form $\mathcal{C}Y$, $Y \in D(M)$ (see Example 2.1), any vector field from $\mathcal{C}D(\pi)$ is tangent to $\mathcal{E}^\infty$. Consequently, either all elements of the coset $[X] = X \bmod \mathcal{C}D(\pi)$, $X \in D(\pi)$, are tangent to $\mathcal{E}^\infty$ or neither of them is. In the first case we say that the coset $[X]$ is *tangent* to $\mathcal{E}^\infty$.

Definition 2.9. An element $[X] = X \bmod \mathcal{C}D(\pi)$, $X \in D(\pi)$, is called a *higher symmetry* of $\mathcal{E}$, if it is tangent to $\mathcal{E}^\infty$.

The set of all higher symmetries forms a Lie algebra over $\mathbb{R}$ and is denoted by $\operatorname{sym}(\mathcal{E})$. We shall usually omit the adjective *higher* in the sequel.

Let us describe the algebra $\operatorname{sym}(\mathcal{E})$ in efficient terms. We start with describing $\operatorname{sym}(\pi)$ as the first step. To do this, note the following. Consider a vector field $X \in D(\pi)$. Then, substituting X into the structural element U_π (see (2.7)), we obtain a field $X^v \in D(\pi)$. The correspondence $U_\pi \colon X \mapsto X^v = X \lrcorner U_\pi$ possesses the following properties:

(i) The field X^v is vertical, i.e., $X^v(C^\infty(M)) = 0$.
(ii) $X^v = X$ for any vertical field.
(iii) $X^v = 0$ if and only if the field X lies in $\mathcal{C}D(\pi)$.

Therefore, we obtain the direct sum decomposition[1]

$$D(\pi) = D^v(\pi) \oplus \mathcal{C}D(\pi),$$

where $D^v(\pi)$ denotes the Lie algebra of vertical fields. A direct corollary of these properties is the following result.

Proposition 2.10. *For any coset $[X] \in \operatorname{sym}(\mathcal{E})$ there exists a unique vertical representative and thus*

$$\operatorname{sym}(\mathcal{E}) = \{X \in D^v(\mathcal{E}) \mid [X, \mathcal{C}D(\mathcal{E})] \subset \mathcal{C}D(\mathcal{E})\}, \tag{2.16}$$

where $\mathcal{C}D(\mathcal{E})$ is spanned by the fields $\mathcal{C}Y$, $Y \in D(M)$.

[1]Note that it is the direct sum of $\mathcal{F}(\pi)$-modules but not of Lie algebras.

Using this result, we shall identify symmetries of $\mathcal{E}$ with vertical vector fields satisfying (2.16).

LEMMA 2.11. *Let $X \in \operatorname{sym}(\pi)$ be a vertical vector field. Then it is completely determined by its restriction onto $\mathcal{F}_0(\pi) \subset \mathcal{F}(\pi)$.*

PROOF. Let X satisfy the conditions of the lemma and $Y \in D(M)$. Then for any $f \in C^\infty(M)$ one has

$$[X, \mathcal{C}Y](f) = X(\mathcal{C}Y(f)) - \mathcal{C}Y(X(f)) = X(Y(f)) = 0$$

and hence the commutator $[X, \mathcal{C}Y]$ is the vertical vector field. On the other hand, $[X, \mathcal{C}Y] \in \mathcal{C}D(\pi)$, because $\mathcal{C}D(\pi)$ is a Lie algebra ideal. Consequently, $[X, \mathcal{C}Y] = 0$.

Note now that in special coordinates we have $D_i(u^j_\sigma) = u^j_{\sigma+1_i}$ for all σ and j. From the above said it follows that

$$X(u^j_{\sigma+1_i}) = D_i\bigl(X(u^j_\sigma)\bigr). \tag{2.17}$$

But X is a vertical derivation and thus is determined by its values at the functions u^j_σ. □

Let now $X_0 \colon \mathcal{F}_0(\pi) \to \mathcal{F}(\pi)$ be a derivation. Then equalities (2.17) allow one to reconstruct locally a vertical derivation $X \in D(\pi)$ satisfying $X\,\big|_{\mathcal{F}_0(\pi)} = X_0$. Obviously, the derivation X thus constructed lies in $\operatorname{sym}(\pi)$ over the neighborhood under consideration. Consider two neighborhoods $\mathcal{U}_1, \mathcal{U}_2 \subset J^\infty(\pi)$ with the corresponding special coordinates in each of them and two symmetries $X^i \in \operatorname{sym}(\pi\,|_{\mathcal{U}_i})$, $i = 1, 2$, arising by the described procedure. But the restrictions of X^1 and X^2 onto $\mathcal{F}_0(\pi\,|_{\mathcal{U}_1 \cap \mathcal{U}_2})$ coincide. Hence, by Lemma 2.11, the field X^1 coincide with X^2 over the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$. In other words, the reconstruction procedure $X_0 \mapsto X$ is a global one. So we have established a one-to-one correspondence between elements of $\operatorname{sym}(\pi)$ and derivations $\mathcal{F}_0(\pi) \to \mathcal{F}(\pi)$.

To complete description of $\operatorname{sym}(\pi)$, note that due to vector bundle structure in $\pi \colon E \to M$, derivations $\mathcal{F}_0(\pi) \to \mathcal{F}(\pi)$ are identified with sections of the pull-back $\pi^*_\infty(\pi)$, or with elements of $\mathcal{F}(\pi, \pi)$.

THEOREM 2.12. *Let $\pi \colon E \to M$ be a vector bundle. Then*:

(i) *The $\mathcal{F}(\pi)$-module $\operatorname{sym}(\pi)$ is in one-to-one correspondence with elements of the module $\mathcal{F}(\pi, \pi)$.*
(ii) *In special coordinates the correspondence $\mathcal{F}(\pi, \pi) \to \operatorname{sym}(\pi)$ is expressed by the formula*

$$\varphi \mapsto \mathit{Э}_\varphi \stackrel{\text{def}}{=} \sum_{j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma}, \tag{2.18}$$

where $\varphi = (\varphi^1, \dots, \varphi^m)$ is the component-wise representation of the section $\varphi \in \mathcal{F}(\pi, \pi)$.

PROOF. The first part of the theorem has already been proved. To prove the second one, it suffices to use equality (2.17). □

DEFINITION 2.10. Let $\pi\colon E \to M$ be a vector bundle.

(i) The field ∂_φ of the form (2.18) is called an *evolutionary vector field* on $J^\infty(\pi)$.
(ii) The section $\varphi \in \mathcal{F}(\pi, \pi)$ is called the *generating section* of the field ∂_φ.

REMARK 2.6. Let $\zeta\colon N \to M$ be an arbitrary smooth fiber bundle and $\xi\colon P \to M$ be a vector bundle. Then it is easy to show that any ζ-vertical vector field X on N can be uniquely lifted up to an $\mathbb{R}$-linear mapping $X^\xi\colon \Gamma(\zeta^*(\xi)) \to \Gamma(\zeta^*(\xi))$ such that

$$X^\xi(f\psi) = X(f)\psi + fX^\xi(\psi), \quad f \in C^\infty(N),\ \psi \in \Gamma(\zeta^*(\xi)). \tag{2.19}$$

In particular, taking π_∞ for ζ, for any evolution derivation ∂_φ we obtain the family of mappings $\partial_\varphi^\xi\colon \mathcal{F}(\pi, \xi) \to \mathcal{F}(\pi, \xi)$ satisfying (2.19).

Consider the mapping $\partial_\varphi^\pi\colon \mathcal{F}(\pi, \pi) \to \mathcal{F}(\pi, \pi)$ and recall that the diagonal element $\rho_0 \in \mathcal{F}_0(\pi, \pi) \subset \mathcal{F}(\pi, \pi)$ is defined (see Example 1.1 on p. 5). As it can be easily seen, the following identity is valid

$$\partial_\varphi^\pi(\rho_0) = \varphi \tag{2.20}$$

which can be taken for the definition of the generating section.

Let ∂_φ, ∂_ψ be two evolutionary derivations. Then, since $\operatorname{sym}(\pi)$ is a Lie algebra and by Theorem 2.12, there exists a unique section $\{\varphi, \psi\}$ satisfying $[\partial_\varphi, \partial_\psi] = \partial_{\{\varphi,\psi\}}$.

DEFINITION 2.11. The section $\{\varphi, \psi\}$ is called the (*higher*) *Jacobi bracket* of the sections $\varphi, \psi \in \mathcal{F}(\pi)$.

PROPOSITION 2.13. *Let $\varphi, \psi \in \mathcal{F}(\pi, \pi)$ be two sections. Then*:

(i) $\{\varphi, \psi\} = \partial_\varphi^\pi(\psi) - \partial_\psi^\pi(\varphi)$.
(ii) *In special coordinates, the Jacobi bracket of φ and ψ is expressed by the formula*

$$\{\varphi, \psi\}^j = \sum_{\alpha,\sigma} \left(D_\sigma(\varphi^\alpha) \frac{\partial \psi^j}{\partial u_\sigma^\alpha} - D_\sigma(\psi^\alpha) \frac{\partial \varphi^j}{\partial u_\sigma^\alpha} \right), \tag{2.21}$$

where superscript j denotes the j-th component of the corresponding section.

PROOF. To prove (i) let us use (2.20):

$$\{\varphi, \psi\} = \partial_{\{\varphi,\psi\}}^\pi(\rho_0) = \partial_\varphi^\pi(\partial_\psi^\pi(\rho_0)) - \partial_\psi^\pi(\partial_\varphi^\pi(\rho_0)) = \partial_\varphi^\pi(\psi) - \partial_\psi^\pi(\varphi).$$

The second statement follows from the first one and from equality (2.18). □

Consider now a nonlinear differential operator $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$ and let φ_Δ be the corresponding section. Then for any $\varphi \in \mathcal{F}(\pi, \pi)$ the section $\partial_\varphi(\varphi_\Delta) \in \mathcal{F}(\pi, \xi)$ is defined and we can set

$$\ell_\Delta(\varphi) \overset{\text{def}}{=} \partial_\varphi(\varphi_\Delta). \tag{2.22}$$

DEFINITION 2.12. The operator $\ell_\Delta\colon \mathcal{F}(\pi,\pi) \to \mathcal{F}(\pi,\xi)$ defined by (2.22) is called the *universal linearization operator* of the operator $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$.

From the definition and equality (2.18) we obtain that for a scalar differential operator

$$\Delta\colon \varphi \mapsto F\Big(x_1,\dots,x_n,\dots,\frac{\partial^{|\sigma|}\varphi^j}{\partial x_\sigma},\dots\Big)$$

one has $\ell_\Delta = (\ell^1_\Delta,\dots,\ell^m_\Delta)$, $m = \dim\pi$, where

$$\ell^\alpha_\Delta = \sum_\sigma \frac{\partial F}{\partial u^\alpha_\sigma} D_\sigma. \tag{2.23}$$

If $\dim\xi = r > 1$ and $\Delta = (\Delta_1,\dots,\Delta_r)$, then

$$\ell_\Delta = \begin{Vmatrix} \ell^1_{\Delta^1} & \ell^2_{\Delta^1} & \dots & \ell^m_{\Delta^1} \\ \ell^1_{\Delta^2} & \ell^2_{\Delta^2} & \dots & \ell^m_{\Delta^2} \\ \dots & \dots & \dots & \dots \\ \ell^1_{\Delta^r} & \ell^2_{\Delta^r} & \dots & \ell^m_{\Delta^r} \end{Vmatrix}. \tag{2.24}$$

In particular, we see that the following statement is valid.

PROPOSITION 2.14. *For any differential operator Δ, its universal linearization is a $\mathcal{C}$-differential operator.*

Now we can describe the algebra $\mathrm{sym}(\mathcal{E})$, $\mathcal{E} \subset J^k(\pi)$ being a formally integrable equation. Let $I(\mathcal{E}) \subset \mathcal{F}(\pi)$ be the ideal of the equation $\mathcal{E}$ (see Subsection 1.1). Then, by definition, ∂_φ is a symmetry of $\mathcal{E}$ if and only if

$$\partial_\varphi(I(\mathcal{E})) \subset I(\mathcal{E}). \tag{2.25}$$

Assume now that $\mathcal{E}$ is given by a differential operator $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$ and locally is described by the system of equations

$$F^1 = 0,\ \dots,\ F^r = 0, \quad F^j \in \mathcal{F}(\pi).$$

Then the functions $F^1,\dots,F^r$ are differential generators of the ideal $I(\mathcal{E})$ and condition (2.25) may be rewritten as

$$\partial_\varphi(F^j) = \sum_{\alpha,\sigma} a^\alpha_\sigma D_\sigma(F^\alpha), \qquad j = 1,\dots,m, \quad a^\alpha_\sigma \in \mathcal{F}(\pi). \tag{2.26}$$

With the use of (2.22), the last equation acquires the form[2]

$$\ell_{F^j}(\varphi) = \sum_{\alpha,\sigma} a^\alpha_\sigma D_\sigma(F^\alpha), \qquad j = 1,\dots,m, \quad a^\alpha_\sigma \in \mathcal{F}(\pi). \tag{2.27}$$

But by Proposition 2.14, the universal linearization is a $\mathcal{C}$-differential operator and consequently can be restricted onto $\mathcal{E}^\infty$ (see Corollary 2.8). It means that we can rewrite equation (2.27) as

$$\ell_{F^j}|_{\mathcal{E}^\infty}(\varphi|_{\mathcal{E}^\infty}) = 0, \qquad j = 1,\dots,m. \tag{2.28}$$

[2]Below we use the notation ℓ_F, $F \in \mathcal{F}(\pi,\xi)$, as a synonym for ℓ_Δ, where $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$ is the operator corresponding to the section F.

Combining these equations with (2.23) and (2.24), we obtain the following fundamental result:

THEOREM 2.15. *Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation and $\Delta = \Delta_{\mathcal{E}}\colon \Gamma(\pi) \to \Gamma(\xi)$ be the operator corresponding to $\mathcal{E}$. Then an evolutionary derivation $\mathcal{Э}_\varphi$, $\varphi \in \mathcal{F}(\pi,\pi)$, is a symmetry of $\mathcal{E}$ if and only if*

$$\ell_{\mathcal{E}}(\bar{\varphi}) = 0, \tag{2.29}$$

where $\ell_{\mathcal{E}}$ and $\bar{\varphi}$ denote restrictions of ℓ_Δ and φ on $\mathcal{E}^\infty$ respectively. In other words,

$$\operatorname{sym}(\mathcal{E}) = \ker \ell_{\mathcal{E}}. \tag{2.30}$$

REMARK 2.7. From the result obtained it follows that higher symmetries of $\mathcal{E}$ can be identified with elements of $\mathcal{F}(\mathcal{E},\pi)$ satisfying equation (2.29). Below we shall say that a *section* $\varphi \in \mathcal{F}(\mathcal{E},\pi)$ *is a symmetry* of $\mathcal{E}$ keeping in mind this identification. Note that due to the fact that $\operatorname{sym}(\mathcal{E})$ is a Lie algebra, for any two symmetries φ, $\psi \in \mathcal{F}(\mathcal{E},\pi)$ their Jacobi bracket $\{\varphi,\psi\}_{\mathcal{E}} = \{\varphi,\psi\} \in \mathcal{F}(\mathcal{E},\pi)$ is well defined and is a symmetry as well.

2.2. Conservation laws. This subsection contains a brief review of the main definitions and facts concerning the theory of conservation laws for nonlinear differential equations. We confine ourselves with main definition and results referring the reader to [**102**] and [**52**] for motivations and proofs.

DEFINITION 2.13. Let $\mathcal{E} \subset J^k(\pi)$, $\pi\colon E \to M$ being a vector bundle, be a differential equation and n be the dimension of the manifold M.

(i) A horizontal $(n-1)$-form $\rho \in \Lambda_h^{n-1}(\mathcal{E})$ on $\mathcal{E}^\infty$ is called a *conserved density* on $\mathcal{E}$, if $d_h\rho = 0$.
(ii) A conserved density ρ is called *trivial*, if $\rho = d_h\rho'$ for some $\rho' \in \Lambda_h^{n-2}(\mathcal{E})$.
(iii) The horizontal cohomology class $[\rho] \in H_h^{n-1}(\mathcal{E})$ of a conserved density ρ is called a *conservation law* on $\mathcal{E}$.

We shall always assume below that the manifold M of independent variables is cohomologically trivial which means triviality of all de Rham cohomology groups $H^i(M)$ except for the group $H^0(M)$.

Note now that the group $H_h^{n-1}(\mathcal{E})$ is the term $E_1^{0,n-1} = E_1^{0,n-1}(\mathcal{E})$ of the spectral sequence associated to the bicomplex $(d_h, d_{\mathcal{C}})$ (see Subsection 1.4 and Remark 2.3 in particular). This fact is not accidental and to clarify it we shall need more information about this spectral sequence. Let us start with the "trivial" case and first introduce preliminary notions and notations.

For any equation $\mathcal{E}$, we shall denote by $\varkappa = \varkappa(\mathcal{E})$ the module $\mathcal{F}(\mathcal{E},\pi)$. In particular, $\varkappa(\pi)$ denotes the module $\varkappa$ in the case $\mathcal{E}^\infty = J^\infty(\pi)$. Let ξ and ζ be two vector bundles over M and $P = \mathcal{F}(\mathcal{E},\xi)$, $Q = \mathcal{F}(\mathcal{E},\zeta)$. Denote by $\mathcal{C}\operatorname{Diff}_l^{\mathrm{alt}}(P,Q)$ the $\mathcal{F}(\mathcal{E})$-module of $\mathbb{R}$-linear mappings

$$\Delta\colon \underbrace{P \otimes \dots \otimes P}_{l \text{ times}} \to Q$$

such that:

(i) Δ is skew-symmetric,

(ii) for any $p_1, \dots, p_{l-1} \in P$, the mapping
$$\Delta_{p_1,\dots,p_{l-1}} : P \to Q, \quad p \mapsto \Delta(p_1, \dots, p_{l-1}, p),$$
is a $\mathcal{C}$-differential operator.

In particular, $\mathcal{C}\operatorname{Diff}(P, Q)$ denotes the module of all $\mathcal{C}$-differential operators acting from P to Q.

Define the complex

$$\begin{aligned} 0 \to \mathcal{C}\operatorname{Diff}(P, \Lambda_h^0(\mathcal{E})) \xrightarrow{d_h^P} \mathcal{C}\operatorname{Diff}(P, \Lambda_h^1(\mathcal{E})) \to \dots \to \mathcal{C}\operatorname{Diff}(P, \Lambda_h^q(\mathcal{E})) \\ \xrightarrow{d_h^P} \mathcal{C}\operatorname{Diff}(P, \Lambda_h^{q+1}(\mathcal{E})) \to \dots \to \mathcal{C}\operatorname{Diff}(P, \Lambda_h^n(\mathcal{E})) \to 0 \end{aligned} \tag{2.31}$$

by setting $d_h^P(\Delta) \overset{\text{def}}{=} d_h \circ \Delta$.

Lemma 2.16. *The above introduced complex* (2.31) *is acyclic at all terms except for the last one. The cohomology group at the n-th term equals the module $\widehat{P} \overset{\text{def}}{=} \hom_{\mathcal{F}(\mathcal{E})}(P, \Lambda_h^n(\mathcal{E}))$.*

Let $\Delta : P \to Q$ be a $\mathcal{C}$-differential operator. Then it generates the cochain mapping
$$\Delta' : (\mathcal{C}\operatorname{Diff}(Q, \Lambda_h^*(\mathcal{E})), d_h^Q) \to (\mathcal{C}\operatorname{Diff}(P, \Lambda_h^*(\mathcal{E})), d_h^P)$$
and consequently the mapping of cohomology groups
$$\Delta^* : \widehat{Q} = \hom_{\mathcal{F}(\mathcal{E})}(Q, \Lambda_h^n(\mathcal{E})) \to \widehat{P} = \hom_{\mathcal{F}(\mathcal{E})}(P, \Lambda_h^n(\mathcal{E})). \tag{2.32}$$

Definition 2.14. The above introduced mapping Δ^* is called the *adjoint operator* to the operator Δ.

In the case $\mathcal{E}^\infty = J^\infty(\pi)$, the local coordinate representation of the adjoint operator is as follows. For the scalar operator $\Delta = \sum_\sigma a_\sigma D_\sigma$ one has
$$\Delta^* = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_\sigma. \tag{2.33}$$
In the multi-dimensional case, $\Delta = \|\Delta_{ij}\|$, the components of the adjoint operator are expressed by
$$(\Delta^*)_{ij} = \Delta_{ji}^*, \tag{2.34}$$
where Δ_{ji}^* are given by (2.33).

Relation between the action of an $\mathcal{C}$-differential $\Delta : P \to Q$ and its adjoint $\Delta^* : \widehat{Q} \to \widehat{P}$ is given by

Proposition 2.17 (Green's formula). *For any elements $p \in P$ and $q \in \widehat{Q}$ there exists an $n-1$-form $\omega \in \Lambda_h^{n-1}(\mathcal{E})$ such that*
$$\langle p, \Delta^*(q) \rangle - \langle \Delta(p), q \rangle = d_h \omega, \tag{2.35}$$
where $\langle R, \widehat{R} \rangle \to \Lambda_h^n(\mathcal{E})$ denotes the natural pairing.

Finally, let us define $\mathcal{F}(\mathcal{E})$-submodules $\mathcal{K}_l(P) \subset \mathcal{C}\operatorname{Diff}^{\text{alt}}_{l-1}(P, \widehat{P})$, $l > 0$, by setting

$$\mathcal{K}_l(P) \stackrel{\text{def}}{=} \{\Delta \in \mathcal{C}\operatorname{Diff}^{\text{alt}}_{l-1}(P, \widehat{P}) \mid \Delta^*_{p_1,\dots,p_{l-2}} = -\Delta_{p_1,\dots,p_{l-2}}, \forall p_1, \dots, p_{l-2} \in P\}.$$

THEOREM 2.18 (one-line theorem). *Let $\pi\colon E \to M$ be a vector bundle over a cohomologically trivial manifold M, $\dim M = n$. Then:*

(i) $E^{0,n}_1(\pi) = H^n_h(E)$.
(ii) $E^{p,n}_1(\pi) = \mathcal{K}_p(\varkappa(\pi))$, $p > 0$.
(iii) $E^{0,0}_1(\pi) = \mathbb{R}$.
(iv) $E^{p,q}_1(\pi) = 0$ *in all other cases.*

Moreover, the following result is valid.

THEOREM 2.19. *The sequence*

$$\Lambda^0_h(\pi) \xrightarrow{d_h} \dots \xrightarrow{d_h} \Lambda^n_h(\pi) \xrightarrow{\mathsf{E}} E^{1,n}_1 \xrightarrow{d^{1,n}_1} E^{2,n}_1 \to \cdots \tag{2.36}$$

where the operator E is the composition

$$\mathsf{E}\colon \Lambda^n_h(\pi) \to H^n_h(\pi) = E^{0,n}_1(\pi) \xrightarrow{d^{0,1}_1} E^{1,n}_1(\pi), \tag{2.37}$$

the first arrow being the natural projection, is exact.

DEFINITION 2.15. Let $\pi\colon E \to M$ be a vector bundle, $\dim M = n$.

(i) The sequence (2.36) is called the *variational complex* of the bundle π.
(ii) The operator E defined by (2.37) is called the *Euler–Lagrange operator.*

It can be shown that for any $\omega \in \Lambda^n_h(\pi)$ one has

$$\mathsf{E}(\omega) = \ell^*_\omega(1), \tag{2.38}$$

from where an explicit formula in local coordinates for E is obtained:

$$\mathsf{E}^j = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ \frac{\partial}{\partial u^j_\sigma}. \tag{2.39}$$

The differentials $d^{p,n}_1$ can also be computed explicitly. In particular, we have

$$d^{1,n}_1(\varphi) = \ell_\varphi - \ell^*_\varphi, \quad \varphi \in E^{1,n}_1(\pi) = \widehat{\varkappa}(\pi). \tag{2.40}$$

Let us now describe the term $E^{p,q}_1(\mathcal{E})$ for a nontrivial equation $\mathcal{E}$. We shall do it for a broad class of equations which is introduced below.

Note first that a well-defined action of $\mathcal{C}$-differential operators $\Delta \in \mathcal{C}\operatorname{Diff}(\mathcal{F}(\mathcal{E}, \mathcal{E})$ on Cartan forms $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$ exists. Namely, for a zero-order operator (i.e., for a function on $\mathcal{E}^\infty$) we set $\Delta(\omega) \stackrel{\text{def}}{=} \Delta \cdot \omega$. If now $\Delta = \sum_\sigma \mathcal{X}_\sigma$, where $\mathcal{X}_\sigma = \mathcal{C}X_{i_1} \circ \dots \circ \mathcal{C}X_{i_s}$, $X_\alpha \in D(M)$, then

$$\Delta(\omega) \stackrel{\text{def}}{=} \sum_{\sigma=(i_1\dots i_s)} \mathrm{L}_{X_{i_1}}(\dots(\mathrm{L}_{X_{i_s}}(\omega))\dots).$$

In general, such a action is not well defined because of the identity

$$\mathrm{L}_{aY}(\omega) = a\mathrm{L}_Y(\omega) + d(a) \wedge \mathrm{i}_Y(\omega).$$

But if $Y = \mathcal{C}X$ and $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$, the second summand vanishes and we obtain the action we seek for.

Let now $\Delta \in \mathcal{C}\,\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E}))$ and $\Delta^1, \dots, \Delta^m$ be the components of this operator. Then we can define the form

$$\omega_\Delta \stackrel{\text{def}}{=} \Delta^1(\omega^1) + \dots + \Delta^m(\omega^m),$$

where $\omega^j = \omega^j_{(0,\dots,0)}$ are the Cartan forms. Thus we obtain the mapping $\mathcal{C}\,\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E})) \to \mathcal{C}\Lambda^1(\mathcal{E})$, $\Delta \mapsto \omega_\Delta$. On the other hand, assume that the equation $\mathcal{E}$ is determined by the operator $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$ and let $P = \mathcal{F}(\mathcal{E}, \xi)$. Then to any operator $\square \in \mathcal{C}\,\mathrm{Diff}(P, \mathcal{F}(\mathcal{E}))$ we can put into correspondence the operator $\square \circ \ell_\mathcal{E} \in \mathcal{C}\,\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E}))$, where $\ell_\mathcal{E}$ is the restriction of ℓ_Δ onto $\mathcal{E}^\infty$. It gives us the mapping $\mathcal{C}\,\mathrm{Diff}(P, \mathcal{F}(\mathcal{E})) \to \mathcal{C}\,\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E}))$. In Chapter 5 it will be shown that the forms $\omega_{\square \circ \ell_\mathcal{E}}$ vanish which means that the sequence

$$0 \to \mathcal{C}\,\mathrm{Diff}(P, \mathcal{F}(\mathcal{E})) \to \mathcal{C}\,\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E})) \to \mathcal{C}\Lambda^1(\mathcal{E}) \to 0 \tag{2.41}$$

is a complex.

DEFINITION 2.16. We say that equation $\mathcal{E}$ is *ℓ-normal* if (2.41) is an exact sequence.

THEOREM 2.20 (two-line theorem). *Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable ℓ-normal equation in a vector bundle $\pi\colon E \to M$ over a cohomologically trivial manifold M, $\dim M = n$. Then*:

(i) *$E_1^{p,q}(\mathcal{E}) = 0$, if $p \geq 1$ and $q \neq n-1, n$.*
(ii) *The differential $d^{0,n-1}\colon E_1^{0,n-1}(\mathcal{E}) \to E_1^{1,n-1}(\mathcal{E})$ is a monomorphism and its image coincides with* $\ker(d^{1,n-1})$.
(iii) *The group $E_1^{1,n-1}(\mathcal{E})$ coincides with* $\ker(\ell^*_\mathcal{E})$.

REMARK 2.8. The theorem has a stronger version, see [**98**], but the one given above is sufficient for our purposes.

REMARK 2.9. The number of nontrivial lines at the top part of the term E_1 relates to the length of the so-called *compatibility complex* for the operator $\ell_\mathcal{E}$ (see [**98, 52**]). For example, for the Yang–Mill equations (see Section 6 of Chapter 1 one has the three-line theorem, [**21**].

DEFINITION 2.17. The elements of $E_1^{1,n-1}(\mathcal{E}) = \ker(\ell^*_\mathcal{E})$ are called *generating sections* of conservation laws.

Theorem 2.20(iii) gives an efficient method to compute generating sections of conservation laws. The following result shows when a generating section corresponds to some conservation law.[3] Let $\ell^*_{\mathcal{E}}(\varphi) = 0$ and the equation $\mathcal{E}$ be given by the operator $\Delta = \Delta_F$. Then $\ell^*_\Delta(\varphi) = \square(F)$ for some $\mathcal{C}$-differential operator $\square$.

PROPOSITION 2.21. *A solution φ of the equation $\ell^*_{\mathcal{E}}(\varphi) = 0$ corresponds to a conservation law of the ℓ-normal equation $\mathcal{E}$, if there exists a $\mathcal{C}$-differential operator ∇ such that $\nabla^* = \nabla$ and the equality*

$$\ell_\varphi + \square^* = \nabla \circ \ell_\Delta$$

is valid being restricted onto $\mathcal{E}^\infty$.

Let us describe the action of symmetries on the space of generating sections. Assume, as above, that $\mathcal{E}$ is given by equations $F = 0$.

PROPOSITION 2.22. *Let ω be a conservation law of an ℓ-normal equation $\mathcal{E}$ and ψ_ω be the corresponding generating section. Then, if $\varphi \in \operatorname{sym}(\mathcal{E})$ is a symmetry, then the generating section*

$$\partial_\varphi(\pi_\omega) + \square^*(\psi_\omega)$$

corresponds to the conservation law $\partial_\varphi(\omega)$, where the operator $\square$ is such that $\partial_\varphi(F) = \square(F)$.

We finish this subsection with a discussion of Euler–Lagrange equations and Nöther symmetries.

DEFINITION 2.18. Let $\pi\colon E \to M$, $\dim M = n$, be a vector bundle and $L = [\omega] \in H^n_h(\pi)$, $\omega \in \Lambda^n_h(\pi)$, be a Lagrangian. The equation $\mathcal{E}_L = \{\mathsf{E}(L) = 0\}$ is called the *Euler–Lagrange equation* corresponding to the Lagrangian L, where E is the Euler–Lagrange operator (2.38).

We say that an evolutionary vector field ∂_φ is a *Nöther symmetry* of L, if $\partial_\varphi(L) = 0$ and denote the Lie algebra of such symmetries by $\operatorname{sym}(L)$. It easy to show that $\operatorname{sym}(L) \subset \operatorname{sym}(\mathcal{E}_L)$.

PROPOSITION 2.23 (Nöther theorem). *To any Nöther symmetry $\partial_\varphi \in \operatorname{sym}(L)$ there corresponds a conservation law of the equation $\mathcal{E}_L$.*

PROOF. In fact, since $\partial_\varphi \in \operatorname{sym}(L)$, one has $\partial_\varphi(\omega) = d_h\rho$ for some $\rho \in \Lambda^{n-1}_h(\pi)$. Then, by Green's formula (2.35), one has

$$\begin{aligned}\partial_\varphi(\omega) - d_h(\rho) = \ell = \omega(\varphi) - d_h(\rho) &= \ell^*_\omega(1)(\varphi) + d_h\theta(\varphi) - d_h(\rho)\\ &= \mathsf{E}(L)(\varphi) + d_h(\theta(\varphi) - \rho) = 0.\end{aligned}$$

Hence, the form $d_h(\theta(\varphi) - \rho)$ vanishes on ${\mathcal{E}^\infty}_L$ and $\eta = \theta(\varphi) - \rho\,|_{{\mathcal{E}^\infty}_L}$ is a desired conserved density. $\square$

We illustrate relations between symmetries and conserved densities by explicit computations for the nonlinear Dirac equations (see Section 5 of Chapter 1).

[3] If $E_1^{2,n-1}(\mathcal{E}) = 0$, then, as it follows from Theorem 2.20(ii), there is a one-to-one correspondence between conservation laws and their generating sections.

EXAMPLE 2.4 (Conservation laws of the Dirac equations). Let us consider the nonlinear Dirac equations with nonvanishing rest mass (case 4 in Section 5 of Chapter 1). Among the symmetries of this equation there are the following ones:

$$\begin{aligned}
V_1 = X_{19} &= u^4\frac{\partial}{\partial u^1} - u^3\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^3} + u^1\frac{\partial}{\partial u^4} \\
&- v^4\frac{\partial}{\partial v^1} + v^3\frac{\partial}{\partial v^2} + v^2\frac{\partial}{\partial v^3} - v^1\frac{\partial}{\partial v^4}, \\
V_2 = X_{20} &= v^1\frac{\partial}{\partial u^1} + v^2\frac{\partial}{\partial u^2} + v^3\frac{\partial}{\partial u^3} + v^4\frac{\partial}{\partial u^4} \\
&- u^1\frac{\partial}{\partial v^1} - u^2\frac{\partial}{\partial v^2} - u^3\frac{\partial}{\partial v^3} - u^4\frac{\partial}{\partial v^4}, \\
V_3 = X_{23} &= v^4\frac{\partial}{\partial u^1} - v^3\frac{\partial}{\partial u^2} - v^2\frac{\partial}{\partial u^3} + v^1\frac{\partial}{\partial u^4} \\
&+ u^4\frac{\partial}{\partial v^1} - u^3\frac{\partial}{\partial v^2} - u^2\frac{\partial}{\partial v^3} + u^1\frac{\partial}{\partial v^4}.
\end{aligned} \tag{2.42}$$

The generators V_1, V_2, V_3 are vertical vector fields on the space $J^0(\pi) = \mathbb{R}^8 \times \mathbb{R}^4 \xrightarrow{\pi} \mathbb{R}^4$ with coordinates $x_1, \dots, x_4$ in the base and $u^1, \dots, v^4$ along the fiber. The fields under consideration are generated by $\partial/\partial u^1$, $\partial/\partial u^2$, $\partial/\partial u^3$, $\partial/\partial u^4$, $\partial/\partial v^1$, $\partial/\partial v^2$, $\partial/\partial v^3$, $\partial/\partial v^4$, i.e.,

$$\pi_* V_j = 0, \quad j = 1, \dots, 3.$$

In fact, we need the prolonged vector fields $V_1^{(1)}$, $V_2^{(1)}$, $V_3^{(1)}$ to $J^1(\pi)$ which can be calculated from (2.42) using formulas (1.34) on p. 26.

Let $L(\underline{u}, \underline{v}, \underline{u}_j, \underline{v}_j)$ be the Lagrangian defined on $J^1(\pi)$ by

$$\begin{aligned}
L = &-u^4v_1^1 + v^4u_1^1 - u^3v_1^2 + v^3u_1^2 - u^2v_1^3 + v^2u_1^3 - u^1v_1^4 + v^1u_1^4 \\
&- v^4v_2^1 - u^4u_2^1 + v^3v_2^2 + u^3u_2^2 - v^2v_2^3 - u^2u_2^3 + v^1v_2^4 + u^1u_2^4 \\
&- u^3v_3^1 + v^3u_3^1 + u^4v_3^2 - v^4u_3^2 - u^1v_3^3 + v^1u_3^3 + u^2v_3^4 + v^2u_4^3 \\
&- u^1v_4^1 + v^1u_4^1 - u^2v_4^2 + v^2u_4^2 - u^3v_4^3 + v^3u_4^3 - u^4v_4^4 + v^4u_4^4 \\
&- K(1 + \frac{1}{2}\lambda^3\epsilon K),
\end{aligned} \tag{2.43}$$

where

$$(\underline{x}, \underline{u}, \underline{v}, \underline{u}_j, \underline{v}_j) = (x_1, \dots, x_4, u^1, \dots, v^4, u_1^1, \dots, u_4^1, \dots, v_1^4, \dots, v_4^4) \tag{2.44}$$

are local coordinates on $J^1(\pi) = \mathbb{R}^{44}$. An easy calculation shows that the Euler–Lagrange equations associated to (2.43), i.e.,

$$\frac{\partial}{\partial x_a}\frac{\partial L}{\partial z_a^A} - \frac{\partial L}{\partial z^A} = 0 \tag{2.45}$$

are just nonlinear the Dirac equations (1.88), see p. 39. In (2.45) we used the notation z^A, $A = 1, \dots, 8$, instead of $u^1, \dots, u^4$, $v^1, \dots, v^4$ and summation convention over $A = 1, \dots, 8$, $a = 1, \dots, 4$, if an index occurs twice.

Let us introduce the form Θ by

$$\Theta = L\omega + (\partial_A^a \lrcorner)\theta^A \wedge \omega_a, \tag{2.46}$$

where

$$\begin{aligned}
&\omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4,\\
&\partial_a = \frac{\partial}{\partial x^a},\ \partial_A = \frac{\partial}{\partial z^A},\ \partial_A^a = \frac{\partial}{\partial z_a^A},\\
&\omega_a = \partial_a \lrcorner\, \omega,\\
&\theta^A = dz^A - z_a^A\, dx^a,
\end{aligned} \tag{2.47}$$

and z_a^A refers to either u_a^j or v_a^j. From (2.45) we derive

$$\begin{aligned}
\Theta = L\omega + (\partial_A^a L)(dz^A) \wedge \omega_a - (\partial_A^a L) z_a^A \omega\\
= \big(L - (\partial_A^a L) z_a^A\big)\omega + (\partial_A^a L)(dz^A) \wedge \omega_a.
\end{aligned} \tag{2.48}$$

Since L defined by (2.43) is linear with respect to z_a^A we derive

$$L - (\partial_A^a L) z_a^A = -K(1 + \frac{1}{2}\lambda^3 \epsilon K). \tag{2.49}$$

We now want to compute the Lie derivatives

$$V_i^{(1)}\Theta,$$

i.e., the Lie derivatives of the form Θ with respect to the vector field $V_i^{(1)}$, $i = 1, 2, 3$. We prove the following

LEMMA 2.24. *The form Θ is V_i-invariant, i.e.,*

$$V_i^{(1)}\Theta = 0, \quad i = 1, 2, 3.$$

PROOF. The proof splits in two parts:

$$1: \quad V_i^{(1)} K(1 + \frac{1}{2}\lambda^3 \epsilon K)\omega = 0,\ i = 1, 2, 3, \tag{2.50}$$

$$2: \quad V_i^{(1)}(\partial_A^a L)\, dz^A \wedge \omega = 0,\ i = 1, 2, 3,\ a = 1, \dots, 4. \tag{2.51}$$

Proof of 1. One has

$$V_i^{(1)} K(1 + \frac{1}{2}\lambda^3 \epsilon K)\omega = V_i^{(1)} \lrcorner (-1 - \lambda^3 \epsilon K) dK \wedge \omega$$

and due to the definition of K (1.89) on p. 39, $dK = 2(u^1 du^1 + u^2 du^2 - u^3 du^3 - u^4 du^4 + v^1 dv^1 + v^2 dv^2 - v^3 dv^3 - v^4 dv^4)$ an easy calculation leads to

$$V_i^{(1)} \lrcorner\, dK = 0, \quad i = 1, 2, 3, \tag{2.52}$$

which completes the proof of part 1.

Proof of 2. In order to prove (2.51), we introduce four 1-forms

$$\begin{aligned}
V_1^* &= (\partial_A^1 L) dz^A = v^4 du^1 + v^3 du^2\\
&\quad + v^2 du^3 + v^1 du^4 - u^4 dv^1 - u^3 dv^2 - u^2 dv^3 - u^1 dv^4,\\
V_2^* &= (\partial_A^2 L) dz^A = -u^4 du^1 + u^3 du^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - u^2du^3 + u^1du^4 - v^4dv^1 + v^3dv^2 - v^2dv^3 + v^1dv^4,\\
V_3^* &= (\partial_A^3 L)dz^A = v^3du^1 - v^4du^2\\
&\quad + v^1du^3 - v^2du^4 - u^3dv^1 + u^4dv^2 - u^1dv^3 + u^2dv^4,\\
V_4^* &= (\partial_A^4 L)dz^A = v^1du^1 + v^2du^2\\
&\quad + v^3du^3 + v^4du^4 - u^1dv^1 - u^2dv^2 - u^3dv^3 - u^4dv^4,
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
dV_1^* &= -2(du^1\wedge dv^4 + du^2\wedge dv^3 + du^3\wedge dv^2 + du^4 d\wedge v^1),\\
dV_2^* &= 2(du^1\wedge du^4 - du^2\wedge du^3 + dv^1\wedge dv^4 - dv^2\wedge dv^3),\\
dV_3^* &= 2(-du^1\wedge dv^3 + du^2\wedge dv^4 - du^3\wedge dv^1 + du^4\wedge dv^2),\\
dV_4^* &= -2(du^1\wedge dv^1 + du^2\wedge dv^2 + du^3\wedge dv^3 + du^4\wedge dv^4). \qquad (2.53)
\end{aligned}
$$

Using (2.42) and (2.53), a somewhat lengthy calculation leads to the following result

$$V_i^{(1)}(V_j^*) = 0, \quad i = 1,2,,3,\ j = 1,\dots,4. \qquad (2.54)$$

This completes the proof of the lemma. □

Now due to the relation

$$(V_i^{(1)})\Theta = (V_i^{(1)}) \lrcorner d\Theta + d(V_i^{(1)} \lrcorner \Theta) = 0, \quad i = 1,2,3, \qquad (2.55)$$

and

$$(V_i^{(1)}) \lrcorner d\Theta = 0, \quad i = 1,2,3, \qquad (2.56)$$

on the "equation manifold", [**95**], we arrive at

$$d(V_i^{(1)} \lrcorner \Theta) = 0, \quad i = 1,2,3 \qquad (2.57)$$

on the "equation manifold". This means that $V_i^{(1)} \lrcorner \Theta$ are conserved currents, i=1,2,3. Combination of (2.42), (2.48), and (2.54) leads to

$$V_i^{(1)} \lrcorner \theta = (V_i^{(1)} \lrcorner V_a^*)\omega_a, \qquad (2.58)$$

i.e., the conserved currents associated to V_1, V_2, V_3 are given by

$$
\begin{aligned}
1:\ & 2\Big(u^4v^4 - u^3v^3 - u^2v^2 + u^1v^1\Big)\, dx_2\wedge x_3\wedge dx_4\\
&- \Big((u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2 - (v^1)^2 - (v^2)^2\\
&+ (v^3)^2 + (v^4)^2\Big)\, dx_1\wedge dx_3\wedge dx_4\\
&+ 2\Big(u^4v^3 + u^3v^4 - u^2v^1 - u^1v^2\Big)\, dx_1\wedge dx_2\wedge dx_4\\
&- 2\Big(u^4v^1 - u^3v^2 - u^2v^3 + u^1v^4\Big)\, dx_1\wedge dx_2\wedge dx_3,\\
2:\ & 2\Big(v^1v^4 + v^2v^3 + u^1u^4 + u^2u^3\Big)\, dx_2\wedge dx_3\wedge dx_4
\end{aligned}
$$

$$
\begin{aligned}
&-2\Big(-u^4v^1+u^3v^2-u^2v^3+u^1v^4\Big)\,dx_1\wedge dx_3\wedge dx_4\\
&+2\Big(v^1v^3-v^2v^4+u^1u^3-u^2u^4\Big)\,dx_1\wedge dx_2\wedge dx_4\\
&-\Big((u^1)^2+(u^2)^2+(u^3)^2+(u^4)^2+(v^1)^2+(v^2)^2\\
&+(v^3)^2+(v^4)^2\Big)\,dx_1\wedge dx_2\wedge dx_3,\\
3:\ &\Big(-(u^1)^2+(u^2)^2+(u^3)^2-(u^4)^2+(v^1)^2-(v^2)^2\\
&-(v^3)^2+(v^4)^2\Big)\,dx_2\wedge dx_3\wedge dx_4\\
&-2\Big(u^4v^4-u^3v^3+u^2v^2+u^1v^1\Big)\,dx_1\wedge dx_3\wedge dx_4\\
&+2\Big(v^3v^4-v^2v^1-u^3u^4+u^1u^2\Big)\,dx_1\wedge dx_2\wedge dx_4\\
&-2\Big(v^1v^4-v^3v^2-u^1u^4+u^2u^3\Big)\,dx_1\wedge dx_2\wedge dx_3.
\end{aligned}
$$

REMARK 2.10. It is possible to derive the conservation laws obtained above by the Nöther theorem 2.23, but we preferred here the explicit way.

3. The Burgers equation

Consider the Burgers equation $\mathcal{E}$

$$u_t = u_{xx} + uu_x \tag{2.59}$$

and choose *internal coordinates* on $\mathcal{E}^\infty$ by setting $u_k = u_{(k,0)}$. Below we compute the complete algebra of higher symmetries for (2.59) using the method described in [**60**] and first published in [**105**].

3.1. Defining equations. Let us rewrite restrictions onto $\mathcal{E}^\infty$ of all basic concepts in this coordinate system.

For the total derivatives we obviously obtain

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i}, \tag{2.60}$$

$$D_t = \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} D_x^i(u_2 + u_0u_1)\frac{\partial}{\partial u_i}. \tag{2.61}$$

The operator of universal linearization for $\mathcal{E}$ is then of the form

$$\ell_{\mathcal{E}} = D_t - u_1 - u_0D_x - D_x^2, \tag{2.62}$$

and, as it follows from Theorem 2.15 on p. 72, an evolutionary vector field

$$\partial_\varphi = \sum_{i=1}^{\infty} D_x^i(\varphi)\frac{\partial}{\partial u_i} \tag{2.63}$$

is a symmetry for $\mathcal{E}$ if and only if the function $\varphi = \varphi(x, t, u_0, \dots, u_k)$ satisfies the equation

$$D_t\varphi = u_1\varphi + u_0 D_x\varphi + D_x^2\varphi, \tag{2.64}$$

where D_t, D_x are given by (2.60), (2.61). Computing $D_x^2\varphi$ we obtain

$$D_x^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + 2\sum_{i=1}^{k} u_{i+1}\frac{\partial^2\varphi}{\partial x\partial u_i} + \sum_{i,j=0}^{k} u_{i+1}u_{j+1}\frac{\partial^2\varphi}{\partial u_i\partial u_j} + \sum_{i=0}^{k} u_{i+2}\frac{\partial\varphi}{\partial u_i},$$

while

$$D_x^i(u_0u_1 + u_3) = \sum_{\alpha=0}^{i}\binom{i}{\alpha}u_\alpha u_{i-\alpha+1} + u_{i+3}.$$

Hence, (2.64) transforms to

$$\begin{aligned}\frac{\partial\varphi}{\partial t} + \sum_{i=1}^{k}\sum_{\alpha=1}^{i}\binom{i}{\alpha}u_\alpha u_{i-\alpha+1}\frac{\partial\varphi}{\partial u_i} &= u_1\varphi + u_0\frac{\partial\varphi}{\partial x} + \frac{\partial^2\varphi}{\partial x^2}\\ &\quad + 2\sum_{i=1}^{k} u_{i+1}\frac{\partial^2\varphi}{\partial x\partial u_i} + \sum_{i,j=0}^{k} u_{i+1}u_{j+1}\frac{\partial^2\varphi}{\partial u_i\partial u_j}.\end{aligned} \tag{2.65}$$

3.2. Higher order terms. Note now that the left-hand side of (2.65) is independent of u_{k+1} while the right-hand one is quadratic in this variable and is of the form

$$u_{k+1}^2\frac{\partial^2\varphi}{\partial u_k^2} + 2u_{k+1}\left(\frac{\partial^2\varphi}{\partial x\partial u_k} + \sum_{i=0}^{k-1} u_{i+1}\frac{\partial^2\varphi}{\partial u_i\partial u_k}\right).$$

It means that

$$\varphi = Au_k + \psi, \tag{2.66}$$

where $A = A(t)$ and $\psi = \psi(t, x, u_0, \dots, u_{k-1})$. Substituting (2.66) into equation (2.65) one obtains

$$\begin{aligned}\dot{A}u_k + \frac{\partial\psi}{\partial t} + \sum_{i=1}^{k-1}\sum_{\alpha=1}^{i}\binom{i}{\alpha}u_\alpha u_{i-\alpha+1}\frac{\partial\psi}{\partial u_i} + \sum_{i=1}^{k}\binom{k}{i}u_i u_{k-i+1}A\\ = u_1(Au_k + \psi) + u_0\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial x^2} + 2\sum_{i=1}^{k-1} u_{i+1}\frac{\partial^2\psi}{\partial x\partial u_i}\\ + \sum_{i,j=0}^{k-1} u_{i+1}u_{j+1}\frac{\partial^2\psi}{\partial u_i\partial u_j},\end{aligned}$$

where $\dot{A} \stackrel{\text{def}}{=} dA/dt$. Here again everything is at most quadratic in u_k, and equating coefficients at u_k^2 and u_k we get

$$\frac{\partial^2\psi}{\partial u_{k-1}^2} = 0, \qquad 2\left(\sum_{i=0}^{k-2} u_{i+1}\frac{\partial^2\psi}{\partial u_i \partial u_{k-1}} + \frac{\partial^2\psi}{\partial x \partial u_{k-1}}\right) = ku_1 A + \dot{A}.$$

Hence,

$$\psi = \frac{1}{2}(ku_0 A + \dot{A}x + \dot{a})u_{k-1} + O[k-2],$$

where $a = a(t)$ and $O[l]$ denotes a function independent of u_i, $i > l$. Thus

$$\varphi = Au_k + \frac{1}{2}(ku_0 A + \dot{A}x + \dot{a})u_{k-1} + O[k-2] \tag{2.67}$$

which gives the "upper estimate" for solutions of (2.64).

3.3. Estimating Jacobi brackets. Let

$$\varphi = \varphi(t, x, u_0, \dots, u_k), \qquad \psi = \psi(t, x, u_0, \dots, u_l)$$

be two symmetries of $\mathcal{E}$. Then their Jacobi bracket restricted onto $\mathcal{E}^\infty$ looks as

$$\{\varphi, \psi\} = \sum_{i=0}^{l} D_x^i(\varphi)\frac{\partial\psi}{\partial u_i} - \sum_{i=0}^{k} D_x^j(\psi)\frac{\partial\varphi}{\partial u_j}. \tag{2.68}$$

Suppose that the function φ is of the form (2.67) and similarly

$$\psi = Bu_l + \frac{1}{2}(lu_0 B + \dot{B}x + \dot{b})u_{l-1} + O[l-2]$$

and let us compute (2.68) for these functions temporary denoting $ku_0A + \dot{A} + a$ and $lu_0B + \dot{B} + b$ by $\bar{A}$ and $\bar{B}$ respectively. Then we have:

$$\begin{aligned}
\{\varphi, \psi\} &= D_x^l(Au_k + \frac{1}{2}\bar{A}u_{k-1})B + \frac{1}{2}D_x^{l-1}(Au_k + \frac{1}{2}\bar{A}u_{k-1})\bar{B} \\
&- D_x^k(Bu_l + \frac{1}{2}\bar{B}u_{l-1})A - \frac{1}{2}D_x^{k-1}(Bu_k + \frac{1}{2}\bar{B}u_{l-1})\bar{A} + O[k+l-1] \\
&= \frac{1}{2}(lD_x(\bar{A})u_{k+l-2} + \bar{A}u_{k+l-1})\bar{B} + \frac{1}{2}(Au_{k+l-1} + \frac{1}{2}\bar{A}u_{k+l-2})B \\
&- \frac{1}{2}(kD_x(\bar{B})u_{k+l-2} + \bar{B}u_{k+l-1})\bar{A} - \frac{1}{2}(Bu_{k+l-1} + \frac{1}{2}\bar{B}u_{k+l-2})A + \\
&\qquad O[k+l-3],
\end{aligned}$$

or in short,

$$\{\varphi, \psi\} = \frac{1}{2}(l\dot{A}B - K\dot{B}A)u_{k+l-2} + O[k+l-3]. \tag{2.69}$$

3.4. Low order symmetries. These computations were done already in Section 3 of Chapter 1 (see equation (1.61)). They can also be done independently taking $k = 2$ and solving equation (2.64) directly. Then one obtains five independent solutions which are

$$\begin{aligned}
\varphi_1^0 &= u_1,\\
\varphi_1^1 &= tu_1 + 1,\\
\varphi_2^0 &= u_2 + u_0u_1,\\
\varphi_2^1 &= tu_2 + (tu_0 + \frac{1}{2}x)u_1 + \frac{1}{2}u_0,\\
\varphi_2^2 &= t^2u_2 + (t^2u_0 + tx)u_1 + tu_0 + x. \qquad (2.70)
\end{aligned}$$

3.5. Action of low order symmetries. Let us compute the action

$$T_i^j \stackrel{\text{def}}{=} \{\varphi_i^j, \bullet\} = \partial_{\varphi_i^j} - \ell_{\varphi_i^j}$$

of symmetries φ_i^j on other symmetries of the equation $\mathcal{E}$.

For φ_1^0 one has

$$T_1^0 = \partial_{u_1} - \ell_{u_1} = \sum_{i\geq 0} u_{i+1}\frac{\partial}{\partial u_i} - D_x = -\frac{\partial}{\partial x}.$$

Hence, if $\varphi = Au_k + O[k-1]$ is a function of the form (2.67), then we obtain

$$T_1^0\varphi = -\frac{1}{2}\dot{A}u_{k-1} + O[k-2].$$

Consequently, if φ is a symmetry, then, since $\mathrm{sym}(\mathcal{E})$ is closed under the Jacobi bracket,

$$(T_1^0)^{k-1}\varphi = \left(-\frac{1}{2}\right)^{k-1}\frac{d^{k-1}A}{dt^{k-1}}u_1 + O[0]$$

is a symmetry as well. But from (2.70) one sees that first-order symmetries are linear in t. Thus, we have the following result:

PROPOSITION 2.25. *If $\varphi = Au_k + O[k-1]$ is a symmetry of the Burgers equation, then A is a k-th degree polynomial in t.*

3.6. Final description. Note that direct computations show that the equation $\mathcal{E}$ possesses a third-order symmetry of the form

$$\varphi_3^0 = u_3 + \frac{3}{2}u_0u_2 + \frac{3}{2}u_0^2 + \frac{3}{4}u_0^2u_1.$$

Using the actions T_2^2 and T_3^0, one can see that

$$((T_2^2)^i \circ (T_3^0 \circ T_2^2)^{k-1})u_1 = \left(-\frac{3}{2}\right)^{k-1}\frac{k!(k-1)!}{(k-i)!}u_k + O[k-1] \qquad (2.71)$$

is a symmetry, since u_1 is the one.

THEOREM 2.26. *The symmetry algebra* $\mathrm{sym}(\mathcal{E})$ *for the Burgers equation* $\mathcal{E} = \{u_t = uu_x + u_{xx}\}$, *as a vector space, is generated by elements of the form*

$$\varphi_k^i = t^i u_k + O[k-1], \quad k \geq 1,\ i = 0, \dots, k,$$

which are polynomial in all variables. For the Jacobi bracket one has

$$\{\varphi_k^i, \varphi_l^j\} = \frac{1}{2}(li - kj)\varphi_{k+l-2}^{i+j-1} + O[k+l-3]. \tag{2.72}$$

The Lie algebra $\mathrm{sym}(\mathcal{E})$ *is simple and has* φ_1^0, φ_2^2, *and* φ_3^0 *as its generators.*

PROOF. It only remains to prove that all φ_k^i are polynomials and that $\mathrm{sym}(\mathcal{E})$ is a simple Lie algebra. The first fact follows from (2.71) and from the obvious observation that coefficients of both T_2^2 and T_3^0 are polynomials.

Let us prove that $\mathrm{sym}(\mathcal{E})$ is a simple Lie algebra. To do this, let us introduce an order in the set $\{\varphi_k^i\}$ defining

$$\Phi_{\frac{k(k+1)}{2}+i} \stackrel{\mathrm{def}}{=} \varphi_k^i.$$

Then any symmetry may be represented as $\sum_{\alpha=1}^{s} \lambda_\alpha \Phi_\alpha$, $\lambda \in \mathbb{R}$.

Let $I \subset \mathrm{sym}(\mathcal{E})$ be an ideal and $\Phi = \Phi_s + \sum_{\alpha=1}^{s-1} \lambda_\alpha \Phi_\alpha$ be its element. Assume that $\Phi_s = \varphi_k^i$ for some $k \geq 1$ and $i \leq k$.

Note now that

$$T_1^1 = \sum_{\alpha \geq 0} D_x^\alpha (tu_1 + 1)\frac{\partial}{\partial u_\alpha} - tD_x = \frac{\partial}{\partial u_0} - t\frac{\partial}{\partial x}$$

and

$$T_2^0 = \sum_{\alpha \geq 0} D_x^\alpha (u_2 + u_0 u_1)\frac{\partial}{\partial u_\alpha} - D_x^2 - u_0 D_x - u_1 = -\frac{\partial}{\partial t}.$$

Therefore,

$$((T_1^1)^{k-1} \circ (T_2^0)^i)\Phi = c\varphi_1^0,$$

where the coefficient c does not vanish. Hence, I contains the function φ_1^0. But due to (2.71) the latter, together with the functions φ_2^2 and φ_3^0, generates the whole algebra. □

Further details on the structure of $\mathrm{sym}(\mathcal{E})$ one can find in [**60**].

4. The Hilbert–Cartan equation

We compute here classical and higher symmetries of the Hilbert–Cartan equation [**2**]. Since higher symmetries happen to depend on arbitrary functions, we consider some special choices of these functions [**38**].

4.1. Classical symmetries. The Hilbert–Cartan equation is in effect an underdetermined system of ordinary differential equations in the sense of Definition 1.10 of Subsection 2.1 in Chapter 1. The number of independent variables, n, is one while the number of dependent variables, m, is two. Local coordinates are given by x, u, v in $J^0(\pi)$, while the order of the equations is two, i.e.,

$$u_x = v_{xx}^2 \tag{2.73}$$

The representative morphism (see Definition 1.6 on p. 6) Φ is given by

$$\Phi_\Delta(x, u, v, u_x, v_x, u_{xx}, v_{xx}) = u_x - v_{xx}^2. \tag{2.74}$$

The total derivative operator D_x is given by the formula

$$D = D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + \cdots \tag{2.75}$$

To construct classical symmetries for (2.73), we start from the vector field X, given by

$$\begin{aligned}X = X(x,u,v)\frac{\partial}{\partial x} &+ U_0(x,u,v)\frac{\partial}{\partial u} + V_0(x,u,v)\frac{\partial}{\partial v}\\
&+ U_1(x,u,x,u_x,v_x)\frac{\partial}{\partial u_x} + V_1(x,u,x,u_x,v_x)\frac{\partial}{\partial v_x}\\
&+ U_2(x,u,x,u_x,v_x,u_{xx},v_{xx})\frac{\partial}{\partial u_{xx}} + V_2(x,u,x,u_x,v_x,u_{xx},v_{xx})\frac{\partial}{\partial v_{xx}}.\end{aligned}$$

The defining relations (1.34) (see p. 26) for U_1, V_1, U_2, V_2 are

$$\begin{aligned}U_1 &= D(U_0) - u_x D(X) = D(U_0 - u_x X) + u_{xx}X,\\
V_1 &= D(V_0) - v_x D(X) = D(V_0 - v_x X) + v_{xx}X,\\
U_2 &= D(U_1) - u_{xx} D(X) = D^2(U_0 - u_x X) + u_{xxx}X,\\
V_2 &= D(V_1) - v_{xx} D(X) = D^2(V_0 - v_x X) + v_{xxx}X.\end{aligned} \tag{2.76}$$

From (2.76) we derive the following explicit expressions for U_1, V_1, U_2, V_2:

$$\begin{aligned}U_1 &= U_{0,x} + U_{0,u}u_x + U_{0,v}v_x - u_x(X_{0,x} + X_{0,u}u_x + X_{0,v}v_x),\\
V_1 &= V_{0,x} + V_{0,u}u_x + V_{0,v}v_x - u_x(X_{0,x} + X_{0,u}u_x + X_{0,v}v_x),\\
U_2 &= U_{0,xx} + 2U_{0,xu}u_x + 2U_{0,xv}v_x + U_{0,uu}u_x^2 + 2U_{0,uv}u_xv_x + U_{0,u}u_{xx}\\
&\quad + U_{0,vv}v_x^2 + U_{0,v}v_{xx} - 2u_{xx}(X_{0,x} + X_{0,u}u_x + X_{0,v}v_x)\\
&\quad - u_x(X_{0,xx} + 2X_{0,xu}u_x + 2X_{0,xv}v_x\\
&\quad + X_{0,uu}u_x^2 + 2X_{0,uv}u_xv_x + X_{0,u}u_{xx} + X_{0,vv}v_x^2 + X_{0,v}v_{xx}),\\
V_2 &= V_{0,xx} + 2V_{0,xu}u_x + 2V_{0,xv}v_x + V_{0,uu}u_x^2 + 2V_{0,uv}u_xv_x + V_{0,u}u_{xx}\\
&\quad + V_{0,vv}v_x^2 + V_{0,v}v_{xx} - 2u_{xx}(X_{0,x} + X_{0,u}u_x + X_{0,v}v_x)\\
&\quad - v_x(X_{0,xx} + 2X_{0,xu}u_x + 2X_{0,xv}v_x + X_{0,uu}u_x^2\\
&\quad + 2X_{0,uv}u_xv_x + X_{0,u}u_{xx} + X_{0,vv}v_x^2 + X_{0,v}v_{xx}).\end{aligned} \tag{2.77}$$

Now the symmetry-condition $X(\Phi_\Delta)\,|_{\mathcal{E}} = 0$ results in

$$U_1 - 2v_{xx}V_2 = \lambda(u_x - v_{xx}^2) \tag{2.78}$$

which is equivalent to

$$U_1 - 2(u_x)^{\frac{1}{2}}V_2 = 0 \bmod \Phi_\Delta = 0, \tag{2.79}$$

which results in

$$\begin{aligned} &U_{0,x} + U_{0,u}u_x + U_{0,v}v_x - u_x(X_{0,x} + X_{0,u}u_x + X_{0,v}v_x) \\ &\quad - \Big(V_{0,xx} + 2V_{0,xu}u_x + 2V_{0,xv}v_x + V_{0,uu}u_x^2 + 2V_{0,uv}u_xv_x + V_{0,u}u_{xx} \\ &\quad + V_{0,vv}v_x^2 + V_{0,v}v_{xx} - 2u_{xx}(X_{0,x} + X_{0,u}u_x + X_{0,v}v_x) \\ &\quad - v_x(X_{0,xx} + 2X_{0,xu}u_x + 2X_{0,xv}v_x + X_{0,uu}u_x^2 + 2X_{0,uv}u_xv_x \\ &\quad + X_{0,u}u_{xx} + X_{0,vv}v_x^2 + X_{0,v}v_{xx})\Big) \cdot 2(u_x)^{1/2} = 0. \end{aligned} \tag{2.80}$$

Equation (2.80) is a polynomial in the "variables" $(u_x)^{1/2}$, v_x, u_{xx}, the coefficients of which should vanish. From this observation we obtain the following system of equations:

$$\begin{aligned} 1 &: & U_{0,x} &= 0, \\ u_x^{1/2} &: & -2V_{0,xx} &= 0, \\ u_x^{1/2}v_x &: & -4V_{0,xv} + 2X_{0,xx} &= 0, \\ u_x^{1/2}u_{xx} &: & -2V_{0,u} &= 0, \\ u_x^{1/2}u_{xx}v_x &: & 2X_{0,u} &= 0, \\ u_x^{1/2}v_x^2 &: & -2V_{0,vv} + 4X_{0,xv} &= 0, \\ u_x^{1/2}v_x^3 &: & 2X_{0,vv} &= 0, \\ u_x &: & U_{0,u} - X_{0,x} - 2V_{0,v} + 4X_{0,x} &= 0, \\ u_xv_x &: & -X_{0,v} + 4X_{0,v} + 2X_{0,v} &= 0, \\ u_x^2 &: & -X_{0,u} + 4X_{0,u} &= 0, \\ u_x^{3/2} &: & -4v_{0,xu} &= 0, \\ u_x^{3/2}v_x &: & -4V_{0,uv} + 4X_{0,xu} &= 0, \\ u_x^{3/2}v_x^2 &: & 4X_{0,uv} &= 0, \\ u_x^{5/2} &: & -2V_{0,uu} &= 0, \\ u_x^{5/2}v_x &: & 2X_{0,uu} &= 0, \\ v_x &: & U_{0,v} &= 0. \end{aligned} \tag{2.81}$$

From system (2.81) we first derive:

$$X_{0,u} = X_{0,v} = 0, \qquad V_{0,uu} = V_{0,uv} = V_{0,vv} = 0 = V_{0,u} = V_{0,xx},$$

$[A_i, A_j]$	A_1	A_2	A_3	A_4	A_5	A_6
A_1	0	0	0	0	A_1	A_3
A_2		0	0	$2A_2$	$-3A_2$	0
A_3			0	A_3	0	0
A_4				0	0	$-A_6$
A_5					0	A_6
A_6						0

FIGURE 2.1. Commutator table for classical symmetries of the Hilbert–Cartan equation

which results in the equality $X(x,u,v) = H(x)$ and in the fact that V_0 is independent of u, being of degree 1 in v and of degree 1 in x, i.e.,

$$X(x,u,v) = H(x), \quad V_0 = a_0 + a_1x + a_2v + a_3xv.$$

Now from the equation labeled by $u_x^{1/2}v_x$ in (2.81) we derive

$$H(x) = a_3x^2 + a_4x + a_5. \tag{2.82}$$

From the equations $U_{0,v} = 0$ and $U_{0,u} + 3X_{0,x} - 2V_{0,v} = 0$ we get

$$U_0 = -(4a_3x + 2a_2 - 3a_5)u + G(x). \tag{2.83}$$

Finally from $U_{0,x} = 0$ we arrive at $a_3 = 0$, $G(x) = a_6$, from which the general solution is obtained as

$$X = a_4x + a_5, \quad U_0 = (2a_2 - 3a_4)u + a_6, \quad V_0 = a_0 + a_1x + a_2v.$$

This results in a 6-dimensional Lie algebra, the generators of which are given by

$$\begin{aligned} A_1 &= \frac{\partial}{\partial x}, \\ A_2 &= \frac{\partial}{\partial u}, \\ A_3 &= \frac{\partial}{\partial v}, \\ A_4 &= 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\ A_5 &= x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}, \\ A_6 &= x\frac{\partial}{\partial v}, \end{aligned}$$

while the commutator table is given on Fig. 2.1.

4.2. Higher symmetries. As a very interesting and completely computable application of the theory of higher symmetries developed in Subsection 2.1, we construct in this section the algebra of higher symmetries for the Hilbert–Cartan equation $\mathcal{E}$

$$u_x - v_{xx}^2 = 0. \tag{2.84}$$

First of all, note that $\mathcal{E}^\infty$ is given by the system of equations:

$$D^i(u_x - v_{xx}^2) = 0, \quad i = 0, 1, \dots \tag{2.85}$$

where D is defined by

$$D = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1}\frac{\partial}{\partial u_k} + \sum_{k=0}^{\infty} v_{k+1}\frac{\partial}{\partial v_k}, \tag{2.86}$$

and $u_k = u_{\underbrace{x\dots x}_{k \text{ times}}}$. So from (2.84) we have

$$\begin{aligned}
D^1F &= u_2 - 2v_2v_3 = 0,\\
D^2F &= u_3 - 2v_3^2 - 2v_2v_4 = 0,\\
D^iF &= u_{1+i} - \sum_{l=0}^{i}\binom{i}{l} v_{2+l}v_{2+i-l} = 0,
\end{aligned}$$

$i = 3, \dots$, with $F(x, u, v, u_1, v_1, u_2, v_2) = u_1 - v_2^2 = 0$.

In order to construct higher symmetries of (2.84), we introduce internal coordinates on $\mathcal{E}^\infty$ which are

$$x, u, v, v_1, v_2, v_3, \cdots \tag{2.87}$$

The total derivative operator restricted to $\mathcal{E}^\infty$, again denoted by D, is given by the following expression

$$\begin{aligned}
D &= \frac{\partial}{\partial x} + v_2^2\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v} + \sum_{i>0} v_{i+1}\frac{\partial}{\partial v_i},\\
D^{(n)} &= \frac{\partial}{\partial x} + v_2^2\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v} + \sum_{i>0}^{n} v_{i+1}\frac{\partial}{\partial v_i}.
\end{aligned} \tag{2.88}$$

Suppose that a vertical vector field $V = \mathcal{Э}_\Phi$ with the generating function Φ,

$$\Phi = \big(f^u(x, u, v, v_1, \dots, v_n), f^v(x, u, v, v_1, \dots, v_n)\big), \tag{2.89}$$

is a higher symmetry of $\mathcal{E}$. We introduce the notation

$$f[v_k] = f(x, u, v, v_1, \dots, v_k). \tag{2.90}$$

Since the vertical vector field V is formally given by

$$V = f^u[v_n]\frac{\partial}{\partial u} + f^v[v_n]\frac{\partial}{\partial v} + f^{v_1}[v_{n+1}]\frac{\partial}{\partial v_1} + f^{v_2}[v_{n+2}]\frac{\partial}{\partial v_2} + \dots, \tag{2.91}$$

we derive the following symmetry conditions from (2.84)

$$\begin{aligned}
D^{(n)}f^u[v_n] - 2v_2f^{v_2}[v_{n+2}] &= 0,\\
D^{(n)}f^v[v_n] - f^{v_1}[v_{n+1}] &= 0,\\
D^{(n+1)}f^{v_1}[v_{n+1}] - f^{v_2}[v_{n+2}] &= 0.
\end{aligned} \tag{2.92}$$

In effect, the second and third equation of (2.92) are just the definitions of $f^{v_1}[v_{n+1}]$ and $f^{v_2}[v_{n+2}]$, due to the evolutionary property of ∂_Φ. We now want to construct the general solution of system (2.92). In order to do so, we first solve the third equation in (2.92) for $f^{v_2}[v_{n+2}]$,

$$f^{v_2}[v_{n+2}] = D^{(n+1)} f^{v_1}[v_{n+1}], \tag{2.93}$$

and the system reduces to

$$\begin{aligned} D^{(n)} f^{u}[v_n] - 2v_2 D^{(n+1)} f^{v_1}[v_{n+1}] &= 0, \\ D^{(n)} f^{v}[v_n] - f^{v_1}[v_{n+1}] &= 0. \end{aligned} \tag{2.94}$$

REMARK 2.11. At this stage it would be possible to solve the last equation for $f^{v_1}[v_{n+1}]$, but we prefer not to do so.

Now (2.94) is a polynomial in v_{n+2} of degree 1 and (2.94) reduces to

$$\begin{aligned} v_{n+2}: &\qquad -2v_2 \frac{\partial f^{v_1}[v_{n+1}]}{\partial v_{n+1}} = 0, \\ 1: &\qquad D^{(n)} f^{u}[v_n] - 2v_2 D^{(n)} f^{v_1}[v_n] = 0, \\ : &\qquad D^{(n)} f^{v}[v_n] - f^{v_1}[v_n] = 0. \end{aligned} \tag{2.95}$$

In (2.95) and below, "v_{n+2} :" refers to the coefficient at v_{n+2} in a particular equation. From (2.95) we arrive, due to the fact that second and third equation are polynomial in v_{n+1}, at

$$\begin{aligned} v_{n+1}: &\qquad \frac{\partial f^{u}[v_n]}{\partial v_n} - 2v_2 \frac{\partial f^{u_1}[v_n]}{\partial v_n} = 0, \\ 1: &\qquad D^{(n-1)} f^{u}[v_n] - 2v_2 D^{(n-1)} f^{v_1}[v_n] = 0, \\ v_{n+1}: &\qquad \frac{\partial f^{v}[v_n]}{\partial v_n} = 0, \\ 1: &\qquad D^{(n-1)} f^{v}[v_n] - f^{v_1}[v_n] = 0. \end{aligned} \tag{2.96}$$

To solve system (2.96), we first note that

$$f^{v}[v_n] = f^{v}[v_{n-1}]. \tag{2.97}$$

By differentiation of the fourth equation in (2.96) twice with respect to v_n, we obtain

$$\frac{\partial^2 f^{v_1}[v_n]}{\partial v_n^2} = 0. \tag{2.98}$$

By consequence, f^{v_1} is linear with respect to v_n, i.e.,

$$f^{v_1}[v_n] = H^1[v_{n-1}] + v_n H^2[v_{n-1}]. \tag{2.99}$$

Now, substitution of (2.97) and (2.99) into (2.96) yields the following system of equations

$$\frac{\partial f^{u}[v_n]}{\partial v_n} - 2v_2 H^2[v_{n-1}] = 0,$$

$$\begin{aligned}D^{(n-1)}f^u[v_n] - 2v_2D^{(n-1)}H^1[v_{n-1}] - 2v_2v_nD^{(n-1)}H^2[v_{n-1}] = 0,\\ D^{(n-1)}f^v[v_{n-1}] - H^1[v_{n-1}] - v_nH^2[v_{n-1}] = 0.\end{aligned} \tag{2.100}$$

We solve the first equation in (2.100) for $f^u[v_n]$, i.e.,

$$f^u[v_n] = 2v_2v_nH^2[v_{n-1}] + H^3[v_{n-1}], \tag{2.101}$$

and from the second and third equation in (2.100) we arrive at

$$\begin{aligned}&2v_3v_nH^2[v_{n-1}] + 2v_2v_nD^{(n-1)}H^2[v_{n-1}] + D^{(n-1)}H^3[v_{n-1}]\\ &\qquad - 2v_2D^{(n-1)}H^1[v_{n-1}] - 2v_2v_nD^{(n-1)}H^2[v_{n-1}] = 0,\\ &D^{(n-1)}f^v[v_{n-1}] - H^1[v_{n-1}] - v_nH^2[v_{n-1}] = 0.\end{aligned} \tag{2.102}$$

Due to cancellation of second and fifth term in the first equation of (2.102) and its polynomial structure with respect to v_n, we obtain a resulting system of four equations:

$$\begin{aligned}v_n:&\qquad 2v_3H^2[v_{n-1}] + \frac{\partial H^3[v_{n-1}]}{\partial v_{n-1}} - 2v_2\frac{\partial H^1[v_{n-1}]}{\partial v_{n-1}} = 0,\\ 1:&\qquad D^{(n-2)}H^3[v_{n-1}] - 2v_2D^{(n-2)}H^1[v_{n-1}] = 0,\\ v_n:&\qquad \frac{\partial f^v[v_{n-1}]}{\partial v_{n-1}} - H^2[v_{n-1}] = 0,\\ 1:&\qquad D^{(n-2)}f^v[v_{n-1}] - H^1[v_{n-1}] = 0.\end{aligned} \tag{2.103}$$

From (2.103) we solve the third equation for $H^2[v_{n-1}]$,

$$H^2[v_{n-1}] = \frac{\partial f^v[v_{n-1}]}{\partial v_{n-1}}, \tag{2.104}$$

and integrate the first one in (2.103):

$$2v_3\frac{\partial f^v[v_{n-1}]}{\partial v_{n-1}} + \frac{\partial H^3[v_{n-1}]}{\partial v_{n-1}} - 2v_2\frac{\partial H^1[v_{n-1}]}{\partial v_{n-1}} = 0, \tag{2.105}$$

which leads to

$$H^3[v_{n-1}] = 2v_2H^1[v_{n-1}] - 2v_3f^v[v_{n-1}] + H^4[v_{n-2}]. \tag{2.106}$$

By obtaining (2.106), we have to put in the requirement $n-1>3$ and we shall return to this case in the next subsection.

We now proceed by substituting the results (2.104) and (2.106) into (2.103), which leads to

$$\begin{aligned}&2v_3H^1[v_{n-1}] + 2v_2D^{(n-2)}H^1[v_{n-1}] - 2v_4f^v[v_{n-1}] - 2v_3D^{(n-2)}f^v[v_{n-1}]\\ &\qquad + D^{(n-2)}H^4[v_{n-2}] - 2v_2D^{(n-2)}H^1[v_{n-1}] = 0,\\ &D^{(n-2)}f^v[v_{n-1}] - H^1[v_{n-1}] = 0.\end{aligned} \tag{2.107}$$

By cancellation of the second and sixth term in the first equation of (2.107), we finally arrive at

$$D^{(n-2)}f^v[v_{n-1}] - H^1[v_{n-1}] = 0,$$

$$D^{(n-2)}H^4[v_{n-2}] - 2v_4 f^v[v_{n-1}] = 0, \tag{2.108}$$

where the first equation in (2.108) can be considered as defining relation for $H^1[v_{n-1}]$, while the second equation determines $f^v[v_{n-1}]$ in terms of an *arbitrary function* $H^4[v_{n-2}]$. The final result can now be obtained by (2.104) and (2.106):

$$\begin{aligned} H^2[v_{n-1}] &= \frac{\partial f^v[v_{n-1}]}{\partial v_{n-1}}, \\ H^3[v_{n-1}] &= 2v_2 H^1[v_{n-1}] - 2v_3 f^v[v_{n-1}] + H^4[v_{n-2}], \end{aligned} \tag{2.109}$$

together with (2.108) and (2.101):

$$\begin{aligned} f^u[v_n] &= 2v_2 v_n \frac{\partial f^v[v_{n-1}]}{\partial v_{n-1}} + 2v_2 H^1[v_{n-1}] - 2v_3 f^v[v_{n-1}] + H^4[v_{n-2}], \\ f^v[v_n] &= f^v[v_{n-1}], \end{aligned} \tag{2.110}$$

whereas in (2.110) $f^v[v_{n-1}]$, $H^1[v_{n-1}]$ are defined by (2.108) in terms of an *arbitrary function* $H^4[v_{n-2}]$! The general result of this section can now be formulated in the following

THEOREM 2.27. *Let H be an arbitrary function of the variables x, u, $v, \ldots, v_{n-2}$, i.e.,*

$$H = H[v_{n-2}], \tag{2.111}$$

and let us define

$$\begin{aligned} f^v[v_{n-1}] &= \frac{1}{2v_4} D^{(n-2)} H[v_{n-2}], \\ f^u[v_n] &= 2v_2 D^{(n-1)} f^v[v_{n-1}] - 2v_3 f^v[v_{n-1}] + H[v_{n-2}]. \end{aligned} \tag{2.112}$$

Then the vector field

$$V = f^u[v_n]\frac{\partial}{\partial u} + f^v[v_{n-1}]\frac{\partial}{\partial v} \tag{2.113}$$

is a higher symmetry of (2.85).

Conversely, given a higher symmetry of (2.85), *then there exists a function H, such that the components f^u, f^v of V are defined by* (2.112).

4.3. Special cases. Due to the restriction $n > 4$ the result (2.109) and (2.110) holds for

$$n = 5, \ldots \tag{2.114}$$

meaning that $H^4[v_{n-2}]$ is a *free* function of x, u, $v, \ldots, v_{n-2}$ and $f^v[v_{n-1}]$ is obtained by (2.109)

$$f^v[v_{n-1}] = \frac{1}{2v_4} D^{(n-2)} H^4[v_{n-2}]. \tag{2.115}$$

From (2.115) and (2.109) it is clear that $f^v[v_{n-1}]$ is linear with respect to the variable v_{n-1} and

$$f^v[v_{n-1}] = \frac{v_{n-1}}{2v_4}\frac{\partial H^4[v_{n-2}]}{\partial v_{n-2}} + \widetilde{f^v}[v_{n-2}]. \tag{2.116}$$

Moreover, the requirement that $f^v[v_{n-1}]$ is independent of v_{n-1} reduces to $H^4[v_{n-2}]$ to be independent of v_{n-2}, i.e.,

$$\frac{\partial f^v[v_{n-1}]}{\partial v_{n-1}} = 0 \Rightarrow H^4[v_{n-2}] = H^4[v_{n-3}]. \tag{2.117}$$

The result (2.117) holds for all $n > 5$.

The results for higher symmetries, or Lie–Bäcklund transformations, for $n < 6$ are obtained by imposing additional conditions on the coefficient f^v of the evolutionary vector field.

The case $n = 5$.

$$\begin{aligned} f^v[v_4] = \frac{1}{2v_4}(&\frac{\partial H^4[v_3]}{\partial x} + v_2^2\frac{\partial H^4[v_3]}{\partial u} + v_1\frac{\partial H^4[v_3]}{\partial v} \\ &+ v_2\frac{\partial H^4[v_3]}{\partial v_1} + v_3\frac{\partial H^4[v_3]}{\partial v_2} + v_4\frac{\partial H^4[v_3]}{\partial v_3}). \end{aligned}$$

The requirement that $f^v[v_4]$ is independent of v_4 now leads to a genuine first order partial differential equation, i.e.,

$$\frac{\partial H^4}{\partial x} + v_2^2\frac{\partial H^4}{\partial u} + v_1\frac{\partial H^4}{\partial v} + v_2\frac{\partial H^4}{\partial v_1} + v_3\frac{\partial H^4}{\partial v_2} = 0, \tag{2.118}$$

and the general solution is given in terms of the invariants of the corresponding vector field

$$U = \frac{\partial}{\partial x} + v_2^2\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v} + v_2\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_2}, \tag{2.119}$$

where the set of invariants is given by

$$\begin{aligned} z_1 &= v_3, \\ z_2 &= v_2 - v_3x, \\ z_3 &= 2v_1 - 2v_2x + v_3x^2, \\ z_4 &= 6v - 6v_1x + 3v_2x^2 - v_3x^3, \\ z_5 &= 3u - 3v_2^2x + 3v_2v_3x^2 - v_3^2x^3. \end{aligned} \tag{2.120}$$

So H^4 is given by

$$H^4 = H^4(z_1, z_2, z_3, z_4, z_5), \tag{2.121}$$

whereas the formulas for f^v and f^u reduce to

$$\begin{aligned} f^u &= H^4 - v_2\frac{\partial H^4}{\partial v_2} - v_3\frac{\partial H^4}{\partial v_3} + v_2v_4\frac{\partial^2 H^4}{\partial v_3^2}, \\ f^v &= \frac{1}{2}\frac{\partial H^4}{\partial v_3}. \end{aligned} \tag{2.122}$$

The case $n = 4$. The requirement the function f^v is independent of v_3 reduces to

$$\frac{\partial^2 H^4}{\partial v_3^2} = 0, \tag{2.123}$$

and (2.118)

$$\frac{\partial H^4}{\partial x} + v_2^2 \frac{\partial H^4}{\partial u} + v_1 \frac{\partial H^4}{\partial v} + v_2 \frac{\partial H^4}{\partial v_1} + v_3 \frac{\partial H^4}{\partial v_2} = 0. \tag{2.124}$$

Substitution of (2.123) into (2.124) immediately leads to the condition

$$\frac{\partial}{\partial v_2} \frac{\partial}{\partial v_3} H^4 = 0, \tag{2.125}$$

i.e.,

$$f^v = f^v(x, u, v, v_1), \tag{2.126}$$

and the result completely reduces to the second order higher symmetries obtained by Anderson [**3**] and [**2**] leading to the 14-dimensional Lie algebra G_2.

5. The classical Boussinesq equation

The classical Boussinesq equation is written as the following system of partial differential equations in $J^3(\pi)$, where $\pi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ with independent variables x, t and u, v for dependent ones:

$$\begin{aligned} u_t &= (uv + \alpha v_{xx})_x = u_x v + u v_x + \alpha v_{xxx}, \\ v_t &= (u + \frac{1}{2} v^2)_x = u_x + v v_x. \end{aligned} \tag{2.127}$$

So in this application $u = (u, v)$ and $(x_1, x_2) = (x, t)$. In order to construct higher symmetries of (2.127), we have to construct solutions of the symmetry condition which are discussed in Section 2. For evolution equations it is custom to choose internal coordinates as $x, t, u, v, u_1, v_1, u_2, v_2, \ldots$, where

$$u_i = \frac{\partial^i u}{\partial x^i}, \quad v_i = \frac{\partial^i v}{\partial x^i}. \tag{2.128}$$

The partial derivative operators D_x and D_t are defined on $\mathcal{E}^\infty$ by

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + \sum_{i>0} u_{i+1} \frac{\partial}{\partial u_i} + \sum_{i>0} v_{i+1} \frac{\partial}{\partial v_i}, \\ D_t &= \frac{\partial}{\partial t} + \sum_{i>0} u_{it} \frac{\partial}{\partial u_i} + \sum_{i>0} v_{it} \frac{\partial}{\partial v_i}, \end{aligned} \tag{2.129}$$

while expressions for u_{it} and v_{it} are derived from (2.127) by

$$u_{it} = D_x^i(u_t), \quad v_{it} = D_x^i(v_t). \tag{2.130}$$

From (2.127) we derive the universal linearization operator as a 2×2 matrix operator by of the form

$$\ell_\Delta = \begin{pmatrix} vD_x + v_1 & \alpha D_x^3 + uD_x + u_1 \\ D_x & vD_x + v_1 \end{pmatrix}. \tag{2.131}$$

To construct higher symmetries for equations (2.127), we start from a vertical vector field of evolutionary type, i.e.,

$$Y \mapsto \partial_Y = \sum_{i=0}^{\infty} D_x^i(Y^u)\frac{\partial}{\partial u_i} + \sum_{i=0}^{\infty} D_x^i(Y^v)\frac{\partial}{\partial v_i}. \tag{2.132}$$

From this and the presentation of the universal linearization operator we derive the condition for $Y = (Y^u, Y^v)$ to be a higher symmetry of (2.127), i.e.,

$$\begin{aligned} &vD_xY^u + v_1Y^u + (\alpha D_x^3 + uD_x + u_1)Y^v = 0, \\ &D_xY^u + (vD_x + v_1)Y^v = 0. \end{aligned} \tag{2.133}$$

It is quite of interest to make some remarks here on the construction of solutions of this overdetermined system of partial differential equations for Y^u, Y^v. Recall that we require Y^u and Y^v to be dependent of a finite number variables. Equations (2.127) are graded, i.e., they admit a scaling symmetry,

$$-x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$

from where we have

$$\begin{aligned} &\deg(x) = -1, \qquad &&\deg(u) = 2, \qquad &&\deg\left(\frac{\partial}{\partial u}\right) = -2, \\ &\deg(t) = -2, &&\deg(v) = 1, &&\deg\left(\frac{\partial}{\partial v}\right) = -1. \end{aligned}$$

Due to the grading of (2.127), equations (2.132) and (2.133) are graded too and we require

$$\begin{aligned} &Y^u \text{ to dependent on } x, t, v, u, v_1, \dots, u_4, v_5, u_5, v_6, \\ &Y^v \text{ to dependent on } x, t, v, u, v_1, \dots, u_4, v_5. \end{aligned}$$

The general solution of (2.133) is then given by the following eight vector fields

$$\partial_{Y_i} = \partial_{(Y_i^u, Y_i^v)}, \qquad i = 1, \dots, 8,$$

where

$$\begin{aligned} Y_1^u &= \alpha v_3 + u_1 v + v_1 u, \\ Y_1^v &= u_1 + v_1 v; \end{aligned}$$

$$\begin{aligned} Y_2^u &= u_1, \\ Y_2^v &= v_1; \end{aligned}$$

$$
\begin{aligned}
Y_3^u &= tu_1,\\
Y_3^v &= tv_1 + 1;\\
Y_4^u &= \frac{1}{2}xu_1 + t(\alpha v_3 + u_1 v + v_1 u) + u,\\
Y_4^v &= \frac{1}{2}xv_1 + t(u_1 + v_1 v) + \frac{1}{2}v;\\
Y_5^u &= \frac{1}{2\alpha}x(\alpha v_3 + u_1 v + v_1 u) + t\big(u_3 + \frac{3}{2}v_3 v + 3v_2 v_1 + \frac{3}{4\alpha}u_1(v^2+2u)\\
&\quad + \frac{3}{2\alpha}v_1 v u\big) + \frac{3}{2}v^2 + \frac{1}{\alpha}vu,\\
Y_5^v &= \frac{1}{2\alpha}x(u_1 + v_1 v) + t\big(v_3 + \frac{3}{2\alpha}u_1 v + \frac{3}{4\alpha}v_1(v^2+2u)\big) + \frac{1}{4\alpha}v^2 + \frac{1}{\alpha}u;\\
Y_6^u &= 2\alpha v_5 + 4u_3 v + v_3(3v^2+5u) + 9u_2 v_1 + 10 v_2 u_1 + 12 v_2 v_1 v\\
&\quad + \frac{1}{\alpha}u_1 v(v^2+6u) + 3v_1^3 + \frac{3}{\alpha}v_1 u(v^2+u),\\
Y_6^v &= 2u_3 + 4v_3 v + 7v_2 v_1 + \frac{3}{\alpha}u_1(v^2+u) + \frac{1}{\alpha}v_1 v(v^2+6u);\\
Y_7^u &= \alpha u_5 + \frac{5}{2}v_5 v + \frac{15}{2}\alpha v_4 v_1 + \frac{5}{2}u_3(v^2+u) + \frac{25}{2}\alpha v_3 v_2\\
&\quad + \frac{5}{4}v_3 v(v^2+5u) + 5u_2 u_1 + \frac{45}{4}u_2 v_1 v + \frac{25}{2}v_2 u_1 v + \frac{5}{2}v_2 v_1(3v^2+5u)\\
&\quad + \frac{75}{8}u_1 v_1^2 + \frac{5}{16\alpha}u_1(v^4 + 12v^2 u + 6u^2) + \frac{15}{4}v_1^3 v + \frac{5}{4}v_1 v u(v^2+3u),\\
Y_7^v &= \alpha v_5 + \frac{5}{2}u_3 v + \frac{5}{2}v_3(v^2+u) + 5u_2 v_1 + 5v_2 u_1 + \frac{35}{4}v_2 v_1 v\\
&\quad + \frac{5}{4}u_1 v(v^2+3u) + \frac{15}{8}v_1^3 + \frac{5}{16}v_1(v^4 + 12v^2 u + 6u^2);\\
Y_8^u &= u_3 + \frac{3}{2}v_3 v + 3v_2 v_1 + \frac{3}{4\alpha}u_1(v^2+2u) + \frac{3}{2\alpha}v_1 v u,\\
Y_8^v &= v_3 + \frac{3}{2\alpha}u_1 v + \frac{3}{4\alpha}v_1(v^2+2u).
\end{aligned}
\tag{2.134}
$$

The Lie algebra structure of these symmetries is constructed by computing the Jacobi brackets of the respective generating functions $Y_i = (Y_i^u, Y_i^v)$. The commutators of the associated vector fields are given then in Fig. 2.2. The generating function Y_9 is defined here by

$$
\begin{aligned}
Y_9^u &= \frac{5}{2}\alpha v_7 + \frac{15}{2}u_5 v + \frac{5}{8}v_5(15v^2+14u) + 25u_4 v_1 + \frac{105}{4}v_4 u_1\\
&\quad + \frac{225}{4}v_4 v_1 v + \frac{175}{4}u_3 v_2 + \frac{25}{4}u_3 v(v^2+3u) + \frac{175}{4}v_3 u_2 + \frac{375}{4}v_3 v_2 v\\
&\quad + \frac{1125}{16}v_3 v_1^2 + \frac{25}{32\alpha}v_3(3v^4 + 30v^2 u + 14u^2) + \frac{75}{2\alpha}u_2 u_1 v
\end{aligned}
$$

$[Y_i, Y_j]$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
Y_1		0	$-Y_2$	$-Y_1$	$-Y_8$	0	0	0
Y_2			0	$-\frac{1}{2}Y_2$	$-\frac{1}{2\alpha}Y_4$	0	0	0
Y_3				$\frac{1}{2}Y_3$	$\frac{1}{\alpha}Y_4$	$4Y_8$	$\frac{5}{4}Y_6$	$\frac{3}{2\alpha}Y_1$
Y_4					$\frac{1}{2}Y_5$	$2Y_6$	$\frac{5}{2}Y_7$	$\frac{3}{2}Y_8$
Y_5						$\frac{4}{\alpha}Y_7$	Y_9	$\frac{3}{4\alpha}Y_6$
Y_6							0	0
Y_7								0
Y_8								

FIGURE 2.2. Commutator table for symmetries of the Boussinesq equation

$$
\begin{aligned}
&+ \frac{25}{16\alpha}u_2v_1(27v^2 + 26u) + \frac{375}{4}v_2^2v_1 + \frac{25}{8\alpha}v_2u_1(15v^2 + 14u) \\
&+ \frac{75}{4\alpha}v_2v_1v(v^2 + 5u) + \frac{125}{4\alpha}u_1^2v_1 + \frac{1125}{16\alpha}u_1v_1^2v \\
&+ \frac{15}{32\alpha^2}u_1v(v^4 + 20v^2u + 30u^2) + \frac{75}{16\alpha}v_1^3(3v^2 + 5u) \\
&+ \frac{75}{32\alpha^2}v_1u(v^4 + 6v^2u + 2u^2),
\end{aligned}
$$

$$
\begin{aligned}
Y_9^v = {}& \frac{5}{2}u_5 + \frac{15}{2}v_5v + 20v_4v_1 + \frac{25}{8\alpha}u_3(3v^2 + 2u) + \frac{125}{4}v_3v_2 \\
&+ \frac{25}{4\alpha}v_3v(v^2 + 3u) + \frac{25}{2\alpha}u_2u_1 + \frac{75}{2\alpha}u_2v_1v + \frac{75}{2\alpha}v_2u_1v \\
&+ \frac{25}{16\alpha}v_2v_1(21v^2 + 22u) + \frac{425}{16\alpha}u_1v_1^2 + \frac{75}{32\alpha^2}u_1(v^4 + 6v^2u + 2u^2) \\
&+ \frac{225}{16\alpha}v_1^3v + \frac{15}{32\alpha^2}v_1v(v^4 + 20v^2u + 30u^2). \qquad (2.135)
\end{aligned}
$$

In order to transform the Lie algebra we introduce

$$
\begin{aligned}
&Z_1 = \alpha Y_5, && Z_0 = Y_4, && Z_{-1} = Y_3, \\
&W_1 = Y_2, && W_2 = \frac{1}{2}Y_1, && W_3 = \frac{1}{2}\alpha Y_8, \\
&W_4 = \frac{3}{8}\alpha Y_6, && W_5 = \frac{3}{2}\alpha Y_7, && W_6 = \frac{3}{2}\alpha^2 Y_9, \qquad (2.136)
\end{aligned}
$$

which results in the Lie algebra structure presented in Fig. 2.3.

It is very interesting to note that the classical Boussinesq equation admits a higher symmetry Z_1 (see (2.134)) which is *local* and which has the property of acting as a *recursion operator* for the (x,t)-independent symmetries of the classical Boussinesq equation, thus giving rise to infinite series of higher symmetries. In Chapter 5 we shall construct the associated recursion operator by deformations of the equation structure of the classical Boussinesq equation.

$[*,*]$	Z_1	Z_0	Z_{-1}	W_1	W_2	W_3	W_4	W_5
Z_1	0	$-\frac{1}{2}Z_1$	$-Z_0$	W_2	W_3	W_4	W_5	W_6
Z_0	0	0	$\frac{1}{2}Z_1$	$\frac{1}{2}W_1$	W_2	$\frac{3}{2}W_3$	$2W_4$	$\frac{5}{2}W_5$
Z_{-1}	0	0	0	0	$\frac{1}{2}W_1$	$\frac{3}{2}W_2$	$3W_3$	$5W_4$
W_1	0	0	0	0	0	0	0	0
W_2	0	0	0	0	0	0	0	0
W_3	0	0	0	0	0	0	0	0
W_4	0	0	0	0	0	0	0	0
W_5	0	0	0	0	0	0	0	0

FIGURE 2.3. Commutator table for symmetries of the Boussinesq equation (2)

CHAPTER 3

Nonlocal theory

The facts exposed in this chapter constitute a formal base to introduce nonlocal variables to the differential setting, i.e., variables of the type $\int \varphi\, dx$, φ being a function on an infinitely prolonged equation. These variables are essential for introducing nonlocal symmetries of PDE as well as for existence of recursion operators. A detailed exposition of this material can be found in [**62, 61**] and [**12**].

1. Coverings

We start with fixing up the setting. To do this, extend the universum of infinitely prolonged equations in the following way. Let $\mathcal{N}$ be a chain of smooth maps $\cdots \to N^{i+1} \xrightarrow{\tau_{i+1,i}} N^i \to \cdots$, i.e., an object of the category $\mathcal{M}^\infty$ (see Chapters 1 and 2), where N^i are smooth finite-dimensional manifolds. As before, let us define the algebra $\mathcal{F}(\mathcal{N})$ of smooth functions on $\mathcal{N}$ as the direct limit of the homomorphisms $\cdots \to C^\infty(N^i) \xrightarrow{\tau_{i+1,i}^*} C^\infty(N^{i+1}) \to \cdots$. Then there exist natural homomorphisms $\tau_{\infty,i}^*\colon C^\infty(N^i) \to \mathcal{F}(\mathcal{N})$ and the algebra $\mathcal{F}(\mathcal{N})$ may be considered to be filtered by the images of these maps. Let us consider calculus (cf. Subsection 1.3 of Chapter 1) over $\mathcal{F}(\mathcal{N})$ agreed with this filtration. We define the category $\mathcal{DM}^\infty$ as follows:

1. The *objects* of the category $\mathcal{DM}^\infty$ are the above introduced chains $\mathcal{N}$ endowed with *integrable distributions* $D_{\mathcal{N}} \subset D(\mathcal{F}(\mathcal{N}))$, where the word "integrable" means that $[D_{\mathcal{N}}, D_{\mathcal{N}}] \subset D_{\mathcal{N}}$.
2. If $\mathcal{N}_1 = \{N_1^i, \tau_{i+1,i}^1\}$, $\mathcal{N}_2 = \{N_2^i, \tau_{i+1,i}^2\}$ are two objects of $\mathcal{DM}^\infty$, then a *morphism* $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ is a system of smooth mappings $\varphi_i\colon N_1^{i+\alpha} \to N_2^i$, where $\alpha \in \mathbb{Z}$ is independent of i, satisfying $\tau_{i+1,i}^2 \circ \varphi_{i+1} = \varphi_i \circ \tau_{i+\alpha+1,i+\alpha}^1$ and such that $\varphi_{*,\theta}(D_{\mathcal{N}_1,\theta}) \subset D_{\mathcal{N}_2,\varphi(\theta)}$ for any point $\theta \in \mathcal{N}_1$.

Definition 3.1. A morphism $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ is called a *covering* in the category $\mathcal{DM}^\infty$, if $\varphi_{*,\theta}|_{D_{\mathcal{N}_1,\theta}}\colon D_{\mathcal{N}_1,\theta} \to D_{\mathcal{N}_2,\varphi(\theta)}$ is an isomorphism for any point $\theta \in \mathcal{N}_1$.

In particular, manifolds $J^\infty(\pi)$ and $\mathcal{E}^\infty$ endowed with the corresponding Cartan distributions are objects of $\mathcal{DM}^\infty$ and we can consider coverings over these objects.

Example 3.1. Let $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ be a differential operator of order $\le k$. Then the system of mappings $\Phi_\Delta^{(l)}\colon J^{k+l}(\pi) \to J^l(\pi')$ (see Definition 1.6

on p. 6) is a morphism of $J^\infty(\pi)$ to $J^\infty(\pi')$. Under unrestrictive conditions of regularity, its image is of the form $\mathcal{E}^\infty$ for some equation $\mathcal{E}$ in the bundle π' while the map $J^\infty(\pi) \to \mathcal{E}^\infty$ is a covering.

DEFINITION 3.2. Let $\varphi' \colon \mathcal{N}' \to \mathcal{N}$ and $\varphi'' \colon \mathcal{N}'' \to \mathcal{N}$ be two coverings.

1. A morphism $\psi \colon \mathcal{N}' \to \mathcal{N}''$ is said to be a *morphism of coverings*, if $\varphi' = \varphi'' \circ \psi$.
2. The coverings φ', φ'' are called *equivalent*, if there exists a morphism $\psi \colon \mathcal{N}' \to \mathcal{N}''$ which is a diffeomorphism.

Assume now that $\varphi \colon \mathcal{N}' \to \mathcal{N}$ is a linear (i.e., vector) bundle and denote by $\mathcal{L}(\mathcal{N}') \subset \mathcal{F}(\mathcal{N}')$ the subset of functions linear along the fibers of the mapping φ.

DEFINITION 3.3. A covering $\varphi \colon \mathcal{N}' \to \mathcal{N}$ is called *linear*, if

1. The mapping φ is a linear bundle.
2. Any element $X \in D(\mathcal{N}')$ preserves $\mathcal{L}(\mathcal{N}')$.

EXAMPLE 3.2. Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation and $\mathcal{E}^\infty$ be its infinite prolongation and $T\mathcal{E}^\infty \to \mathcal{E}^\infty$ be its tangent bundle. Denote by $\tau^v \colon V\mathcal{E}^\infty \to \mathcal{E}^\infty$ the subbundle whose sections are π_∞-vertical vector fields. Obviously, any Cartan form $\omega_f = d_{\mathcal{C}}(f)$, $f \in \mathcal{F}(\mathcal{E}^\infty)$ (see (2.13) on p. 66) can be understood as a fiber-wise linear function on $V\mathcal{E}^\infty$:

$$\omega_f(Y) \stackrel{\text{def}}{=} Y \lrcorner\, \omega_f, \quad Y \in \Gamma(\tau^v), \tag{3.1}$$

and any function $\varphi \in \mathcal{L}(V\mathcal{E}^\infty)$ is a linear combination of the above ones (with coefficients in $\mathcal{F}(\mathcal{E})$).

Take the Cartan distribution $\mathcal{C}$ for the distribution $D_{\mathcal{E}^\infty}$ and let us define the action of any vector field Z lying in this distribution on the functions of the form (3.1) by

$$Z(\omega_f) \stackrel{\text{def}}{=} \mathrm{L}_Z \omega_f.$$

Since any Z under consideration is (at least locally) of the form $Z = \sum_i f_i \mathcal{C}X_i$, $X \in D(M)$, $f_i \in \mathcal{F}(\mathcal{E})$, one has

$$Z(\omega_f) = \mathrm{L}_{\sum_i f_i \mathcal{C}X_i} \omega_f = \sum_i \left(f \mathrm{L}_{\mathcal{C}X_i} d_{\mathcal{C}} f + df_i \wedge \mathrm{i}_{\mathcal{C}X_i}(d_{\mathcal{C}} f) \right)$$
$$= \sum_i d_{\mathcal{C}}(\mathcal{C}X_i f) = \sum_i f_i \omega_{\mathcal{C}X_i(f)}.$$

But defined on linear functions, you obtain a vector field $\widetilde{Z}$ on the entire manifold $V\mathcal{E}^\infty$. Obviously, the distribution spanned by all $\widetilde{Z}$ is integrable and projects to the Cartan distribution on $\mathcal{E}^\infty$ isomorphically. Thus we obtain a linear covering structure in $\tau^v \colon V\mathcal{E}^\infty \to \mathcal{E}^\infty$ which is called the (*even*) *Cartan covering*.

REMARK 3.1. In Chapter 6 we shall introduce a similar construction where the functions ω_f will play the role of *odd* variables. This explains the adjective *even* in the above definition.

If the equation $\mathcal{E} \subset J^k(\pi)$ is locally presented in the form $\mathcal{E} = \{F = 0\}$, then the object $V\mathcal{E}^\infty$ is isomorphic to the infinite prolongation of the equation

$$\begin{cases} F = 0, \\ \sum_{j,\sigma} \dfrac{\partial F}{\partial u_\sigma^j} w_\sigma^j = 0, \end{cases} \tag{3.2}$$

where $w_\sigma^j \overset{\text{def}}{=} \omega_{u_\sigma^j}$. Thus, $V\mathcal{E}^\infty$ corresponds to the initial equation together with its linearization.

Let $\mathcal{N}$ be an object of $\mathcal{DM}^\infty$ and W be a smooth manifold. Consider the projection $\tau_W \colon \mathcal{N} \times W \to \mathcal{N}$ to the first factor. Then we can make a covering of τ_W by lifting the distribution $D_\mathcal{N}$ to $\mathcal{N} \times W$ in a trivial way.

DEFINITION 3.4. A covering $\tau \colon \mathcal{N}' \to \mathcal{N}$ is called *trivial*, if it is equivalent to the covering τ_W for some W.

Let again $\varphi' \colon \mathcal{N}' \to \mathcal{N}$, $\varphi'' \colon \mathcal{N}'' \to \mathcal{N}$ be two coverings. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{N}' \times_\mathcal{N} \mathcal{N}'' \arrow[r, "\varphi''^*(\varphi')"] \arrow[d, "\varphi'^*(\varphi'')"'] & \mathcal{N}'' \arrow[d, "\varphi''"] \\ \mathcal{N}' \arrow[r, "\varphi'"] & \mathcal{N} \end{tikzcd}$$

where

$$\mathcal{N}' \times_\mathcal{N} \mathcal{N}'' = \{\, (\theta', \theta'') \in \mathcal{N}' \times \mathcal{N}'' \mid \varphi'(\theta') = \varphi''(\theta'') \,\}$$

while $\varphi'^*(\varphi'')$, $\varphi''^*(\varphi')$ are the natural projections. The manifold $\mathcal{N}' \times_\mathcal{N} \mathcal{N}''$ is supplied with a natural structure of an object of $\mathcal{DM}^\infty$ and the mappings $(\varphi')^*(\varphi'')$, $(\varphi'')^*(\varphi')$ become coverings.

DEFINITION 3.5. The composition

$$\varphi' \times_\mathcal{N} \varphi'' = \varphi' \circ \varphi'^*(\varphi'') = \varphi'' \circ \varphi''^*(\varphi') \colon \mathcal{N}' \times_\mathcal{N} \mathcal{N}'' \to \mathcal{N}$$

is called the *Whitney product* of the coverings φ' and φ''.

DEFINITION 3.6. A covering is said to be *reducible*, if it is equivalent to a covering of the form $\varphi \times_\mathcal{N} \tau$, where τ is a trivial covering. Otherwise it is called *irreducible.*

From now on, all coverings under consideration will be assumed to be smooth fiber bundles. The fiber dimension is called the *dimension of the covering* φ under consideration and is denoted by $\dim \varphi$.

PROPOSITION 3.1. *Let $\mathcal{E} \subset J^k(\pi)$ be an equation in the bundle $\pi\colon E \to M$ and $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a smooth fiber bundle. Then the following statements are equivalent*:

1. *The bundle φ is equipped with a structure of a covering.*
2. *There exists a connection $\mathcal{C}^\varphi$ in the bundle $\pi_\infty \circ \varphi\colon \mathcal{N} \to M$, $\mathcal{C}^\varphi\colon X \mapsto X^\varphi$, $X \in D(M)$, $X^\varphi \in D(\mathcal{N})$, such that*
   (a) *$[X^\varphi, Y^\varphi] = [X, Y]^\varphi$, i.e., $\mathcal{C}^\varphi$ is flat, and*
   (b) *any vector field X^φ is projectible to $\mathcal{E}^\infty$ under φ_* and $\varphi_*(X^\varphi) = \mathcal{C}X$, where $\mathcal{C}$ is the Cartan connection on $\mathcal{E}^\infty$.*

The proof reduces to the check of definitions.

Using this result, we shall now obtain coordinate description of coverings. Namely, let $x_1, \dots, x_n, u^1, \dots, u^m$ be local coordinates in $J^0(\pi)$ and assume that internal coordinates in $\mathcal{E}^\infty$ are chosen. Suppose also that over the neighborhood under consideration the bundle $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ is trivial with the fiber W and $w^1, w^2, \dots, w^s, \dots$ are local coordinates in W. The functions w^j are called *nonlocal coordinates* in the covering φ. The connection $\mathcal{C}^\varphi$ puts into correspondence to any partial derivative $\partial/\partial x_i$ the vector field $\mathcal{C}^\varphi(\partial/\partial x_i) = \tilde{D}_i$. By Proposition 3.1, these vector fields are to be of the form

$$\tilde{D}_i = D_i + X_i^v = D_i + \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}, \quad i = 1, \dots, n, \tag{3.3}$$

where D_i are restrictions of total derivatives to $\mathcal{E}^\infty$, and satisfy the conditions

$$\begin{aligned}[\tilde{D}_i, \tilde{D}_i] &= [D_i, D_j] + [D_i, X_j^v] + [X_i^v, D_j] + [X_i^v, X_j^v] \\ &= [D_i, X_j^v] + [X_i^v, D_j] + [X_i^v, X_j^v] = 0 \end{aligned} \tag{3.4}$$

for all $i, j = 1, \dots, n$.

We shall now prove a number of facts that simplify checking of triviality and equivalence of coverings.

PROPOSITION 3.2. *Let $\varphi_1\colon \mathcal{N}_1 \to \mathcal{E}^\infty$ and $\varphi_2\colon \mathcal{N}_2 \to \mathcal{E}^\infty$ be two coverings of the same dimensions $r < \infty$. They are equivalent if and only if there exists a submanifold $X \subset \mathcal{N}_1 \times_{\mathcal{E}^\infty} \mathcal{N}_2$ such that*

1. *The equality* $\operatorname{codim} X = r$ *holds.*
2. *The restrictions $\varphi_1^*(\varphi_2)\,|_X$ and $\varphi_2^*(\varphi_1)\,|_X$ are surjections.*
3. *One has $(D_{\mathcal{N}_1 \times_{\mathcal{E}^\infty} \mathcal{N}_2})_\theta \subset T_\theta X$ for any point $\theta \in X$.*

PROOF. In fact, if $\psi\colon \mathcal{N}_1 \to \mathcal{N}_2$ is an equivalence, then its graph

$$G_\psi = \{\, (y, \psi(y)) \mid y \in \mathcal{N}_1 \,\}$$

is the needed manifold X. Conversely, if X is a manifold satisfying the assupmtions of the proposition, then the correspondence

$$y \mapsto \varphi_1^*(\varphi_2)\big((\varphi_1^*(\varphi_2))^{-1}(y) \cap X\big)$$

is an equivalence. $\square$

Submanifolds X satisfying assumption (3) of the previous proposition are called *invariant.*

PROPOSITION 3.3. *Let $\varphi_1 : \mathcal{N}_1 \to \mathcal{E}^\infty$ and $\varphi_2 : \mathcal{N}_2 \to \mathcal{E}^\infty$ be two irreducible coverings of the same dimension $r < \infty$. Assume that the Whitney product of φ_1 and φ_2 is reducible and there exists an invariant submanifold X in $\mathcal{N}_1 \times_{\mathcal{E}^\infty} \mathcal{N}_2$ of codimension r. Then φ_1 and φ_2 are equivalent almost everywhere.*

PROOF. Since φ_1 and φ_2 are irreducible, X is to be mapped surjectively almost everywhere by $\varphi_1^*(\varphi_2)$ and $\varphi_2^*(\varphi_1)$ to $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively (otherwise, their images would be invariant submanifolds). Hence, the coverings are equivalent by Proposition 3.2. $\square$

COROLLARY 3.4. *If φ_1 and φ_2 are one-dimensional coverings over $\mathcal{E}^\infty$ and their Whitney product is reducible, then they are equivalent.*

PROPOSITION 3.5. *Let $\varphi : \mathcal{N} \to \mathcal{E}^\infty$ be a covering and $\mathcal{U} \subset \mathcal{E}^\infty$ be a domain such that the the manifold $\tilde{\mathcal{U}} = \varphi^{-1}(\mathcal{U})$ is represented in the form $\mathcal{U} \times \mathbb{R}^r$, $r \le \infty$, while $\varphi|_{\tilde{\mathcal{U}}}$ is the projection to the first factor. Then the covering φ is locally irreducible if the system*

$$D_1^\varphi(f) = 0, \dots, D_n^\varphi(f) = 0 \tag{3.5}$$

has constant solutions only.

PROOF. Suppose that there exists a solution $f \neq \text{const}$ of (3.5). Then, since the only solutions of the system

$$D_1(f) = 0, \dots, D_n(f) = 0,$$

where D_i is the restriction of the i-th total derivative to $\mathcal{E}^\infty$, are constants, f depends on one nonlocal variable w^α at least. Without loss of generality, we may assume that $\partial f/\partial w^1 \neq 0$ in a neighborhood $\mathcal{U}' \times V$, $\mathcal{U}' \subset \mathcal{U}$, $V \subset \mathbb{R}^r$. Define the diffeomorphism $\psi : \mathcal{U}' \subset \mathcal{U} \to \psi(\mathcal{U}' \subset \mathcal{U})$ by setting

$$\psi(\dots, x_i, \dots, p_\sigma^j, \dots, w^\alpha, \dots) = (\dots, x_i, \dots, p_\sigma^j, \dots, f, w^2, \dots, w^\alpha, \dots).$$

Then $\psi_*(D_i^\varphi) = D_i + \sum_{\alpha > 1} X_i^\alpha \partial/\partial w^\alpha$ and consequently φ is reducible.

Let now φ be a reducible covering, i.e., $\varphi = \varphi' \times_{\mathcal{E}^\infty} \tau$, where τ is trivial. Then, if f is a smooth function on the total space of the covering τ, the function $f^* = \big(\tau^*(\varphi')\big)^*(f)$ is a solution of (3.5). Obviously, there exists an f such that $f^* \neq \text{const}$. $\square$

2. Nonlocal symmetries and shadows

Let $\mathcal{N}$ be an object of $\mathcal{DM}^\infty$ with the integrable distribution $\mathcal{P} = \mathcal{P}_\mathcal{N}$. Define

$$D_\mathcal{P}(\mathcal{N}) = \{\, X \in D(\mathcal{N}) \mid [X, \mathcal{P}] \subset \mathcal{P} \,\}$$

and set $\operatorname{sym} \mathcal{N} = D_\mathcal{P}(\mathcal{N})/\mathcal{P}_\mathcal{N}$. Obviously, $D_\mathcal{P}(\mathcal{N})$ is a Lie $\mathbb{R}$-algebra and D is its ideal. Elements of the Lie algebra $\operatorname{sym} \mathcal{N}$ are called *symmetries* of the object $\mathcal{N}$.

DEFINITION 3.7. Let $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering. A *nonlocal φ-symmetry* of $\mathcal{E}$ is an element of $\operatorname{sym}\mathcal{N}$. The Lie algebra of such symmetries is denoted by $\operatorname{sym}_\varphi \mathcal{E}$.

EXAMPLE 3.3. Consider the even Cartan covering $\tau^v\colon V\mathcal{E}^\infty \to \mathcal{E}^\infty$ (see Example 3.2) and a symmetry $X \in \operatorname{sym}\mathcal{E}$ of the equation $\mathcal{E}$. Then we can define a vector field X^e on $V\mathcal{E}^\infty$ by setting $X^e(f) = X(f)$ for any function $f \in \mathcal{F}(\mathcal{E})$ and

$$X^e(\omega_f) = \mathrm{L}_X(d_\mathcal{C} f) = d_\mathcal{C}(Xf) = \omega_{Xf}.$$

Then, by obvious reasons, $X^e \in \operatorname{sym}_{\tau^v}\mathcal{E}$ and $\tau^v_* X^e = X$. In other, words X^e is a nonlocal symmetry which is obtained by lifting the corresponding higher symmetry of $\mathcal{E}$ to $V\mathcal{E}^\infty$.

On the other hand, we can define a field X^o by $X^o(f) = 0$ and

$$X^o(\omega_f) = \mathrm{i}_X(d_\mathcal{C} f) = X(f).$$

Again, X^o is a nonlocal symmetry in τ^v, but as a vector field it is τ^v-vertical. So, in a sense, this symmetry is "purely nonlocal".

Due to identities $[\mathrm{L}_X, \mathrm{L}_Y] = \mathrm{L}_{[X,Y]}$, $[\mathrm{L}_X, \mathrm{i}_Y] = \mathrm{i}_{[X,Y]}$, and $[\mathrm{i}_X, \mathrm{i}_Y] = 0$, we have

$$[X^e, Y^e] = [X,Y]^e, \quad [X^e, Y^o] = [X,Y]^e, \quad [X^o, Y^o] = 0.$$

A base for computation of nonlocal symmetries is the given by following two results.

THEOREM 3.6. *Let $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering. The algebra $\operatorname{sym}_\varphi \mathcal{E}$ is isomorphic to the Lie algebra of vector fields X on $\mathcal{N}$ such that*

1. *The field X is vertical, i.e., $X(\varphi^*(f)) = 0$ for any function $f \in C^\infty(M) \subset \mathcal{F}(\mathcal{E})$.*
2. *The identities $[X, D_i^\varphi] = 0$ hold for all $i = 1, \dots, n$.*

PROOF. Note that the first condition means that in coordinate representation the coefficients of the field X at all $\partial/\partial x_i$ vanish. Hence the intersection of the set of vertical fields with D vanish. On the other hand, in any coset $[X] \in \operatorname{sym}_\varphi \mathcal{E}$ there exists one and only one vertical element X^v. In fact, let X be an arbitrary element of $[X]$. Then $X^v = X - \sum_i a_i D_i^\varphi$, where a_i is the coefficient of X at $\partial/\partial x_i$. □

THEOREM 3.7. *Let $\varphi\colon \mathcal{N} = \mathcal{E}^\infty \times \mathbb{R}^r \to \mathcal{E}^\infty$ be the covering locally determined by the fields*

$$D_i^\varphi = D_i + \sum_{\alpha=1}^r X_i^\alpha \frac{\partial}{\partial w^\alpha}, \quad i = 1, \dots, n, \quad X_i^\alpha \in \mathcal{F}(\mathcal{N}),$$

where $w^1, w^2, \dots$ are coordinates in $\mathbb{R}^r$ (nonlocal variables). Then any nonlocal φ-symmetry of the equation $\mathcal{E} = \{F = 0\}$ is of the form

$$\tilde{\partial}_{\psi,a} = \tilde{\partial}_\psi + \sum_{\alpha=1}^r a^\alpha \frac{\partial}{\partial w^\alpha}, \tag{3.6}$$

where $\psi = (\psi^1, \dots, \psi^m)$, $a = (a^1, \dots, a^r)$, ψ^i, $a^\alpha \in \mathcal{F}(\mathcal{N})$ *are functions satisfying the conditions*

$$\tilde{\ell}_F(\psi) = 0, \tag{3.7}$$

$$D_i^\varphi(a^\alpha) = \tilde{\partial}_{\psi,a}(X_i^\alpha) \tag{3.8}$$

while

$$\tilde{\partial}_\psi = \sum_{j,\sigma} D_\sigma^\varphi(\psi) \frac{\partial}{\partial u_\sigma^j} \tag{3.9}$$

and $\tilde{\ell}_F$ *is obtained from* ℓ_F *by changing total derivatives* D_i *for* D_i^φ*.*

Proof. Let $X \in \operatorname{sym}_\varphi \mathcal{E}$. Using Theorem 3.6, let us write down the field X in the form

$$X = \sum_{\sigma,j}{}' b_\sigma^j \frac{\partial}{\partial u_\sigma^j} + \sum_{\alpha=1}^r a^\alpha \frac{\partial}{\partial w^\alpha}, \tag{3.10}$$

where "prime" over the first sum means that the summation extends on internal coordinates in $\mathcal{E}^\infty$ only. Then, equating to zero the coefficient at $\partial/\partial u_\sigma^j$ in the commutator $[X, D_i^\varphi]$, we obtain the following equations

$$D_i^\varphi(b_\sigma^j) = \begin{cases} b_{\sigma i}^j, & \text{if } u_{\sigma i}^j \text{ is an internal coordinate,} \\ X(u_{\sigma i}^j) & \text{otherwise.} \end{cases}$$

Solving these equations, we obtain that the first summand in (3.10) is of the form $\tilde{\partial}_\psi$, where ψ satisfies (3.7). $\square$

Comparing the result obtained with the description of local symmetries (see Theorem 2.15 on p. 72), we see that in the nonlocal setting an additional obstruction arises represented by equation (3.8). Thus, in general, not every solution of (3.7) corresponds to a nonlocal φ-symmetry. We call vector fields $\tilde{\partial}_\psi$ of the form (3.9), where ψ satisfies equation (3.7), *φ-shadows*. In the next subsection it will be shown that for any φ-shadow $\tilde{\partial}_\psi$ there exists a covering $\varphi' \colon \mathcal{N}' \to \mathcal{N}$ and a nonlocal $\varphi \circ \varphi'$-symmetry S such that $\varphi'_*(S) = \tilde{\partial}_\psi$.

3. Reconstruction theorems

Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation. Let us first establish relations between horizontal cohomology of $\mathcal{E}$ (see Definition 2.7 on p. 65) and coverings over $\mathcal{E}^\infty$. All constructions below are realized in a local chart $\mathcal{U} \subset \mathcal{E}^\infty$.

Let us consider a horizontal 1-form $\omega = \sum_{i=1}^n X_i\, dx_i \in \Lambda_h^1(\mathcal{E})$ and define on the space $\mathcal{E}^\infty \times \mathbb{R}$ the vector fields

$$D_i^\omega = D_i + X_i \frac{\partial}{\partial w}, \quad X_i \in \mathcal{F}(\mathcal{E}), \tag{3.11}$$

where w is a coordinate along $\mathbb{R}$. By direct computations, one can easily see that the conditions $[D_i^\omega, D_j^\omega] = 0$ are fulfilled if and only if $d_h\omega = 0$. Thus, (3.11) determines a covering structure in the bundle $\varphi \colon \mathcal{E}^\infty \times \mathbb{R} \to \mathcal{E}^\infty$ and this covering is denoted by φ^ω. It is also obvious that the coverings φ^ω and $\varphi^{\omega'}$ are equivalent if and only if the forms ω and ω' are cohomologous, i.e., if $\omega - \omega' = d_h f$ for some $f \in \mathcal{F}(\mathcal{E})$.

DEFINITION 3.8. A covering over $\mathcal{E}^\infty$ constructed by means of elements of $H_h^1(\mathcal{E})$ is called *Abelian.*

Let $[\omega_1], \dots, [\omega^\alpha], \dots$ be an $\mathbb{R}$-basis of the vector space $H_h^1(\mathcal{E})$. Let us define the covering $\mathfrak{a}_{1,0} \colon \mathcal{A}^1(\mathcal{E}) \to \mathcal{E}^\infty$ as the Whitney product of all φ^{ω_α}. It can be shown that the equivalence class of $\mathfrak{a}_{1,0}$ does not depend on the basis choice. Now, literary in the same manner as it was done in Definition 2.7 for $\mathcal{E}^\infty$, we can define horizontal cohomology for $\mathcal{A}^1(\mathcal{E})$ and construct the covering $\mathfrak{a}_{2,1} \colon \mathcal{A}^2(\mathcal{E}) \to \mathcal{A}^1(\mathcal{E})$, etc.

DEFINITION 3.9. The inverse limit of the chain

$$\cdots \to \mathcal{A}^k(\mathcal{E}) \xrightarrow{\mathfrak{a}_{k,k-1}} \mathcal{A}^{k-1}(\mathcal{E}) \to \cdots \to \mathcal{A}^1(\mathcal{E}) \xrightarrow{\mathfrak{a}_{1,0}} \mathcal{E}^\infty \tag{3.12}$$

is called the *universal Abelian covering* of the equation $\mathcal{E}$ and is denoted by $\mathfrak{a} \colon \mathcal{A}(\mathcal{E}) \to \mathcal{E}^\infty$.

Obviously, $H_h^1(\mathcal{A}(\mathcal{E})) = 0$.

THEOREM 3.8 (see [**43**]). *Let $\mathfrak{a} \colon \mathcal{A}(\mathcal{E}) \to \mathcal{E}^\infty$ be the universal Abelian covering over the equation $\mathcal{E} = \{F = 0\}$. Then any $\mathfrak{a}$-shadow reconstructs up to a nonlocal $\mathfrak{a}$-symmetry, i.e., for any solution $\psi = (\psi^1, \dots, \psi^m)$, $\psi^j \in \mathcal{F}(\mathcal{A}(\mathcal{E}))$, of the equation $\tilde{\ell}_F(\psi) = 0$ there exists a set of functions $a = (a_{\alpha,i})$, where $a_{\alpha,i} \in \mathcal{F}(\mathcal{A}(\mathcal{E}))$, such that $\tilde{\partial}_{\psi,a}$ is a nonlocal $\mathfrak{a}$-symmetry.*

PROOF. Let $w^{j,\alpha}$, $j \le k$, be nonlocal variables in $\mathcal{A}^k(\mathcal{E})$ and assume that the covering structure in $\mathfrak{a}$ is determined by the vector fields $D_i^{\mathfrak{a}} = D_i + \sum_{j,\alpha} X_i^{j,\alpha} \partial/\partial w^{j,\alpha}$, where, by construction, $X_i^{j,\alpha} \in \mathcal{F}(\mathcal{A}^{j-1}(\mathcal{E}))$, i.e., the functions $X_i^{j,\alpha}$ do not depend on $w^{k,\alpha}$ for all $k \ge j$.

Our aim is to prove that the system

$$D_i^{\mathfrak{a}}(a_{j,\alpha}) = \tilde{\partial}_{\psi,a}(X_i^{j,\alpha}) \tag{3.13}$$

is solvable with respect to $a = (a_{j,\alpha})$ for any $\psi \in \ker \tilde{\ell}_F$. We do this by induction on j. Note that

$$[D_i^{\mathfrak{a}}, \tilde{\partial}_{\psi,a}] = \sum_{j,\alpha} \big(D_i^{\mathfrak{a}}(a_{j,\alpha}) - \tilde{\partial}_{\psi,a}(X_i^{j,\alpha})\big) \frac{\partial}{\partial w^{j,\alpha}}$$

for any set of functions $(a_{j,\alpha})$. Then for $j = 1$ one has $[D_i^{\mathfrak{a}}, \tilde{\partial}_{\psi,a}](X_k^{1,\alpha}) = 0$, or

$$D_i^{\mathfrak{a}}\big(\tilde{\partial}_{\psi,a}(X_k^{1,\alpha})\big) = \tilde{\partial}_{\psi,a}\big(D_i^{\mathfrak{a}}(X_k^{1,\alpha})\big),$$

since $X_k^{1,\alpha}$ are functions on $\mathcal{E}^\infty$.

But from the construction of the covering $\mathfrak{a}$ one has the following equality:

$$D_i^{\mathfrak{a}}(X_k^{1,\alpha}) = D_k^{\mathfrak{a}}(X_i^{1,\alpha}),$$

and we finally obtain

$$D_i^{\mathfrak{a}}\big(\partial_\psi(X_k^{1,\alpha})\big) = D_k^{\mathfrak{a}}\big(\partial_\psi(X_i^{1,\alpha})\big).$$

Note now that the equality $H_h^1(\mathcal{A}(\mathcal{E})) = 0$ implies existence of functions $a_{1,\alpha}$ satisfying

$$D_i^{\mathfrak{a}}(a_{1,\alpha}) = \partial_\psi(X_i^{1,\alpha}),$$

i.e., equation (3.13) is solvable for $j = 1$.

Assume now that solvability of (3.13) was proved for $j < s$ and the functions $(a_{1,\alpha}, \dots, a_{j-1,\alpha})$ are some solutions. Then, since $[D_i^{\mathfrak{a}}, \tilde{\partial}_{\psi,a}]\big|_{\mathcal{A}^{j-1}(\mathcal{E})} = 0$, we obtain the needed $a_{j,\alpha}$ literally repeating the proof for the case $j = 1$. □

Let now $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be an arbitrary covering. The next result shows that any φ-shadow is reconstructable.

THEOREM 3.9 (see also [**44**]). *For any φ-shadow, i.e., for any solution $\psi = (\psi^1, \dots, \psi^m)$, $\psi^j \in \mathcal{F}(\mathcal{N})$, of the equation $\tilde{\ell}_F(\psi) = 0$, there exists a covering $\varphi_\psi\colon \mathcal{N}_\psi \xrightarrow{\bar{\psi}} \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ and a φ_ψ-symmetry S_ψ, such that $S_\psi|_{\mathcal{E}^\infty} = \tilde{\partial}_\psi|_{\mathcal{E}^\infty}$.*

PROOF. Let locally the covering φ be represented by the vector fields

$$D_i^\varphi = D_i + \sum_{\alpha=1}^{r} X_i^\alpha \frac{\partial}{\partial w^\alpha},$$

$r \le \infty$ being the dimension of φ. Consider the space $\mathbb{R}^\infty$ with the coordinates w_l^α, $\alpha = 1, \dots, r$, $l = 0, 1, 2, \dots$, $w_0^\alpha = w^\alpha$, and set $\mathcal{N}_\psi = \mathcal{N} \times \mathbb{R}^\infty$ with

$$D_i^{\varphi_\psi} = D_i + \sum_{l,\alpha} \left(\tilde{\partial}_\psi + S_w\right)^l (X_i^\alpha) \frac{\partial}{\partial w_l^\alpha}, \tag{3.14}$$

where

$$\tilde{\partial}_\psi = {\sum_{\sigma,k}}' D_\sigma^\varphi(\psi^k) \frac{\partial}{\partial u_\sigma^k}, \quad S_w = \sum_{\alpha,l} w_{l+1}^\alpha \frac{\partial}{\partial w_l^\alpha} \tag{3.15}$$

and "prime", as before, denotes summation over internal coordinates.

Set $S_\psi = \tilde{\partial}_\psi + S_w$. Then

$$\begin{aligned}
[S_\psi, D_i^{\varphi_\psi}] = {\sum_{\sigma,k}}' \tilde{\partial}_\psi(\bar{u}_{\sigma i}^k) \frac{\partial}{\partial u_\sigma^k} + \sum_{l,\alpha} \left(\tilde{\partial}_\psi + S_w\right)^{l+1} (X_i^\alpha) \frac{\partial}{\partial w_l^\alpha} \\
- {\sum_{\sigma,k}}' D_i^{\varphi_\psi}(D_\sigma^\varphi(\psi^k)) \frac{\partial}{\partial u_\sigma^k} - \sum_{l,\alpha} \left(\tilde{\partial}_\psi + S_w\right)^{l+1} (X_i^\alpha) \frac{\partial}{\partial w_l^\alpha} \\
= {\sum_{\sigma,k}}' \left(\tilde{\partial}_\psi(\bar{u}_{\sigma i}^k) - D_{\sigma i}^\varphi(\psi^k)\right) \frac{\partial}{\partial u_\sigma^k} = 0.
\end{aligned}$$

Here, by definition, $\bar{u}^k_{\sigma i} = D^{\varphi}_i(u^k_\sigma)\,|_{\mathcal{N}}$.

Now, using the above proved equality, one has

$$[D^{\varphi_\psi}_i, D^{\varphi_\psi}_j] = \sum_{l,\alpha}\Big(D^{\varphi_\psi}_j\big(\tilde{\partial}_\psi + S_w\big)^l(X^\alpha_j) - D^{\varphi_\psi}_j\big(\tilde{\partial}_\psi + S_w\big)^l(X^\alpha_i)\Big)\frac{\partial}{\partial w^\alpha_l}$$
$$= \sum_{l,\alpha}\big(\tilde{\partial}_\psi + S_w\big)^l\big(D^{\varphi_\psi}_i(X^\alpha_j) - D^{\varphi_\psi}_j(X^\alpha_i)\big)\frac{\partial}{\partial w^\alpha_l} = 0,$$

since $D^{\varphi_\psi}_i(X^\alpha_j) - D^{\varphi_\psi}_j(X^\alpha_i) = D^{\varphi}_i(X^\alpha_j) - D^{\varphi}_j(X^\alpha_i) = 0$. $\square$

Let now $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering and $\varphi'\colon \mathcal{N}' \xrightarrow{\bar{\varphi}} \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ be another one. Then, by obvious reasons, any φ-shadow ψ is a φ'-shadow as well. Applying the construction of Theorem 3.9 to both φ and φ', we obtain two coverings, φ_ψ and φ'_ψ respectively.

LEMMA 3.10. *The following commutative diagram of coverings*

$$\begin{array}{ccccc}
\mathcal{N}'_\psi & \longrightarrow & \mathcal{N}_\psi & & \\
\Big\downarrow{\scriptstyle \bar{\psi}'} & & \Big\downarrow{\scriptstyle \bar{\psi}} & & \\
\mathcal{N}' & \xrightarrow{\ \bar{\varphi}\ } & \mathcal{N} & \xrightarrow{\ \varphi\ } & \mathcal{E}^\infty
\end{array}$$

takes place. Moreover, if S_ψ and S'_ψ are nonlocal symmetries corresponding in $\mathcal{N}_\psi$ and $\mathcal{N}'_\psi$ constructed by Theorem 3.9, then $S'_\psi\Big|_{\mathcal{F}(\mathcal{N}_\psi)} = S_\psi$.

PROOF. It suffices to compare expressions (3.14) and (3.15) for the coverings $\mathcal{N}_\psi$ and $\mathcal{N}'_\psi$. $\square$

As a corollary of Theorem 3.9 and of the previous lemma, we obtain the following result.

THEOREM 3.11. *Let $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$, where $\mathcal{E} = \{\, F = 0 \,\}$, be an arbitrary covering and $\psi_1, \dots, \psi_s \in \mathcal{F}(\mathcal{N})$ be solutions of the equation $\tilde{\ell}_F(\psi) = 0$. Then there exists a covering $\varphi_\Psi\colon \mathcal{N}_\Psi \to \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ and φ_Ψ-symmetries $S_{\psi_1}, \dots, S_{\psi_s}$, such that $S_{\psi_s}\,|_{\mathcal{E}^\infty} = \tilde{\partial}_{\psi_i}\,|_{\mathcal{E}^\infty}$, $i = 1, \dots, s$.*

PROOF. Consider the section ψ_1 and the covering $\varphi_{\psi_1}\colon \mathcal{N}_{\psi_1} \xrightarrow{\bar{\varphi}_{\psi_1}} \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ together with the symmetry S_{ψ_1} constructed in Theorem 3.9. Then ψ_2 is a φ_{ψ_1}-shadow and we can construct the covering

$$\varphi_{\psi_1,\psi_2}\colon \mathcal{N}_{\psi_1,\psi_2} \xrightarrow{\bar{\varphi}_{\psi_1,\psi_2}} \mathcal{N}_{\psi_1} \xrightarrow{\varphi_{\psi_1}} \mathcal{E}^\infty$$

with the symmetry S_{ψ_2}. Applying this procedure step by step, we obtain the series of coverings

$$\mathcal{N}_{\psi_1,\dots,\psi_s} \xrightarrow{\bar{\varphi}_{\psi_1,\dots,\psi_s}} \mathcal{N}_{\psi_1,\dots,\psi_{s-1}} \xrightarrow{\bar{\varphi}_{\psi_1,\dots,\psi_{s-1}}} \dots \xrightarrow{\bar{\varphi}_{\psi_1,\psi_2}} \mathcal{N}_{\psi_1} \xrightarrow{\bar{\varphi}_{\psi_1}} \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$$

with the symmetries $S_{\psi_1}, \dots, S_{\psi_s}$. But ψ_1 is a $\varphi_{\psi_1,\dots,\psi_s}$-shadow and we can construct the covering $\varphi_{\psi_1}\colon \mathcal{N}^{(1)}_{\psi_1} \to \mathcal{N}_{\psi_1,\dots,\psi_s} \to \mathcal{E}^\infty$ with the symmetry $S^{(1)}_{\psi_1}$ satisfying $S^{(1)}_{\psi_1}\Big|_{\mathcal{F}(\mathcal{N}_{\psi_1})} = S_{\psi_1}$ (see Lemma 3.10), etc. Passing to the inverse limit, we obtain the covering $\mathcal{N}_\Psi$ we need. □

4. Nonlocal symmetries of the Burgers equation

Consider the Burgers equation $\mathcal{E}$ given by

$$u_t = u_{xx} + uu_x \tag{3.16}$$

and choose *internal coordinates* on $\mathcal{E}^\infty$ by setting $u = u_0 = u_{(0,0)}$, $u_k = u_{(k,0)}$. Below we use the method described in [**60**]. The Lie algebra of higher symmetries of the Burgers equation is well known and is described in Section 3 of Chapter 2.

The total derivative operators D_x, D_t are given by

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i}, \\ D_t &= \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} D_x^i(u_2 + uu_1)\frac{\partial}{\partial u_i}. \end{aligned} \tag{3.17}$$

We now start from the only one existing conservation law for Burgers equation, i.e.,

$$D_t(2u) = D_x(u^2 + 2u_1). \tag{3.18}$$

From (3.18) we introduce the new formal variable p by defining its partial derivatives as follows:

$$p_x = 2u, \quad p_t = u^2 + 2u_1, \tag{3.19}$$

which is in a formal sense equivalent to

$$p = \int (2u)\, dx, \tag{3.20}$$

from which we have p is a nonlocal variable. Note at this moment that (3.18) is just the compatibility condition on p_x, p_t. We can now put the question: What are symmetries of equation $\widetilde{\mathcal{E}}$ which is defined by

$$\begin{aligned} u_t &= u_{xx} + uu_x, \\ p_x &= 2u, \\ p_t &= (u^2 + 2u_x). \end{aligned} \tag{3.21}$$

In effect (3.21) is a system of partial differential equations for two dependent variables, u and p, as functions of x and t. The infinite prolongation of

$\widetilde{\mathcal{E}}$, denoted by $\widetilde{\mathcal{E}^\infty}$, admits internal coordinates x, t, u, p, u_1, $u_2, \dots$, while the total derivative operators $\widetilde{D}_x$ and $\widetilde{D}_t$ are given by

$$\widetilde{D}_x = \frac{\partial}{\partial x} + 2u\frac{\partial}{\partial p} + \sum_{k=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i},$$
$$\widetilde{D}_t = \frac{\partial}{\partial t} + (u^2 + 2u_1)\frac{\partial}{\partial p} + \sum_{k=0}^{\infty} D_x^i(u_2 + uu_1)\frac{\partial}{\partial u_i}. \tag{3.22}$$

In order to search for higher symmetries, we search for vertical vector fields with generating function $\varphi = (\varphi^u, \varphi^p)$, where φ^u, φ^p are functions dependent on the internal coordinates x, t, u, p, u_1, $u_2 \dots$.

The remarkable result is a symmetry $\mathcal{Э}_\varphi$ whose generating function $\varphi = (\varphi^u, \varphi^v)$ is

$$\varphi^u = \Big(-2\frac{\partial g(x,t)}{\partial x} + g(x,t)u\Big) e^{-p/4}$$
$$\varphi^p = -4g(x,t)e^{-p/4}, \tag{3.23}$$

where $g(x,t)$ is an arbitrary solution to the heat equation

$$\frac{\partial g(x,t)}{\partial t} - \frac{\partial^2 g(x,t)}{\partial x^2} = 0. \tag{3.24}$$

If we now contract the vector field $\mathcal{Э}_\varphi$, φ given by (3.23), with the Cartan one-form associated to the nonlocal variable p, i.e.,

$$d_{\mathcal{C}}(u) = du - u_x dx - (u_x x + uu_x)dt, \tag{3.25}$$

we obtain an additional condition to $\widetilde{\mathcal{E}}$, (3.21), i.e.,

$$-2\frac{\partial g(x,t)}{\partial x} + g(x,t)u = 0, \tag{3.26}$$

or equivalently,

$$u = 2(g(x,t))^{-1}\frac{\partial g(x,t)}{\partial x}. \tag{3.27}$$

Substitution of (3.27) into (3.16) yields the fact that *any* function $u(x,t)$ of the form (3.27), where $g(x,t)$ is a solution of the heat equation (3.24), is a solution of Burgers equation (3.16). Note that (3.27) is the well-known *Cole–Hopf transformation.*

This rather simple example of the notion of nonlocal symmetry indicates its significance in the study of geometrical structures of partial differential equations. Further applications of the nonlocal theory, which are more intricate, will be treated in the next sections.

5. Nonlocal symmetries of the KDV equation

In order to demonstrate how to handle calulations concerning the construction of nonlocal symmetries and the calculation of Lie brackets of the corresponding vertical vector fields, or equivalently, the associated Jacobi bracket of the generating functions, we discuss these features for the KdV equation

$$u_t = uu_x + u_{xxx}. \tag{3.28}$$

The infinite prolongation of (3.28), denoted by $\mathcal{E}^\infty$, is given as

$$\begin{aligned} u_t &= uu_x + u_{xxx}, \\ u_{xt} &= D_x(uu_x + u_{xxx}) = u_x^2 + uu_{xx} + u_{xxxx}, \\ u_{x\dots xt} &= D_x \dots D_x(uu_x + u_{xxx}), \end{aligned}$$

where total partial derivative operators D_x and D_t are given with respect to the internal coordinates x, t, u, u_x, u_{xx}, $u_{xxx}, \dots$ as

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \dots, \\ D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{xxt} \frac{\partial}{\partial u_{xx}} + \dots \end{aligned}$$

Classical symmetries of KdV Equation are given by

$$\begin{aligned} \overline{V}_1 &= -\frac{\partial}{\partial x}, \\ \overline{V}_2 &= -\frac{\partial}{\partial t}, \\ \overline{V}_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ \overline{V}_4 &= -x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}, \end{aligned}$$

or equivalently, the generating functions associated to them, given by

$$\begin{aligned} V_1^u &= u_x, \\ V_2^u &= uu_x + u_{xxx}, \\ V_3^u &= 1 - tu_x, \\ V_4^u &= xu_x + 3t(uu_x + u_{xxx}) + 2u. \end{aligned}$$

Associated to (3.28), we can construct conservation laws A_x, A_t such that

$$D_t(A_x) = D_x(A_t), \tag{3.29}$$

which leads to

$$\begin{aligned} A_x^1 &= u, \\ A_t^1 &= \frac{1}{2}u^2 + u_{xx}, \\ A_x^2 &= u^2, \end{aligned}$$

$$A_t^2 = \frac{2}{3}u^3 - u_x^2 + 2uu_{xx}. \tag{3.30}$$

A few higher conservation laws are given by

$$\begin{aligned}
A_x^3 &= u^3 - 3u_x^2,\\
A_t^3 &= \frac{3}{4}(u^4 + 4u^2u_{xx} - 8uu_x^2 + 4u_{xx}^2 - 8u_xu_{xxx}),\\
A_x^4 &= u^4 - 12uu_x^2 + \frac{36}{5}u_{xx}^2,\\
A_t^4 &= \frac{4}{5}u^5 + 4u^3u_{xx} - 18u^2u_x^2 - 24uu_xu_{xxx} + 12u_x^2u_{xx},\\
&\quad + \frac{96}{5}uu_{xx}^2 + \frac{72}{5}u_{xx}u_{xxxx} - \frac{36}{5}u_{xxx}^2.
\end{aligned} \tag{3.31}$$

We now introduce nonlocal variables associated to two of conservation laws (3.30) in the form

$$\begin{aligned}
p_1 &= \int u\,dx,\\
p_3 &= \int (u^2)\,dx.
\end{aligned} \tag{3.32}$$

We also introduce the grading to the polynomial functions on the KdV equation by setting

$$[x] = -1, \quad [t] = -3, \quad [u] = 2, \quad [u_x] = 3, \quad [u_t] = 5, \dots \tag{3.33}$$

Then the nonlocal variables p_1 and p_3 are of degree

$$[p_1] = 1, \quad [p_3] = 3.$$

In order to study nonlocal symmetries of the KdV equation, we consider the augmented system

$$\begin{aligned}
u_t &= uu_x + u_{xxx},\\
(p_1)_x &= u,\\
(p_1)_t &= \frac{1}{2}u^2 + u_{xx},\\
(p_3)_x &= u^2,\\
(p_3)_t &= \frac{2}{3}u^3 - u_x^2 + 2uu_{xx}.
\end{aligned} \tag{3.34}$$

We note here that system (3.34) is in effect a system of partial differential equations in *three* dependent variables u, p_1, p_3 and two independent variables x, t. We choose internal coordinates on $\mathcal{E}^\infty \times \mathbb{R}^2$ as

$$x,\ t,\ u,\ p_1,\ p_3,\ u_x,\ u_{xx},\ u_{xxx},\ u_{xxxx},\ u_{xxxxx}, \dots, \tag{3.35}$$

while the total derivative operators $\overline{D}_x, \overline{D}_t$ are given as

$$\overline{D}_x = D_x + u\frac{\partial}{\partial p_1} + u^2\frac{\partial}{\partial p_3},$$

$$\overline{D}_t = D_t + \left(\frac{1}{2}u^2 + u_{xx}\right)\frac{\partial}{\partial p_1} + \left(\frac{2}{3}u^3 - u_x^2 + 2uu_{xx}\right)\frac{\partial}{\partial p_3}. \tag{3.36}$$

A vertical vector field V on $\mathcal{E}^\infty \times \mathbb{R}^2$ has as its generating functions V^u, V^{p_1}, V^{p_3}. The symmetry conditions resulting from (3.34) are

$$\begin{aligned}\overline{D}_t V^u &= V^u u_x + u\overline{D}_x V^u + \overline{D}_x^3 V^u,\\ \overline{D}_x V^{p_1} &= V^u,\\ \overline{D}_x V^{p_3} &= 2uV^u.\end{aligned} \tag{3.37}$$

For the vertical vector fields $V_1, \dots, V_4$ we derive from this after a short computation

$$\begin{aligned}
V_1^u &= u_x, & V_2^u &= uu_x + u_{xxx},\\
V_1^{p_1} &= u, & V_2^{p_1} &= \frac{1}{2}u^2 + u_{xx},\\
V_1^{p_3} &= u^2, & V_2^{p_3} &= \frac{2}{3}u^3 + 2uu_{xx} - u_x^2,\\
V_3^u &= 1 - tu_x, & V_4^u &= xu_x + 3t(uu_x + u_{xxx}) + 2u,\\
V_3^{p_1} &= x - tu, & V_4^{p_1} &= xu + 3t\left(\frac{1}{2}u^2 + u_{xx}\right) + p_1,\\
V_3^{p_3} &= 2p_1 - tu^2, & V_4^{p_3} &= xu^2 + 3t\left(\frac{2}{3}u^3 + 2uu_{xx} - u_x^2\right) + 3p_3.
\end{aligned} \tag{3.38}$$

It is a well-known fact [**80**] that the KdV equation (3.28) admits the *Lenard recursion operator* for higher symmetries, i.e.,

$$L = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}. \tag{3.39}$$

From this we have

$$\begin{aligned}
L(V_1^u) &= V_2^u,\\
L(V_2^u) &= V_5^u = u_{xxxxx} + \frac{5}{3}u_{xxx}u + \frac{10}{3}u_{xx}u_x + \frac{5}{6}u_x u^2,\\
L(V_3^u) &= \frac{2}{3}u + \frac{1}{3}xu_x + t(uu_x + u_{xxx}) = \frac{1}{3}V_4^u.
\end{aligned} \tag{3.40}$$

We now compute the action of the Lenard recursion operator L on the generating function V_4^u of the symmetry V_4. The result is

$$\begin{aligned}
V_5^u = L(V_4^u) &= x(u_{xxx} + uu_x)\\
&+ 3t\left(u_{xxxxx} + \frac{5}{3}u_{xxx}u + \frac{10}{3}u_{xx}u_x + \frac{5}{6}u_x u^2\right) + 4u_{xx} + \frac{4}{3}u^2 + \frac{1}{3}u_x p_1.
\end{aligned} \tag{3.41}$$

It is a straightforward check that V_5^u satisfies the first condition of (3.37), i.e.,

$$\overline{D}_t(V_5^u) = V_5^u u_x + u\overline{D}_x V_5^u + \overline{D}_x^3 V_5^u. \tag{3.42}$$

The component $V_5^{p_1}$ can be computed directly from the second condition in (3.37), i.e.,

$$\overline{D}_x(V_5^{p_1}) = V_5^u, \tag{3.43}$$

which readily leads to

$$V_5^{p_1} = x\left(u_{xx} + \frac{1}{2}u^2\right) + 3t\left(u_{xxxx} + \frac{5}{3}u_{xx}u + \frac{5}{6}u_x^2 + \frac{5}{18}u^3\right) + 3u_x + \frac{1}{3}up_1 + \frac{1}{2}p_3. \tag{3.44}$$

The construction of the component $V_5^{p_3}$, which should result from the third condition in (3.37), i.e.,

$$\overline{D}_x(V_5^{p_3}) = 2uV_5^u, \tag{3.45}$$

causes a problem:

It is impossible to derive a formula for $V_5^{p_3}$ in this setting.

The way out of this problem is to augment system (3.34) once more with the nonlocal variable p_5 resulting from

$$\begin{aligned} (p_5)_x &= u^3 - 3u_x^2, \\ (p_5)_t &= \frac{3}{4}(u^4 + 4u^2u_{xx} - 8uu_x^2 + 4u_{xx}^2 - 8u_xu_{xxx}), \end{aligned} \tag{3.46}$$

or equivalently

$$p_5 = \int (u^3 - 3u_x^2)\,dx, \tag{3.47}$$

and extending total derivative operators $\overline{D}_x$, $\overline{D}_t$ to

$$\begin{aligned} \widetilde{D}_x &= \overline{D}_x + (u^3 - 3u_x^2)\frac{\partial}{\partial p_5}, \\ \widetilde{D}_t &= \overline{D}_t + \frac{3}{4}(u^4 + 4u^2u_{xx} - 8uu_x^2 + 4u_{xx}^2 - 8u_xu_{xxx})\frac{\partial}{\partial p_5}. \end{aligned} \tag{3.48}$$

Within this once more augmented setting, i.e., having a system of partial differential equations for u, p_1, p_3, and p_5, it is posssible to solve the symmetry condition for p_3, (3.34):

$$\widetilde{D}_x(V_5^{p_3}) = 2uV_5^u, \tag{3.49}$$

the result being the vertical vector field V_5 whose generating functions are given by (3.41), (3.44), and from (3.49) we obtain

$$\begin{aligned} V_5^u &= x(u_{xxx} + uu_x) + 3t\left(u_{xxxxx} + \frac{5}{3}u_{xxx}u + \frac{10}{3}u_{xx}u_x + \frac{5}{6}u_xu^2\right) \\ &\quad + 4u_{xx} + \frac{4}{3}u^2 + \frac{1}{3}u_xp_1, \\ V_5^{p_1} &= x\left(u_{xx} + \frac{1}{2}u^2\right) + 3t\left(u_{xxxx} + \frac{5}{3}u_{xx}u + \frac{5}{6}u_x^2 + \frac{5}{18}u^3\right) \end{aligned}$$

$$
\begin{aligned}
&+ 3u_x + \frac{1}{3}up_1 + \frac{1}{2}p_3, \\
V_5^{p_3} &= 2x\left(uu_{xx} - \frac{1}{2}u_x^2 + \frac{1}{3}u^3\right) \\
&+ 6t\left(uu_{xxxx} - u_x u_{xxx} + \frac{1}{2}u_{xx}^2 + \frac{5}{3}u^2 u_{xx} + \frac{5}{24}u^5\right) \\
&+ 6uu_x + \frac{1}{3}u^2 p_1 + \frac{5}{3}p_5. \qquad (3.50)
\end{aligned}
$$

The outline above indicates that we are working in effect in an augmented system of partial differential equations in which *all* nonlocal variables associated to *all* conservation laws for the KdV equation are incorporated (cf. Theorem 3.8).

The computation of Lie brackets of vertical vector fields, or equivalently, the computation of the Jacobi brackets for the associated generating functions, is to be carried out in this augmented setting. To demonstrate this, we want to compute the Lie bracket of the symmetry V_1 and the *nonlocal symmetry* V_5 with the generating functions

$$
\begin{aligned}
V_1^u &= u_x, \\
V_5^u &= x(u_{xxx} + uu_x) + 3t\left(u_{xxxxx} + \frac{5}{3}u_{xxx}u + \frac{10}{3}u_{xx}u_x + \frac{5}{6}u_x u^2\right) \\
&+ 4u_{xx} + \frac{4}{3}u^2 + \frac{1}{3}u_x p_1. \qquad (3.51)
\end{aligned}
$$

The associated Jacobi bracket $\{V_5^u, V_1^u\}$ is defined as

$$
\widetilde{V}^u = \{V_5^u, V_1^u\} = \partial_{V_5}(V_1^u) - \partial_{V_1}(V_5^u), \qquad (3.52)
$$

which, using in this computation the equality $V_1^{p_1} = u$, results in

$$
\widetilde{V}^u = u_{xxx} + uu_x = V_2^u.
$$

In a similar way the Jacobi bracket $\{V_5^u, V_2^u\}$ equals

$$
\{V_5^u, V_2^u\} = 3\left(u_{xxxxx} + \frac{5}{3}u_{xxx}u + \frac{10}{3}u_{xx}u_x + \frac{5}{6}u_x u^2\right),
$$

which is just the generating function of the classical first higher symmetry of the KdV equation.

REMARK 3.2. The functions V_i^u, $i = 1, \ldots, 5$, are just the so-called *shadows* (see the previous section) of the symmetries V_i, $i = 1, \ldots, 5$, in the augmented setting, including all nonlocal variables.

6. Symmetries of the massive Thirring model

We shall establish higher and nonlocal symmetries of the so-called massive Thirring model [**32**], which is defined as the following system $\mathcal{E}^0$ of partial differential equations defined on $J^1(\pi)$, where $\pi\colon \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^2$ is the trivial bundle with the coordinates u_1, v_1, u_2, v_2 in the fiber (unknown functions) and x, t in the base (independent variables):

$$\begin{aligned}
-\frac{\partial u_1}{\partial x}+\frac{\partial u_1}{\partial t}&=mv_2-(u_2^2+v_2^2)v_1,\\
\frac{\partial u_2}{\partial x}+\frac{\partial u_2}{\partial t}&=mv_1-(u_1^2+v_1^2)v_2,\\
\frac{\partial v_1}{\partial x}-\frac{\partial v_1}{\partial t}&=mu_2-(u_2^2+v_2^2)u_1,\\
-\frac{\partial v_2}{\partial x}-\frac{\partial v_2}{\partial t}&=mu_1-(u_1^2+v_1^2)u_2.
\end{aligned}\tag{3.53}$$

For this system of equations we choose internal coordinates on $\mathcal{E}^1$ as x, t, u_1, v_1, u_2, v_2, $u_{1,1}$, $v_{1,1}$, $u_{2,1}$, $v_{2,1}$, while internal coordinates on $\mathcal{E}^4$ are chosen as x, t, u_1, v_1, u_2, $v_2,\dots,u_{1,4}$, $v_{1,4}$, $u_{2,4}$, $v_{2,4}$, where $u_{i,j}$, $v_{i,j}$ refer to $\partial^j u_i/\partial x^j$, $\partial^j v_i/\partial x^j$, $i=1,2$, $j=1,\dots,4$. In a similar way coordinates can be choosen on $\mathcal{E}^\infty$.

6.1. Higher symmetries. According to Theorem 2.15 on p. 72, we construct higher symmetries (symmetries of order 2) by constructing vertical vector fields $\mathsf{Э}_\varphi$, where the generating functions φ^{u_1}, φ^{v_1}, φ^{u_2}, φ^{v_2} depend on the local variables x, t, u_1, v_1, u_2, v_2, $u_{1,1}$, $v_{1,1}$, $u_{2,1}$, $v_{2,1}$, $u_{1,2}$, $v_{1,2}$, $u_{2,2}$, $v_{2,2}$ [**41**]. The symmetry condition then is

$$\begin{aligned}
-D_x\varphi^{u_1}+D_t\varphi^{u_1}&=m\varphi^{v_2}-2(u_2\varphi^{u_2}+v_2\varphi^{v_2})v_1+(u_2^2+v_2^2)\varphi^{v_1},\\
D_x\varphi^{u_2}+D_t\varphi^{u_2}&=m\varphi^{v_1}-2(u_1\varphi^{u_2}+v_1\varphi^{v_2})v_2+(u_1^2+v_1^2)\varphi^{v_2},\\
D_x\varphi^{v_1}-D_t\varphi^{v_1}&=m\varphi^{u_2}-2(u_2\varphi^{u_2}+v_2\varphi^{v_2})u_1+(u_2^2+v_2^2)\varphi^{u_1},\\
-D_x\varphi^{v_2}-D_t\varphi^{v_2}&=m\varphi^{u_1}-2(u_1\varphi^{u_2}+v_1\varphi^{v_2})u_2+(u_1^2+v_1^2)\varphi^{u_2}.
\end{aligned}\tag{3.54}$$

The result then is the existence of four symmetries $X_1,\dots,X_4$ of order 1 the generating functions of which, $\varphi_i^{u_1}$, $\varphi_i^{v_1}$, $\varphi_i^{u_2}$, $\varphi_i^{v_2}$, $i=1,\dots,4$, are given as

$$\begin{aligned}
\varphi_1^{u_1}&=\frac{1}{2}(-mv_2+v_1(u_2^2+v_2^2)),\\
\varphi_1^{v_1}&=\frac{1}{2}(mu_2-u_1(u_2^2+v_2^2)),\\
\varphi_1^{u_2}&=\frac{1}{2}(2u_{2,1}-mv_1+v_2(u_1^2+v_1^2)),\\
\varphi_1^{v_2}&=\frac{1}{2}(2v_{2,1}+mu_1-u_2(u_1^2+v_1^2)),
\end{aligned}$$

$$\begin{aligned}
\varphi_2^{u_1}&=\frac{1}{2}(2u_{1,1}+mv_2-v_1(u_2^2+v_2^2)),\\
\varphi_2^{v_1}&=\frac{1}{2}(2v_{1,1}-mu_2+u_1(u_2^2+v_2^2)),\\
\varphi_2^{u_2}&=\frac{1}{2}(mv_1-v_2(u_1^2+v_1^2)),
\end{aligned}$$

$$
\begin{aligned}
\varphi_2^{v_2} &= \frac{1}{2}(-mu_1 + u_2(u_1^2 + v_1^2)),\\
\varphi_3^{u_1} &= u_{1,1}(x+t) + mv_2x + \frac{1}{2}u_1 - v_1(u_2^2 + v_2^2)x,\\
\varphi_3^{v_1} &= v_{1,1}(x+t) - mu_2x + \frac{1}{2}v_1 + u_1(u_2^2 + v_2^2)x,\\
\varphi_3^{u_2} &= u_{2,1}(-x+t) + mv_1x - \frac{1}{2}u_2 - v_2(u_1^2 + v_1^2)x,\\
\varphi_3^{v_2} &= v_{2,1}(-x+t) + mu_1x - \frac{1}{2}v_2 + u_2(u_1^2 + v_1^2)x,\\
\varphi_4^{u_1} &= v_1,\\
\varphi_4^{v_1} &= -u_1,\\
\varphi_4^{u_2} &= v_2,\\
\varphi_4^{v_2} &= -u_2.
\end{aligned} \tag{3.55}
$$

Thus in effect, the fields X_1, X_2, X_3 are of the first order, while X_4 is of order zero.

In order to find symmetries of higher order, we take great advantage of the fact that the massive Thirring model is a graded system, as is the case with all equations possessing a scaling symmetry, i.e.,

$$
\begin{aligned}
&\deg(x) = \deg(t) = -2,\\
&\deg(u_1) = \deg(v_1) = \deg(u_2) = \deg(v_2) = 1,\\
&\deg(m) = 2,\ \deg\left(\frac{\partial u_1}{\partial x}\right) = 3, \dots
\end{aligned} \tag{3.56}
$$

Due to this grading, all equations in (3.53) are of degree three; the total derivative operators D_x, D_t are graded too as is the symmetry condition

$$\partial_\varphi(\mathcal{E}^0) = 0 \bmod \mathcal{E}^3. \tag{3.57}$$

The solutions of (3.57) are graded too. Note that the fields $X_1, \dots, X_4$ are of degrees 2, 2, 0, 0 respectively.

We now introduce the following notation:

$[u]$ refers to u_1, v_1, u_2, v_2,
$[u]_x$ refers to $u_{1,1}$, $v_{1,1}$, $u_{2,1}$, $v_{2,1}$,
....

In our search for higher symmetries we are not constructing the general solution of the overdetermined system of partial differential equations for the generating functions φ^{u_1}, φ^{v_1}, φ^{u_2}, φ^{v_2}, resulting from (3.57).

We are just looking for those (x,t)-independent functions which are of degree five; so the presentation of these functions is as follows:

$$\varphi^* = [u]_{xx} + ([u]^2 + [m])[u]_x + ([u]^5 + [m][u]^3 + [m]^2[u]). \tag{3.58}$$

Using the presentation above, we derive two higher symmetries, X_5 and X_6 of degree 4 and order 2, whose generating functions are given as

$$
\begin{aligned}
\varphi_5^{u_1} &= \frac{1}{4}(2u_{2,1}(-m+2v_1v_2) - 4v_{2,1}u_2v_1 - mv_2(R_1+R_2) \\
&\quad - 2mv_1R + v_1(R_2^2+2R_1R_2)), \\
\varphi_5^{v_1} &= \frac{1}{4}(2v_{2,1}(-m+2u_1u_2) - 4u_{2,1}v_2u_1 + mu_2(R_1+R_2) \\
&\quad + 2mu_1R - u_1(R_2^2+2R_1R_2)), \\
\varphi_5^{u_2} &= \frac{1}{4}(-4v_{2,2} + 2u_{1,1}(-m+2u_1u_2) + 4u_{2,1}(R_1+R_2) + 4v_{1,1}u_2v_1 \\
&\quad - mv_1(R_1+R_2) - 2mv_2R + v_2(R_1^2+2R_1R_2)), \\
\varphi_5^{v_2} &= \frac{1}{4}(4u_{2,2} + 2v_{1,1}(-m+2v_1v_2) + 4v_{2,1}(R_1+R_2) + 4u_{1,1}v_2u_1 \\
&\quad + mu_1(R_1+R_2) + 2mu_2R - u_2(R_1^2+2R_1R_2)), \\[1ex]
\varphi_6^{u_1} &= \frac{1}{4}(4v_{1,2} + 2u_{2,1}(-m+2u_1u_2) + 4u_{1,1}(R_1+R_2) + 4v_{2,1}u_1v_2 \\
&\quad + mv_2(R_1+R_2) + 2mv_1R + v_1(R_2^2+2R_1R_2)), \\
\varphi_6^{v_1} &= \frac{1}{4}(-4u_{1,2} + 2v_{2,1}(-m+2v_1v_2) + 4v_{1,1}(R_1+R_2) + 4u_{2,1}u_2v_1 \\
&\quad - mu_2(R_1+R_2) - 2mu_1R + u_1(R_2^2+2R_1R_2)), \\
\varphi_6^{u_2} &= \frac{1}{4}(2u_{1,1}(-m+2v_1v_2) - 4v_{1,1}u_1v_2 + mv_1(R_1+R_2) \\
&\quad + 2mv_2R - v_2(R_1^2+2R_1R_2)), \\
\varphi_6^{v_2} &= \frac{1}{4}(2v_{1,1}(-m+2u_1u_2) - 4u_{1,1}u_2v_1 - mu_1(R_1+R_2) \\
&\quad - 2mu_2R + u_2(R_1^2+2R_1R_2)),
\end{aligned}
\tag{3.59}
$$

whereas in (3.59)

$$
R_1 = u_1^2+v_1^2, \quad R_2 = u_2^2+v_2^2, \quad R = u_1u_2+v_1v_2.
$$

For third order higher symmetries the representation of the generating functions, whose degree is seven, is

$$
\begin{aligned}
\varphi^* &= [u]_{xxx} + ([u]^2+[m])[u]_{xx} + [u][u]_x^2 \\
&\quad + ([u]^4+[m][u]^2+[m]^2)[u]_x \\
&\quad + ([u]^7+[m][u]^5+[m]^2[u]^3+[m]^3[u]).
\end{aligned}
$$

After a massive computation, we arrive at the existence of higher symmetries X_7 and X_8 of degree 6 and order 3, given by

$$
\begin{aligned}
\varphi_7^{u_1} &= \frac{1}{8}(8u_{2,2}u_2v_1 + 4v_{2,2}(2v_1v_2-m) - 4u_{2,1}^2v_1 \\
&\quad + 4u_{2,1}(m(R_1+R_2+v_1^2+v_2^2) - 3v_1v_2(R_1+R_2)) - 4v_{2,1}^2v_1
\end{aligned}
$$

$$
\begin{aligned}
&+ 4v_{2,1}(-m(u_1v_1 + u_2v_2) + 3u_2v_1(R_1 + R_2)) + 4u_{1,1}mR \\
&- 2m^2v_1(R_1 + R_2) - 4v_2m^2R + 4v_1mR(R_1 + 2R_2) \\
&+ v_2m(R_1^2 + 4R_1R_2 + R_2^2) - v_1(R_2^3 + 6R_2^2R_1 + 3R_2R_1^2)), \\
\varphi_7^{v_1} &= \frac{1}{8}(-8v_{2,2}v_2u_1 - 4u_{2,2}(2u_1u_2 - m) + 4v_{2,1}^2u_1 \\
&+ 4v_{2,1}(m(R_1 + R_2 + u_1^2 + u_2^2) - 3u_1u_2(R_1 + R_2)) \\
&+ 4u_{2,1}^2u_1 + 4u_{2,1}(-m(u_1v_1 + u_2v_2) + 3v_2u_1(R_1 + R_2)) + 4v_{1,1}mR \\
&+ 2m^2u_1(R_1 + R_2) + 4u_2m^2R - 4u_1mR(R_1 + 2R_2) \\
&- u_2m(R_1^2 + 4R_1R_2 + R_2^2) + u_1(R_2^3 + 6R_2^2R_1 + 3R_2R_1^2)), \\
\varphi_7^{u_2} &= \frac{1}{8}(8u_{2,3} + 12v_{2,2}(R_1 + R_2) + 8u_{1,2}u_1v_2 + 4v_{1,2}(2v_1v_2 - m) \\
&- 12u_{2,1}^2v_2 + 24u_{2,1}v_{2,1}u_2 + 2u_{2,1}(10mR - 3R_1^2 - 12R_1R_2 - 3R_2^2) \\
&+ 12v_{2,1}^2v_2 + 24v_{2,1}u_{1,1}u_1 + 24v_{2,1}v_{1,1}v_1 + 8u_{1,1}^2v_2 \\
&+ 4u_{1,1}(m(R_1 + R_2 + u_1^2 + u_2^2) - 3u_1u_2(R_1 + R_2)) + 8v_{1,1}^2v_2 \\
&+ 4v_{1,1}(m(u_1v_1 + u_2v_2) - 3u_2v_1(R_1 + R_2)) - 4m^2v_1R \\
&- 2m^2v_2(R_1 + R_2) + mv_1(R_2^2 + 4R_1R_2 + R_1^2) + 4mv_2R(R_2 + 2R_1) \\
&- v_2(R_1^3 + 6R_1^2R_2 + 3R_1R_2^2)), \\
\varphi_7^{v_2} &= \frac{1}{8}(8v_{2,3} - 12u_{2,2}(R_1 + R_2) + 8v_{1,2}u_2v_1 - 4u_{1,2}(2u_1u_2 - m) \\
&- 12v_{2,1}^2u_2 - 24u_{2,1}v_{2,1}v_2 + 2v_{2,1}(10mR - 3R_1^2 - 12R_1R_2 - 3R_2^2) \\
&+ 12u_{2,1}^2u_2 + 24u_{2,1}v_{1,1}v_1 + 24u_{2,1}u_{1,1}u_1 - 8v_{1,1}^2u_2 \\
&+ 4v_{1,1}(m(R_1 + R_2 + v_1^2 + v_2^2) - 3v_1v_2(R_1 + R_2)) - 8u_{1,1}^2u_2 \\
&+ 4u_{1,1}(m(u_1v_1 + u_2v_2)) - 3v_2u_1(R_1 + R_2) + 4m^2u_1R \\
&+ 2m^2u_2(R_1 + R_2) - mu_1(R_2^2 + 4R_1R_2 + R_1^2) \\
&- 4mu_2R(R_2 + 2R_1) + u_2(R_1^3 + 6R_1^2R_2 + 3R_1R_2^2)). \qquad (3.60)
\end{aligned}
$$

The vector field associated to $\varphi_8 = (\varphi_8^{u_1}, \varphi_8^{v_1}, \varphi_8^{u_2}, \varphi_8^{v_2})$ can be derived from φ_7 by the transformation

$$
T\colon \begin{cases} u_1 \mapsto u_2, \quad v_1 \mapsto v_2, \quad u_2 \mapsto u_1, \quad v_2 \mapsto v_1, \\ \partial/\partial x \mapsto -\partial/\partial x, \\ R_1 \mapsto R_2, \quad R_2 \mapsto R_1, \quad R \mapsto R \end{cases} \qquad (3.61)
$$

in the following way:

$$
\begin{aligned}
\varphi_8^{u_1} &= -T(\varphi_7^{u_2}), & \varphi_8^{v_1} &= -T(\varphi_7^{v_2}), \\
\varphi_8^{u_2} &= -T(\varphi_7^{u_1}), & \varphi_8^{v_2} &= -T(\varphi_7^{v_1}). \qquad (3.62)
\end{aligned}
$$

The Lie bracket of vector fields can be computed by calculation of the Jacobi bracket of the associated generating functions:

$$[X_i, X_j]^l = X_i(X_j^l) - X_j(X_i^l), \quad l = u_1, \dots, v_2; \quad i, j = 1, \dots, 8, \tag{3.63}$$

where $X_i = \mathcal{D}_{\varphi_i}$, which results in the following nonzero commutators:

$$\begin{aligned}
&[\mathcal{D}_{\varphi_1}, \mathcal{D}_{\varphi_3}] = \mathcal{D}_{\varphi_1},\\
&[\mathcal{D}_{\varphi_2}, \mathcal{D}_{\varphi_3}] = -\mathcal{D}_{\varphi_2},\\
&[\mathcal{D}_{\varphi_3}, \mathcal{D}_{\varphi_5}] = -2\mathcal{D}_{\varphi_5} - \frac{m^2}{2}\mathcal{D}_{\varphi_4},\\
&[\mathcal{D}_{\varphi_3}, \mathcal{D}_{\varphi_6}] = 2\mathcal{D}_{\varphi_6} - \frac{m^2}{2}\mathcal{D}_{\varphi_4},\\
&[\mathcal{D}_{\varphi_3}, \mathcal{D}_{\varphi_7}] = -3\mathcal{D}_{\varphi_7} + \frac{m^2}{2}(\mathcal{D}_{\varphi_1} + \mathcal{D}_{\varphi_2}),\\
&[\mathcal{D}_{\varphi_3}, \mathcal{D}_{\varphi_8}] = 3\mathcal{D}_{\varphi_8} - \frac{m^2}{2}(\mathcal{D}_{\varphi_1} + \mathcal{D}_{\varphi_2}).
\end{aligned} \tag{3.64}$$

Transformation of the vector fields $\mathcal{D}_{\varphi_1}, \dots, \mathcal{D}_{\varphi_8}$ by

$$\begin{aligned}
Y_1 &= \mathcal{D}_{\varphi_1},\\
Y_2 &= \mathcal{D}_{\varphi_2},\\
Y_3 &= \mathcal{D}_{\varphi_3},\\
Y_4 &= \mathcal{D}_{\varphi_4},\\
Y_5 &= \mathcal{D}_{\varphi_5} + \frac{m^2}{4}\mathcal{D}_{\varphi_4},\\
Y_6 &= \mathcal{D}_{\varphi_6} - \frac{m^2}{4}\mathcal{D}_{\varphi_4},\\
Y_7 &= \mathcal{D}_{\varphi_7} - \frac{m^2}{2}\mathcal{D}_{\varphi_1} - \frac{m^2}{4}\mathcal{D}_{\varphi_2},\\
Y_8 &= \mathcal{D}_{\varphi_8} - \frac{m^2}{4}\mathcal{D}_{\varphi_1} - \frac{m^2}{2}\mathcal{D}_{\varphi_2},
\end{aligned} \tag{3.65}$$

then leads to the following commutator table presented on Fig. 3.1.

Note that from (3.64) and (3.65) we see that $[Y_i, Y_j] = 0$, $i, j = 1, 2, 5, 6, 7, 8$, while Y_3 is the scaling symmetry.

6.2. Nonlocal symmetries. Here we shall discuss nonlocal symmetries of the massive Thirring model [**41**]. In order to find nonlocal variables for the system

$$\begin{aligned}
-\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial t} &= mv_2 - (u_2^2 + v_2^2)v_1,\\
\frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial t} &= mv_1 - (u_1^2 + v_1^2)v_2,\\
\frac{\partial v_1}{\partial x} - \frac{\partial v_1}{\partial t} &= mu_2 - (u_2^2 + v_2^2)u_1,
\end{aligned}$$

$[*,*]$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
Y_1	0	0	Y_1	0	0	0	0	0
Y_2	0	0	$-Y_2$	0	0	0	0	0
Y_3	0	0	0	0	$-2Y_5$	$2Y_6$	$-3Y_7$	$3Y_8$
Y_4	0	0	0	0	0	0	0	0
Y_5	0	0	0	0	0	0	0	0
Y_6	0	0	0	0	0	0	0	0
Y_7	0	0	0	0	0	0	0	0
Y_8	0	0	0	0	0	0	0	0

FIGURE 3.1. Commutator table for symmetries of the massive Thirring model

$$-\frac{\partial v_2}{\partial x} - \frac{\partial v_2}{\partial t} = mu_1 - (u_1^2 + v_1^2)u_2, \tag{3.66}$$

we first have to construct conservation laws, i.e., sets (A_i^x, A_i^t) satisfying the condition

$$D_t(A_i^x) = D_x(A_i^t),$$

from which we can introduce nonlocal variables.

6.2.1. *Construction of nonlocal symmetries.* To construct conservation laws, we take great advantage of the grading of system (3.66).

Since

$$\deg(x) = \deg(t) = -2,$$

we start from two arbitrary polynomials A^x, A^t with respect to the variables $u_1, \dots, v_2, u_{1,1}, \dots, v_{2,1}, \dots$ such that the degree with respect to the grading is just k, $k = 1, \dots$

It should be noted here that to get rid of trivial conservation laws, we are making computations modulo total derivatives: this means in practice that we start from a general polynomial A_0^x of degree $k-2$ (with respect to the grading), and eliminate resulting constants in A_0^x by equating terms in the expression

$$A^x - D_x(A_0^x).$$

to zero. This procedure is quite effective and has been used in several applications. Another way to arrive at conservation laws here, is to start from symmetries and to apply the Nöther theorem (Theorem 2.23).

The result is the following number of conservation laws, (A_i^x, A_i^t), $i = 1, \dots, 4$:

$$A_1^x = \frac{1}{2}(u_1 v_{1,1} - u_{1,1} v_1 + u_2 v_{2,1} - u_{2,1} v_2),$$
$$A_1^t = \frac{1}{2}(u_1 v_{1,1} - u_{1,1} v_1 - u_2 v_{2,1} + u_{2,1} v_2 + R_1 R_2),$$

$$
\begin{aligned}
A_2^x &= \frac{1}{2}(u_1v_{1,1} - u_{1,1}v_1 - u_2v_{2,1} + u_{2,1}v_2 + R_1R_2 - 2mR),\\
A_2^t &= \frac{1}{2}(u_1v_{1,1} - u_{1,1}v_1 + u_2v_{2,1} - u_{2,1}v_2),\\
A_3^x &= \frac{1}{2}(R_1 + R_2),\\
A_3^t &= \frac{1}{2}(R_1 - R_2),\\
A_4^x &= \frac{1}{2}x(u_1v_{1,1} - u_{1,1}v_1 - u_2v_{2,1} + u_{2,1}v_2 + R_1R_2 - 2mR)\\
&\quad + \frac{1}{2}t(u_1v_{1,1} - u_{1,1}v_1 + u_2v_{2,1} - u_{2,1}v_2),\\
A_4^t &= \frac{1}{2}x(u_1v_{1,1} - u_{1,1}v_1 + u_2v_{2,1} - u_{2,1}v_2)\\
&\quad + \frac{1}{2}t(u_1v_{1,1} - u_{1,1}v_1 - u_2v_{2,1} + u_{2,1}v_2 + R_1R_2),
\end{aligned}
\tag{3.67}
$$

where in (3.67) we have

$$R_1 = u_1^2 + v_1^2, \quad R_2 = u_2^2 + v_2^2, \quad R = u_1u_2 + v_1v_2.$$

We now formally introduce variables the p_0, p_1, p_2 by

$$
\begin{aligned}
p_0 &= \int A_3^x\,dx = \frac{1}{2}\int (R_1 + R_2)\,dx,\\
p_1 &= \int (A_1^x + A_2^x)\,dx = \int (u_1v_{1,1} - u_{1,1}v_1 + \frac{1}{2}R_1R_2 - mR)\,dx,\\
p_2 &= \int (A_1^x - A_2^x)\,dx = \int (u_2v_{2,1} - u_{2,1}v_2 - \frac{1}{2}R_1R_2 + mR)\,dx.
\end{aligned}
\tag{3.68}
$$

Note that p_0, p_1, p_2 are of degree 0, 2, 2 respectively (see (3.56)).

We now arrive from these nonlocal variables to the following augmented system of partial differential equations

$$
\begin{aligned}
-u_{1,1} + u_{1t} &= mv_2 - (u_2^2 + v_2^2)v_1,\\
u_{2,1} + u_{2t} &= mv_1 - (u_1^2 + v_1^2)v_2,\\
v_{1,1} - v_{1t} &= mu_2 - (u_2^2 + v_2^2)u_1,\\
-v_{2,1} - v_{2t} &= mu_1 - (u_1^2 + v_1^2)u_2,\\
(p_0)_x &= \frac{1}{2}(R_1 + R_2),\\
(p_0)_t &= \frac{1}{2}(R_1 - R_2),\\
(p_1)_x &= u_1v_{1,1} - u_{1,1}v_1 + \frac{1}{2}R_1R_2 - mR,\\
(p_1)_t &= u_1v_{1,1} - u_{1,1}v_1 + \frac{1}{2}R_1R_2,
\end{aligned}
$$

$$(p_2)_x = u_2 v_{2,1} - u_{2,1} v_2 - \frac{1}{2} R_1 R_2 + mR,$$
$$(p_2)_t = -u_2 v_{2,1} + u_{2,1} v_2 + \frac{1}{2} R_1 R_2. \tag{3.69}$$

We want to construct nonlocal higher symmetries of (3.53) which are just higher symmetries of (3.69) (see Section 2). In effect we shall just construct the shadows of nonlocal symmetries, as discussed in Section 2. For a more detailed exposition of the construction we refer to the construction the nonlocal symmetries of the KdV equation in Section 5.

To construct nonlocal symmetries of (3.53), we start from a vertical vector field Z of degree 2 and of polynomial degree one with respect to x, t. So the generating functions $Z^{u_1}, \dots, Z^{v_2}$ are of degree 3. The total derivative operators $\overline{D}_x$, $\overline{D}_t$ are given by (3.70):

$$\overline{D}_x = D_x + (p_0)_x \frac{\partial}{\partial p_0} + (p_1)_x \frac{\partial}{\partial p_1} + (p_2)_x \frac{\partial}{\partial p_2},$$
$$\overline{D}_t = D_t + (p_0)_t \frac{\partial}{\partial p_0} + (p_1)_t \frac{\partial}{\partial p_1} + (p_2)_t \frac{\partial}{\partial p_2}, \tag{3.70}$$

while the symmetry condition for the generating functions $Z^{u_1}, \dots, Z^{v_2}$ is

$$-\overline{D}_x(Z^{u_1}) + \overline{D}_t(Z^{u_1}) = mZ^{v_2} - v_1(2u_2 Z^{u_2} + 2v_2 Z^{v_2}) - R_2 Z^{v_1},$$
$$\overline{D}_x(Z^{u_2}) + \overline{D}_t(Z^{u_2}) = mZ^{v_1} - v_2(2u_1 Z^{u_1} + 2v_1 Z^{v_1}) - R_1 Z^{v_2},$$
$$\overline{D}_x(Z^{v_1}) - \overline{D}_t(Z^{v_1}) = mZ^{u_2} - u_1(2u_2 Z^{u_2} + 2v_2 Z^{v_2}) - R_2 Z^{u_1},$$
$$-\overline{D}_x(Z^{v_2}) - \overline{D}_t(Z^{v_2}) = mZ^{u_1} - u_2(2u_1 Z^{u_1} + 2v_1 Z^{v_1}) - R_1 Z^{u_2}. \tag{3.71}$$

Application of these conditions does lead to a number of equations for the generating functions $Z^{u_1}, \dots, Z^{v_2}$.

The result is the existence of two nonlocal higher symmetries ∂_{Z_1} and ∂_{Z_2}, where the generating functions $Z_1 = (Z_1^{u_1}, Z_1^{v_1}, Z_1^{u_2}, Z_1^{v_2})$ and $Z_2 = (Z_2^{u_1}, Z_2^{v_1}, Z_2^{u_2}, Z_2^{v_2})$ are given by

$$Z_1^{u_1} = v_1 p_2 + x(-2\Phi_5^{u_1} - m^2 v_1) + t(2\Phi_5^{u_1}) + \frac{1}{2} m u_2,$$
$$Z_1^{v_1} = -u_1 p_2 + x(-2\Phi_5^{v_1} + m^2 u_1) + t(2\Phi_5^{v_1}) + \frac{1}{2} m v_2,$$
$$Z_1^{u_2} = v_2 p_2 + x(-2\Phi_5^{u_2} - m^2 v_2) + t(2\Phi_5^{u_2}) + \frac{3}{2} m u_1 + 3v_{2,1},$$
$$- \frac{3}{2} R_1 u_2 - \frac{1}{2} R_2 u_2,$$
$$Z_1^{v_2} = -u_2 p_2 + x(-2\Phi_5^{v_2} + m^2 u_2) + t(2\Phi_5^{v_2}) + \frac{3}{2} m v_1 - 3u_{2,1},$$
$$- \frac{3}{2} R_1 v_2 - \frac{1}{2} R_2 v_2,$$
$$Z_2^{u_1} = v_1 p_1 + x(-2\Phi_6^{u_1} + m^2 v_1) + t(-2\Phi_6^{u_1}) + \frac{3}{2} m u_2 - 3v_{1,1}$$

$$
\begin{aligned}
&-\frac{3}{2}R_2u_1-\frac{1}{2}R_1u_1,\\
Z_2^{v_1}&=-u_1p_1+x(-2\Phi_6^{v_1}-m^2u_1)+t(-2\Phi_6^{v_1})+\frac{3}{2}mv_2+3u_{1,1}\\
&-\frac{3}{2}R_2v_1-\frac{1}{2}R_1v_1,\\
Z_2^{u_2}&=v_2p_1+x(-2\Phi_6^{u_2}+m^2v_2)+t(-2\Phi_6^{u_2})+\frac{1}{2}mu_1,\\
Z_2^{v_2}&=-u_2p_1+x(-2\Phi_6^{v_2}-m^2u_2)+t(-2\Phi_6^{v_2})+\frac{1}{2}mv_1. \qquad (3.72)
\end{aligned}
$$

The components $Z_1^{p0},\dots,Z_1^{p2}$, $Z_2^{p0},\dots,Z_2^{p2}$ can be obtained from the invariance of the equations

$$
\begin{aligned}
(p_0)_x&=\frac{1}{2}(R_1+R_2),\\
(p_1)_x&=u_1v_{1,1}-u_{1,1}v_1+\frac{1}{2}R_1R_2-mR,\\
(p_2)_x&=u_2v_{2,1}-u_{2,1}v_2-\frac{1}{2}R_1R_2+mR. \qquad (3.73)
\end{aligned}
$$

6.2.2. *Action of nonlocal symmetries.* In order to derive the action of the nonlocal symmetries ∂_{Z_1}, ∂_{Z_2} on the symmetries $\varphi_1,\dots,\varphi_6$, we have to extend the Lie bracket of vector fields in a way analogous to (3.52). This is in effect, as has been demonstrated for the KdV equation in previous Section 5, where we extended the Jacobi bracket to the nonlocal variables, i.e., u versus u, p, in this situation from u_1, v_1, u_2, v_2 to p_1, p_2. Since the nonlocal variable p_0 does not take part in the presentation of the vector fields $\varphi_1,\dots,\varphi_6$, Z_1, Z_2, we discard in this subsection the nonlocal variable p_0, see (3.68).

The extended Lie bracket of the evolutionary vector fields ∂_{Z_i}, $i=1,2$, and $\partial_{\varphi_1},\dots,\partial_{\varphi_6}$ is obtained from the extended Jacobi bracket for the generating functions, which is given by

$$
\{Z_i,\varphi_j\}^w=\partial_{Z_i}(\varphi_j^w)-\partial_{\varphi_j}(Z_i^w), \qquad (3.74)
$$

where in (3.74), $i=1,2$, $j=1,\dots,6$, $w=u_1,\dots,v_2$.

Since the generating functions φ_j^w are local, we do not need to compute the components $Z_1^{p_1}$, $Z_1^{p_2}$, $Z_2^{p_1}$, $Z_2^{p_2}$, in order to calculate the first term in the right-hand side of (3.74)). The calculation of the second term in the righ-thand side of (3.74) however does require the components $\varphi_1^{p_1}$, $\varphi_1^{p_2},\dots,\varphi_6^{p_1},\varphi_6^{p_2}$. These components result from the invariance of the partial differential equations (3.73) for the variables p_1, p_2, leading to the equations

$$
\begin{aligned}
\overline{D}_x(\varphi_j^{p_1})&=\partial_{\varphi_j}\Big(u_1v_{1,1}-u_{1,1}v_1+\frac{1}{2}R_1R_2-mR\Big),\\
\overline{D}_x(\varphi_j^{p_2})&=\partial_{\varphi_j}\Big(u_2v_{2,1}-u_{2,1}v_2-\frac{1}{2}R_1R_2+mR\Big). \qquad (3.75)
\end{aligned}
$$

From this we obtain the generating functions in the nonlocal, augmented setting u_1, v_1, u_2, v_2, p_1, p_2:

$$
\begin{aligned}
\Phi_1^{u_1} &= \frac{1}{2}(-mv_2 + v_1(u_2^2 + v_2^2)),\\
\Phi_1^{v_1} &= \frac{1}{2}(mu_2 - u_1(u_2^2 + v_2^2)),\\
\Phi_1^{u_2} &= \frac{1}{2}(2u_{2,1} - mv_1 + v_2(u_1^2 + v_1^2)),\\
\Phi_1^{v_2} &= \frac{1}{2}(2v_{2,1} + mu_1 - u_2(u_1^2 + v_1^2)),\\
\Phi_1^{p_1} &= -\frac{1}{2}mR,\\
\Phi_1^{p_2} &= -v_2u_{2,1} + u_2v_{2,1} + \frac{1}{2}mR - \frac{1}{2}R_1R_2,\\
\Phi_2^{u_1} &= \frac{1}{2}(2u_{1,1} + mv_2 - v_1(u_2^2 + v_2^2)),\\
\Phi_2^{v_1} &= \frac{1}{2}(2v_{1,1} - mu_2 + u_1(u_2^2 + v_2^2)),\\
\Phi_2^{u_2} &= \frac{1}{2}(mv_1 - v_2(u_1^2 + v_1^2)),\\
\Phi_2^{v_2} &= \frac{1}{2}(-mu_1 + u_2(u_1^2 + v_1^2)),\\
\Phi_2^{p_1} &= -v_1u_{1,1} + u_1v_{1,1} - \frac{1}{2}mR + \frac{1}{2}R_1R_2,\\
\Phi_2^{p_2} &= \frac{1}{2}mR,\\
\Phi_3^{u_1} &= u_{1,1}(x+t) + mv_2x + \frac{1}{2}u_1 - v_1(u_2^2 + v_2^2)x,\\
\Phi_3^{v_1} &= v_{1,1}(x+t) - mu_2x + \frac{1}{2}v_1 + u_1(u_2^2 + v_2^2)x,\\
\Phi_3^{u_2} &= u_{2,1}(-x+t) + mv_1x - \frac{1}{2}u_2 - v_2(u_1^2 + v_1^2)x,\\
\Phi_3^{v_2} &= v_{2,1}(-x+t) + mu_1x - \frac{1}{2}v_2 + u_2(u_1^2 + v_1^2)x,\\
\Phi_3^{p_1} &= \frac{1}{2}(x+t)(2u_1v_{1,1} - 2v_1u_{1,1} + R_1R_2) - tmR + p_1,\\
\Phi_3^{p_2} &= \frac{1}{2}(x+t)(-2u_2v_{2,1} + 2v_2u_{2,1} + R_1R_2) + tmR - p_2,\\
\Phi_4^{u_1} &= v_1,\\
\Phi_4^{v_1} &= -u_1,\\
\Phi_4^{u_2} &= v_2,\\
\Phi_4^{v_2} &= -u_2,
\end{aligned}
$$

$$
\begin{aligned}
&\Phi_4^{p_1} = 0,\\
&\Phi_4^{p_2} = 0
\end{aligned}
\tag{3.76}
$$

and similar for Φ_5, Φ_6

$$
\begin{aligned}
\Phi_5^{u_1} &= \frac{1}{4}(2u_{2,1}(-m + 2v_1v_2) - 4v_{2,1}u_2v_1 - mv_2(R_1 + R_2)\\
&\quad - 2mv_1R + v_1(R_2^2 + 2R_1R_2)),\\
\Phi_5^{v_1} &= \frac{1}{4}(2v_{2,1}(-m + 2u_1u_2) - 4u_{2,1}v_2u_1 + mu_2(R_1 + R_2)\\
&\quad + 2mu_1R - u_1(R_2^2 + 2R_1R_2)),\\
\Phi_5^{u_2} &= \frac{1}{4}(-4v_{2,2} + 2u_{1,1}(-m + 2u_1u_2) + 4u_{2,1}(R_1 + R_2)\\
&\quad + 4v_{1,1}u_2v_1 - mv_1(R_1 + R_2) - 2mv_2R + v_2(R_1^2 + 2R_1R_2)),\\
\Phi_5^{v_2} &= \frac{1}{4}(-4u_{2,2} + 2v_{1,1}(-m + 2v_1v_2) + 4v_{2,1}(R_1 + R_2) + 4u_{1,1}v_2u_1\\
&\quad + mu_1(R_1 + R_2) + 2mu_2R - u_2(R_1^2 + 2R_1R_2)),\\
\Phi_5^{p_1} &= -\frac{1}{2}mv_1u_{2,1} + \frac{1}{2}mu_1v_{2,1} - \frac{1}{4}mR(R_1 + R_2) + \frac{1}{4}m^2(R_1 + R_2),\\
\Phi_5^{p_2} &= u_{2,2}u_2 + v_{2,2}v_2 - u_{2,1}^2 - v_{2,1}^2 - \frac{1}{2}mu_2v_{1,1} + mv_1u_{2,1}\\
&\quad + \frac{1}{2}mv_2u_{1,1} - mu_1v_{2,1} - u_{2,1}v_2(R_2 + 2R_1) + v_{2,1}u_2(R_2 + 2R_1)\\
&\quad - \frac{1}{4}m^2(R_1 + R_2) + \frac{3}{4}mR(R_1 + R_2) + \frac{1}{2}R_1R_2(R_1 + R_2),\\
\Phi_6^{u_1} &= \frac{1}{4}(4v_{1,2} + 2u_{2,1}(-m + 2u_1u_2) + 4u_{1,1}(R_1 + R_2)\\
&\quad + 4v_{2,1}u_1v_2 + mv_2(R_1 + R_2) + 2mv_1R + v_1(R_2^2 + 2R_1R_2)),\\
\Phi_6^{v_1} &= \frac{1}{4}(-4u_{1,2} + 2v_{2,1}(-m + 2v_1v_2) + 4v_{1,1}(R_1 + R_2)\\
&\quad + 4u_{2,1}u_2uv_1 - mu_2(R_1 + R_2) - 2mu_1R + u_1(R_2^2 + 2R_1R_2)),\\
\Phi_6^{u_2} &= \frac{1}{4}(+2u_{1,1}(-m + 2v_1v_2) - 4v_{1,1}u_1v_2 + mv_1(R_1 + R_2) + 2mv_2R\\
&\quad - v_2(R_1^2 + 2R_1R_2)),\\
\Phi_6^{v_2} &= \frac{1}{4}(+2v_{1,1}(-m + 2u_1u_2) - 4u_{1,1}u_2v_1 - mu_1(R_1 + R_2)\\
&\quad - 2mu_2R + u_2(R_1^2 + 2R_1R_2)),\\
\Phi_6^{p_1} &= -u_{1,2}u_1 - v_{1,2}v_1 + v_{1,1}^2 + u_{1,1}^2 - \frac{1}{2}mu_1v_{2,1} + mv_2u_{1,1}\\
&\quad + \frac{1}{2}mv_1u_{2,1} - mu_2v_{1,1} - u_{1,1}v_1(R_1 + 2R_2) + v_{1,1}u_1(R_1 + 2R_2)
\end{aligned}
$$

$$+\frac{1}{4}m^2(R_1+R_2)-\frac{3}{4}mR(R_1+R_2)-\frac{1}{2}R_1R_2(R_1+R_2),$$
$$\Phi_6^{p_2}=-\frac{1}{2}mv_2u_{1,1}+\frac{1}{2}mu_2v_{1,1}+\frac{1}{4}mR(R_1+R_2)-\frac{1}{4}m^2(R_1+R_2). \quad (3.77)$$

The $\partial/\partial p_1$-component of $\boldsymbol{\vartheta}_{Z_1}$ and the $\partial/\partial p_2$-component of $\boldsymbol{\vartheta}_{Z_2}$ are given by

$$\begin{aligned}
Z_1^{p_1}&=\frac{1}{2}(x-t)(-2mu_1v_{2,1}+2mv_1u_{2,1}-(-m^2+mR)(R_1+R_2)\\
&-\frac{1}{2}m(u_1v_2-u_2v_1),\\
Z_2^{p_2}&=\frac{1}{2}(x+t)(-2mu_2v_{1,1}+2mv_2u_{1,1}+(+m^2-mR)(R_1+R_2)\\
&+\frac{1}{2}m(u_1v_2-u_2v_1).
\end{aligned} \quad (3.78)$$

Computation of the Jacobi brackets (3.74) then leads to the following commutators for the evolutionary vector fields:

$$\begin{aligned}
[\vartheta_{Z_1},\vartheta_{\Phi_1}]&=-\frac{1}{2}m^2\vartheta_{\Phi_4}-2\vartheta_{\Phi_5},\\
[\vartheta_{Z_2},\vartheta_{\Phi_1}]&=\frac{1}{2}m^2\vartheta_{\Phi_4},\\
[\vartheta_{Z_1},\vartheta_{\Phi_2}]&=-\frac{1}{2}m^2\vartheta_{\Phi_4},\\
[\vartheta_{Z_2},\vartheta_{\Phi_2}]&=\frac{1}{2}m^2\vartheta_{\Phi_4}-2\vartheta_{\Phi_6},\\
[\vartheta_{Z_1},\vartheta_{\Phi_3}]&=\vartheta_{Z_1},\\
[\vartheta_{Z_2},\vartheta_{\Phi_3}]&=\vartheta_{Z_2},\\
[\vartheta_{Z_1},\vartheta_{\Phi_4}]&=0,\\
[\vartheta_{Z_2},\vartheta_{\Phi_4}]&=0,\\
[\vartheta_{Z_1},\vartheta_{\Phi_5}]&=4\vartheta_{\Phi_7}-2m^2\vartheta_{\Phi_1}-m^2\vartheta_{\Phi_2},\\
[\vartheta_{Z_2},\vartheta_{\Phi_5}]&=m^2\vartheta_{\Phi_1},\\
[\vartheta_{Z_1},\vartheta_{\Phi_6}]&=m^2\vartheta_{\Phi_2},\\
[\vartheta_{Z_2},\vartheta_{\Phi_6}]&=4\vartheta_{\Phi_8}-m^2\vartheta_{\Phi_1}-2m^2\vartheta_{\Phi_2},\\
[\vartheta_{Z_1},\vartheta_{Z_2}]&=-2m^2\vartheta_{\Phi_3}.
\end{aligned} \quad (3.79)$$

Transformation of the vector fields by

$$\begin{aligned}
Y_1&=\vartheta_{\Phi_1},\\
Y_2&=\vartheta_{\Phi_2},\\
Y_3&=\vartheta_{\Phi_3},\\
Y_4&=\vartheta_{\Phi_4},\\
Y_5&=\vartheta_{\Phi_5}+\frac{m^2}{4}\vartheta_{\Phi_4},
\end{aligned}$$

$[*,*]$	Z_1	Z_2	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
Z_1	0	$-2m^2Y_3$	$-2Y_5$	$-\frac{m^2}{2}Y_4$	Z_1	0	$4Y_7$	m^2Y_2	$*$	$*$
Z_2		0	$\frac{m^2}{2}Y_4$	$-2Y_6$	$-Z_2$	0	m^2Y_1	$4Y_8$	$*$	$*$
Y_1			0	0	Y_1	0	0	0	0	0
Y_2				0	$-Y_2$	0	0	0	0	0
Y_3					0	0	$-2Y_5$	$2Y_6$	$-3Y_7$	$3Y_8$
Y_4						0	0	0	0	0
Y_5							0	0	0	0
Y_6								0	0	0
Y_7									0	0
Y_8										0

FIGURE 3.2. Commutator table for nolocal symmetries of the massive Thirring model

$$
\begin{aligned}
Y_6 &= \partial_{\Phi_6} - \frac{m^2}{4}\partial_{\Phi_4},\\
Y_7 &= \partial_{\Phi_7} - \frac{m^2}{2}\partial_{\Phi_1} - \frac{m^2}{4}\partial_{\Phi_2},\\
Y_8 &= \partial_{\Phi_7} - \frac{m^2}{4}\partial_{\Phi_1} - \frac{m^2}{2}\partial_{\Phi_2},
\end{aligned}
\tag{3.80}
$$

leads us to the following commutator table presented on Fig. 3.2.

From the commutator table we conclude that Z_1 acts as a generating *recursion operator* on the hierarchy $Y = (Y_1, Y_5, \dots)$ while Z_2 acts as a generating *recursion operator* on the hierarchy $\hat{Y} = (Y_2, Y_6, \dots)$. The action of Z_1 on Y_2, Y_6 is of a decreasing nature just as Z_2 acts on Y_1, Y_5. We expect that the vector fields Z_1, Z_2 generate a hierarchy of commuting higher symmetries.

REMARK 3.3. In (3.78), only those components of Z_1 and Z_2 are given that are necessary to compute the Jacobi bracket of the generating functions, i.e., for Z_1 the $\partial/\partial p_1$- and for Z_2 the $\partial/\partial p_2$-component

$$
\{Z_1, Z_2\} = -2m^2Y_3. \tag{3.81}
$$

We should mention here that Z_1 does not admit a $\partial/\partial p_2$-component, while Z_2 does not admit a $\partial/\partial p_1$-component in this formulation. The associated components can be obtained after introduction of nonlocal variables arising from higher conservation laws, a situation similar to the nonlocal symmetries of the KdV equation, Section 5.

7. Symmetries of the Federbush model

We present here results of symmetry computations for the Federbush model. The Federbush model is described by the matrix system of equations

$$\begin{pmatrix} i(\partial/\partial t+\partial/\partial x) & -m(s) \\ -m(s) & i(\partial/\partial t-\partial/\partial x) \end{pmatrix} \begin{pmatrix} \Psi_{s,1} \\ \Psi_{s,2} \end{pmatrix} = 4\pi s\lambda \begin{pmatrix} |\Psi_{-s,2}|^2\Psi_{s,1} \\ |\Psi_{-s,1}|^2\Psi_{s,2} \end{pmatrix}, \tag{3.82}$$

where in (3.82) $s=\pm 1$ and $\Psi_s(x,t)$ are two component complex-valued functions $\mathbb{R}^2 \to \mathbb{C}$.

Suppressing the factor 4π from now on (we set $\lambda' = 4\pi\lambda$) and introducing the eight variables u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4 by

$$\begin{aligned} &\Psi_{+1,1} = u_1 + iv_1, \qquad &&\Psi_{+1,2} = u_2 + iv_2, \qquad &&m(+1) = m_1, \\ &\Psi_{-1,1} = u_3 + iv_3, \qquad &&\Psi_{-1,2} = u_4 + iv_4, \qquad &&m(-1) = m_2, \end{aligned} \tag{3.83}$$

equation (3.82) is rewritten as a system of eight nonlinear partial differential equations for the component functions $u_1, \ldots, v_4$, i.e.,

$$\begin{aligned} u_{1,t} + u_{1,x} - m_1 v_2 &= \lambda(u_4^2+v_4^2)v_1, \\ -v_{1,t} - v_{1,x} - m_1 u_2 &= \lambda(u_4^2+v_4^2)u_1, \\ u_{2,t} - u_{2,x} - m_1 v_1 &= -\lambda(u_3^2+v_3^2)v_2, \\ -v_{2,t} + v_{2,x} - m_1 u_1 &= -\lambda(u_3^2+v_3^2)u_2, \\ u_{3,t} + u_{3,x} - m_2 v_4 &= -\lambda(u_2^2+v_2^2)v_3, \\ -v_{3,t} - v_{3,x} - m_2 u_4 &= -\lambda(u_2^2+v_2^2)u_3, \\ u_{4,t} - u_{4,x} - m_2 v_3 &= \lambda(u_2^2+v_2^2)v_4, \\ -v_{4,t} + v_{4,x} - m_2 u_3 &= \lambda(u_2^2+v_2^2)u_4. \end{aligned} \tag{3.84}$$

The contents of this section is strongly related to a number of papers [**42, 36, 92**] and references therein.

7.1. Classical symmetries. The symmetry condition (2.29) on p. 72 leads to the following five classical symmetries

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} + \frac{1}{2}(u_1\frac{\partial}{\partial u_1} + v_1\frac{\partial}{\partial v_1} - u_2\frac{\partial}{\partial u_2} - v_2\frac{\partial}{\partial v_2} \\ &\quad + u_3\frac{\partial}{\partial u_3} + v_3\frac{\partial}{\partial v_3} - u_4\frac{\partial}{\partial u_4} - v_4\frac{\partial}{\partial v_4}), \\ V_4 &= -v_1\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial v_1} - v_2\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial v_2}, \\ V_5 &= -v_3\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial u_4} + u_4\frac{\partial}{\partial v_4}. \end{aligned} \tag{3.85}$$

Associated to these classical symmetries, we construct in a straightforward way the conservation laws (C_x^i, C_t^i), satisfying

$$D_x(C_t^i) - D_t(C_x^i) = 0, \tag{3.86}$$

i.e.,

$$\begin{aligned}
C_x^1 &= u_{1x}v_1 - u_1v_{1x} + u_{2x}v_2 - u_2v_{2x} + u_{3x}v_3 - u_3v_{3x} + u_{4x}v_4 - u_4v_{4x},\\
C_t^1 &= -u_{1x}v_1 + u_1v_{1x} + u_{2x}v_2 - u_2v_{2x} - u_{3x}v_3 + u_3v_{3x} + u_{4x}v_4 - u_4v_{4x}\\
&\quad + \lambda(R_1R_4 - R_2R_3),\\
C_x^2 &= -u_{1x}v_1 + u_1v_{1x} + u_{2x}v_2 - u_2v_{2x} - u_{3x}v_3 + u_3v_{3x} + u_{4x}v_4 - u_4v_{4x}\\
&\quad + 2m_1(u_1u_2 + v_1v_2) + 2m_2(u_3u_4 + v_3v_4) + \lambda(R_1R_4 - R_2R_3),\\
C_t^2 &= u_{1x}v_1 - u_1v_{1x} + u_{2x}v_2 - u_2v_{2x} + u_{3x}v_3 - u_3v_{3x} + u_{4x}v_4 - u_4v_{4x},\\
C_x^3 &= xC_x^2 + tC_x^1,\\
C_t^3 &= xC_t^2 + tC_t^1,\\
C_x^4 &= R_1 + R_2,\\
C_t^4 &= -R_1 + R_2,\\
C_x^5 &= R_3 + R_4,\\
C_t^5 &= -R_3 + R_4.
\end{aligned} \tag{3.87}$$

In (3.87) we used the notations

$$R_1 = u_1^2 + v_1^2, \quad R_2 = u_2^2 + v_2^2, \quad R_3 = u_3^2 + v_3^2, \quad R_4 = u_4^2 + v_4^2. \tag{3.88}$$

7.2. First and second order higher symmetries. We now construct first and second order higher symmetries of the Federbush model. In obtaining the results, we observe the remarkable fact of the existence of first order higher symmetries, which are not equivalent to classical symmetries.

The results for first order symmetries are

$$\begin{aligned}
X_1 &= \frac{\lambda}{2}v_1R_4\frac{\partial}{\partial u_1} - \frac{\lambda}{2}u_1R_4\frac{\partial}{\partial v_1} + \frac{\lambda}{2}v_2R_4\frac{\partial}{\partial u_2} - \frac{\lambda}{2}u_2R_4\frac{\partial}{\partial v_2}\\
&\quad + \frac{1}{2}m_2v_4\frac{\partial}{\partial u_3} - \frac{1}{2}m_2u_4\frac{\partial}{\partial v_3}\\
&\quad + \frac{1}{2}(2u_{4x} + m_2v_3 + \lambda v_4(R_1 + R_2))\frac{\partial}{\partial u_4}\\
&\quad + \frac{1}{2}(2v_{4x} - m_2u_3 - \lambda u_4(R_1 + R_2))\frac{\partial}{\partial v_4},\\
X_2 &= \frac{\lambda}{2}v_1R_3\frac{\partial}{\partial u_1} - \frac{\lambda}{2}u_1R_3\frac{\partial}{\partial v_1} + \frac{\lambda}{2}v_2R_3\frac{\partial}{\partial u_2} - \frac{\lambda}{2}u_2R_3\frac{\partial}{\partial v_2}\\
&\quad + \frac{1}{2}(2u_{3x} - m_2v_4 + \lambda v_3(R_1 + R_2))\frac{\partial}{\partial u_3}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(2v_{3x}+m_2u_4-\lambda u_3(R_1+R_2))\frac{\partial}{\partial v_3}\\
&-\frac{1}{2}m_2v_3\frac{\partial}{\partial u_4}+\frac{1}{2}m_2u_3\frac{\partial}{\partial v_4},\\
X_3=&\ \frac{1}{2}m_1v_2\frac{\partial}{\partial u_1}-\frac{1}{2}m_1u_2\frac{\partial}{\partial v_1}\\
&+\frac{1}{2}(2u_{2x}+m_1v_1-\lambda v_2(R_3+R_4))\frac{\partial}{\partial u_2}\\
&+\frac{1}{2}(2v_{2x}-m_1u_1+\lambda u_2(R_3+R_4))\frac{\partial}{\partial v_2}\\
&-\frac{\lambda}{2}v_3R_2\frac{\partial}{\partial u_3}+\frac{\lambda}{2}u_3R_2\frac{\partial}{\partial v_3}-\frac{\lambda}{2}v_4R_2\frac{\partial}{\partial u_4}+\frac{\lambda}{2}u_4R_2\frac{\partial}{\partial v_4},\\
X_4=&\ \frac{1}{2}(2u_{1x}-m_1v_2-\lambda v_1(R_3+R_4))\frac{\partial}{\partial u_1}\\
&+\frac{1}{2}(2v_{1x}+m_1u_2+\lambda u_1(R_3+R_4))\frac{\partial}{\partial v_1}\\
&-\frac{1}{2}m_1v_1\frac{\partial}{\partial u_2}+\frac{1}{2}m_1u_1\frac{\partial}{\partial v_2}\\
&-\frac{\lambda}{2}v_3R_1\frac{\partial}{\partial u_3}+\frac{\lambda}{2}u_3R_1\frac{\partial}{\partial v_3}-\frac{\lambda}{2}v_4R_1\frac{\partial}{\partial u_4}+\frac{\lambda}{2}u_4R_1\frac{\partial}{\partial v_4}.
\end{aligned}
$$

Recall that two symmetries, X and Y are *equivalent* (we use the notation $\doteq$), see Chapter 2, if their exist functions $f, g \in \mathcal{F}(\mathcal{E})$ such that

$$X = Y + fD_x + gD_t, \tag{3.89}$$

where D_x, D_t are the total derivative operators.

From this one notes that

$$
\begin{aligned}
X_2+X_4 &\doteq -\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right),\\
X_1+X_3 &\doteq -\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right).
\end{aligned}
\tag{3.90}
$$

We did find these first order higher symmetries of the Federbush model using the following grading of the model:

$$\deg(x)=\deg(t)=-2, nonumber \tag{3.91}$$

$$\deg(\frac{\partial}{\partial x})=\deg(\frac{\partial}{\partial t})=2, nonumber \tag{3.92}$$

$$\deg(u_1)=\cdots=\deg(v_4)=1, nonumber \tag{3.93}$$

$$\deg(\frac{\partial}{\partial u_1})=\cdots=\deg(\frac{\partial}{\partial v_4})=-1, nonumber \tag{3.94}$$

$$\deg(m_1)=\deg(m_2)=2. \tag{3.95}$$

In order to find first order higher symmetries which are equivalent to the vector field V_3 (3.85), we searched for a vertical vector field of the following presentation:

$$V = xH_1 + tH_2 + C, \tag{3.96}$$

where H_1, H_2 are combinations of the vector fields V_4, V_5, $X_1, \dots, X_4$, while C is a correction of an appropriate degree.

From (3.96) and condition (3.91) we obtain two additional first order higher symmetries X_5, X_6, i.e.,

$$\begin{aligned} X_5 &= x(X_1 - X_2) + t(X_1 + X_2) - \frac{1}{2}\left(u_3\frac{\partial}{\partial u_3} + v_3\frac{\partial}{\partial v_3} - u_4\frac{\partial}{\partial u_4} - v_4\frac{\partial}{\partial v_4}\right), \\ X_6 &= x(X_3 - X_4) + t(X_3 + X_4) - \frac{1}{2}\left(u_1\frac{\partial}{\partial u_1} + v_1\frac{\partial}{\partial v_1} - u_2\frac{\partial}{\partial u_2} - v_2\frac{\partial}{\partial v_2}\right). \end{aligned} \tag{3.97}$$

Note that

$$X_5 + X_6 \doteq -V_3. \tag{3.98}$$

In order to construct second order higher symmmetries of the Federbush model, we searched for a vector field V, whose defining functions $V^{u_1}, \dots, V^{v_4}$ are dependent on the variables $u_1, \dots,\ v_4, \dots,\ u_{1xx}, \dots, v_{4xx}$. Due to the above introduced grading (3.91) the presentation of the defining functions $V^{u_1}, \dots, V^{v_4}$ is of the folowing structure:

$$V^* = [u]_{xx} + ([u]^2 + [m])[u]_x + ([u]^5 + [m][u]^3 + [m]^2[u]) \tag{3.99}$$

whereas in (3.99)

$[u]$	refers to	$u_1, \dots, v_4$,
$[u]_x$	refers to	$u_{1x}, \dots, v_{4x}$,
$[u]_{xx}$	refers to	$u_{1xx}, \dots, v_{4xx}$,
$[m]$	refers to	m_1, m_2.

From presentation (3.99) and the symmetry condition we derive an overdetermined system of partial differential equations. The solution of this system leads to four second-order higher symmetries of the Federbush model, $X_7, \dots, X_{10}$, i.e.:

$$\begin{aligned} X_7^{u_1} &= \frac{\lambda}{2}v_1K_7,\ X_7^{v_1} = -\frac{\lambda}{2}u_1K_7,\ X_7^{u_2} = \frac{\lambda}{2}v_2K_7,\ X_7^{v_2} = -\frac{\lambda}{2}u_2K_7, \\ X_7^{u_3} &= \frac{1}{4}m_2\big(2u_{4x} + \lambda v_4(R_1 + R_2)\big),\ X_7^{v_3} = \frac{1}{4}m_2\big(2v_{4x} - \lambda u_4(R_1 + R_2)\big), \\ X_7^{u_4} &= \frac{1}{4}\big(- 4v_{4xx} + 2\lambda u_4(R_1 + R_2)_x + 4\lambda u_{4x}(R_1 + R_2) + 2m_2u_{3x} \\ &\quad + \lambda m_2v_3(R_1 + R_2) + \lambda^2v_4(R_1 + R_2)^2\big), \\ X_7^{v_4} &= \frac{1}{4}\big(4u_{4xx} + 2\lambda v_4(R_1 + R_2)_x + 4\lambda v_{4x}(R_1 + R_2) + 2m_2v_{3x} \\ &\quad - \lambda m_2u_3(R_1 + R_2) - \lambda^2u_4(R_1 + R_2)^2\big), \\ X_8^{u_1} &= \frac{\lambda}{2}v_1K_8,\ X_8^{v_1} = -\frac{\lambda}{2}u_1K_8,\ X_8^{u_2} = \frac{\lambda}{2}v_2K_8,\ X_8^{v_2} = -\frac{\lambda}{2}u_2K_8, \end{aligned}$$

$$
\begin{aligned}
X_8^{u_3} &= \frac{1}{4}\big(-4v_{3xx} + 2\lambda u_3(R_1+R_2)_x + 4\lambda u_{3x}(R_1+R_2) - 2m_2u_{4x}\\
&\quad - \lambda m_2 v_4(R_1+R_2) + \lambda^2 v_3(R_1+R_2)^2\big),\\
X_8^{v_3} &= \frac{1}{4}\big(4u_{3xx} + 2\lambda v_3(R_1+R_2)_x + 4\lambda v_{3x}(R_1+R_2) - 2m_2v_{4x}\\
&\quad + \lambda m_2 u_4(R_1+R_2) - \lambda^2 u_3(R_1+R_2)^2\big),\\
X_8^{u_4} &= \frac{1}{4}m_2\big(-2u_{3x} - \lambda v_3(R_1+R_2)\big),\\
X_8^{v_4} &= \frac{1}{4}m_2\big(-2v_{3x} + \lambda u_3(R_1+R_2)\big),\\
X_9^{u_1} &= \frac{1}{4}m_1\big(2u_{2x} - \lambda v_2(R_3+R_4)\big),\\
X_9^{v_1} &= \frac{1}{4}m_1\big(2v_{2x} + \lambda u_2(R_3+R_4)\big),\\
X_9^{u_2} &= \frac{1}{4}\big(-4v_{2xx} - 2\lambda u_2(R_3+R_4)_x - 4\lambda u_{2x}(R_3+R_4) + 2m_1u_{1x}\\
&\quad - \lambda m_1 v_1(R_3+R_4) + \lambda^2 v_2(R_3+R_4)^2\big),\\
X_9^{v_2} &= \frac{1}{4}\big(4u_{2xx} - 2\lambda v_2(R_3+R_4)_x - 4\lambda v_{2x}(R_3+R_4) + 2m_1v_{1x}\\
&\quad + \lambda m_1 u_1(R_3+R_4) - \lambda^2 u_2(R_3+R_4)^2\big),\\
X_9^{u_3} &= \frac{\lambda}{2}v_3K_9,\ X_9^{v_3} = -\frac{\lambda}{2}u_3K_9,\ X_9^{u_4} = \frac{\lambda}{2}v_4K_9,\ X_9^{v_4} = -\frac{\lambda}{2}u_4K_9,\\
X_{10}^{u_1} &= \frac{1}{4}\big(-4v_{1xx} - 2\lambda u_1(R_3+R_4)_x - 4\lambda u_{1x}(R_3+R_4) - 2m_1u_{2x}\\
&\quad + \lambda m_1 v_2(R_3+R_4) + \lambda^2 v_1(R_3+R_4)^2\big),\\
X_{10}^{v_1} &= \frac{1}{4}\big(4u_{1xx} - 2\lambda v_1(R_3+R_4)_x - 4\lambda v_{1x}(R_3+R_4) - 2m_1v_{2x}\\
&\quad - \lambda m_1 u_2(R_3+R_4) - \lambda^2 u_1(R_3+R_4)^2\big),\\
X_{10}^{u_2} &= \frac{1}{4}m_1\big(-2u_{1x} + \lambda v_1(R_3+R_4)\big),\\
X_{10}^{v_2} &= \frac{1}{4}m_1\big(-2v_{1x} - \lambda u_1(R_3+R_4)\big),\\
X_{10}^{u_3} &= \frac{\lambda}{2}v_3K_{10},\ X_{10}^{v_3} = -\frac{\lambda}{2}u_3K_{10},\\
X_{10}^{u_4} &= \frac{\lambda}{2}v_4K_{10},\ X_{10}^{v_4} = -\frac{\lambda}{2}u_4K_{10},
\end{aligned}
\tag{3.100}
$$

whereas in (3.100)

$$
\begin{aligned}
K_7 &= 2u_{4x}v_4 - 2u_4v_{4x} + m_2(u_3u_4 + v_3v_4) + \lambda R_4(R_1+R_2),\\
K_8 &= 2u_{3x}v_3 - 2u_3v_{3x} - m_2(u_3u_4 + v_3v_4) + \lambda R_3(R_1+R_2),\\
K_9 &= -2u_{2x}v_2 + 2u_2v_{2x} - m_1(u_1u_2 + v_1v_2) + \lambda R_2(R_3+R_4),
\end{aligned}
$$

$$K_{10} = -2u_{1x}v_1 + 2u_1v_{1x} + m_1(u_1u_2 + v_1v_2) + \lambda R_1(R_3 + R_4). \tag{3.101}$$

The Lie bracket for vertical vector fields V_i, $i \in \mathbb{N}$, defined by

$$V_i = V_i^{u_1}\frac{\partial}{\partial u_1} + V_i^{v_1}\frac{\partial}{\partial v_1} + \cdots + V_i^{u_4}\frac{\partial}{\partial u_4} + V_i^{v_4}\frac{\partial}{\partial v_4}, \tag{3.102}$$

is given by

$$[V_i, V_j]^\alpha = V_i(V_j^\alpha) - V_j(V_i^\alpha), \quad \alpha = u_1, \ldots, v_4. \tag{3.103}$$

The commutators of the associated vector fields V_4, V_5, $X_1, \ldots, X_4$, X_5, X_6, $X_7, \ldots, X_{10}$ are given by the following nonzero commutators:

$$\begin{aligned}
[X_1, X_5] &= -X_1, \\
[X_2, X_5] &= X_2, \\
[X_3, X_6] &= -X_3, \\
[X_4, X_6] &= X_4, \\
[X_5, X_7] &= 2X_7 - \frac{1}{2}m_2^2V_5, \\
[X_5, X_8] &= -2X_8 + \frac{1}{2}m_2^2V_5, \\
[X_6, X_9] &= 2X_9 - \frac{1}{2}m_1^2V_4, \\
[X_6, X_{10}] &= -2X_{10} + \frac{1}{2}m_1^2V_4.
\end{aligned} \tag{3.104}$$

We now transform the vector fields by

$$\begin{aligned}
Y_0^+ &= V_4, & Y_0^- &= V_5, \\
Y_1^+ &= X_3, & Y_1^- &= X_1, \\
Y_{-1}^+ &= X_4, & Y_{-1}^- &= X_2, \\
Y_2^+ &= X_9 - \frac{1}{4}m_1^2V_4, & Y_2^- &= X_7 - \frac{1}{4}m_2^2V_5, \\
Y_{-2}^+ &= X_{10} - \frac{1}{4}m_1^2V_4, & Y_{-2}^- &= X_8 - \frac{1}{4}m_2^2V_5, \\
Z_0^+ &= X_6, & Z_0^- &= X_5.
\end{aligned} \tag{3.105}$$

From (3.103) and (3.105) we obtain a direct sum of two Lie algebras: each "+"-denoted element commutes with any "−"-denoted element and

$$[Z_0, Y_i] = iY_i, \quad [Y_i, Y_j] = 0, \qquad i, j = -2, \ldots, 2. \tag{3.106}$$

In (3.106) Z_0, Y_i, where $i = -2, \ldots, 2$, are assumed to have the same upper sign, $+$ or $-$.

7.3. Recursion symmetries. We shall now construct four (x,t)-dependent higher symmetries which act, by the Lie bracket for vertical vector fields, as recursion operators on the above constructed (x,t)-independent vector fields $X_1,\dots,X_4$, $X_7,\dots,X_{10}$. We are motivated by the results for the massive Thirring model, which were discussed in Subsections 6.1 and 6.2, and the results of Subsection 7.2, leading to the direct sum of two Lie algebras, each of which having a similar structure to the Lie algebra for the massive Thirring model. So we are forced to search for *nonlocal* higher symmetries, including the nonlocal variables (3.87) associated to the vector fields V_1, V_2 in (3.85).

Surprisingly, carrying through the huge computations, the nonlocal variables dropped out automatically from intermediate results, finally leading to *local* (x,t)-dependent higher symmetries. So, for simplicity we shall discuss the search for creating and annihilating symmetries, assuming from the beginning that they are local.

The formulation of creating and annihilating symmetries will follow from the Lie brackets of these symmetries with $Y_i^{\pm}$, meaning going up or down in the hierarchy. The symmetries Y_0^+, Y_0^- are of degree 0, Y_1^+, Y_{-1}^+, Y_1^-, Y_{-1}^- are of degree 2, while the symmetries Y_2^+, Y_{-2}^+, Y_2^-, Y_{-2}^- are of degree 4, see (3.105).

We now search for an (x,t)-dependent higher symmetry of second order, linear with respect to x, t, and of degree 2, i.e., for a vector field V of the form

$$V = xH_1 + tH_2 + C^*, \tag{3.107}$$

where H_1, H_2 are higher symmetries of degree four and, due to the fact that m_1, m_2 are of degree two, H_1, H_2 are assumed to be linear with respect to Y_0^+, $Y_0^-,\dots,Y_2^+$, Y_{-2}^+, Y_2^-, Y_{-2}^-, while V in (3.107) has to satisfy the symmetry condition. From these conditions we obtained the following result.

The symmetry condition is satisfied under the special assumption for V, (3.107), leading to the following four higher symmetries:

$$\begin{aligned}
X_{11} &= x\left(-Y_{-2}^+ + \frac{1}{4}m_1^2Y_0^+\right) + t\left(Y_{-2}^+ + \frac{1}{4}m_1^2Y_0^+\right) + C_{11},\\
X_{12} &= x\left(Y_2^+ - \frac{1}{4}m_1^2Y_0^+\right) + t\left(Y_2^+ + \frac{1}{4}m_1^2Y_0^+\right) + C_{12},\\
X_{13} &= x\left(-Y_{-2}^- + \frac{1}{4}m_2^2Y_0^-\right) + t\left(Y_{-2}^- + \frac{1}{4}m_2^2Y_0^-\right) + C_{13},\\
X_{14} &= x\left(Y_{-2}^- - \frac{1}{4}m_2^2Y_0^-\right) + t\left(Y_{-2}^- + \frac{1}{4}m_2^2Y_0^-\right) + C_{14}.
\end{aligned}\tag{3.108}$$

where in (3.108) the functions $C_{11},\dots,C_{14}$ are given by the following expressions

$$C_{11} = \frac{1}{2}\Big(2v_{1x} + m_1u_2 + \lambda u_1(R_3 + R_4)\Big)\frac{\partial}{\partial u_1}$$

$$
\begin{aligned}
&+\frac{1}{2}\Big(-2u_{1x}+m_1v_2+\lambda v_1(R_3+R_4)\Big)\frac{\partial}{\partial v_1},\\
C_{12}&=\frac{1}{2}\Big(-2v_{2x}+m_1u_1-\lambda u_2(R_3+R_4)\Big)\frac{\partial}{\partial u_2}\\
&+\frac{1}{2}\Big(2u_{2x}+m_1v_1-\lambda v_2(R_3+R_4)\Big)\frac{\partial}{\partial v_2},\\
C_{13}&=\frac{1}{2}\Big(2v_{3x}+m_2u_4-\lambda u_3(R_1+R_2)\Big)\frac{\partial}{\partial u_3}\\
&+\frac{1}{2}\Big(-2u_{3x}+m_2v_4-\lambda v_3(R_1+R_2)\Big)\frac{\partial}{\partial v_3},\\
C_{14}&=\frac{1}{2}\Big(-2v_{4x}+m_2u_3+\lambda u_4(R_1+R_2)\Big)\frac{\partial}{\partial u_4}\\
&+\frac{1}{2}\Big(2u_{4x}+m_2v_3+\lambda v_4(R_1+R_2)\Big)\frac{\partial}{\partial v_4}.
\end{aligned}
\tag{3.109}
$$

From (3.108) and (3.109) we define

$$
Z^+_{-1}=X_{11},\quad Z^+_1=X_{12},\quad Z^-_{-1}=X_{13},\quad Z^-_1=X_{14}. \tag{3.110}
$$

Computation of the commutators of Z^+_{-1}, Z^+_1, Z^-_{-1}, Z^-_1 and $Y^\pm_i$, where $i=-2,\dots,2$, leads to the following result:

$$
\begin{aligned}
&[Z^+_{-1},Y^+_2]=-\frac{1}{2}m_1^2Y^+_1, && [Z^+_1,Y^+_2]=Y^+_3,\\
&[Z^+_{-1},Y^+_1]=\frac{1}{4}m_1^2Y^+_0, && [Z^+_1,Y^+_1]=Y^+_2,\\
&[Z^+_{-1},Y^+_0]=0, && [Z^+_1,Y^+_0]=0,\\
&[Z^+_{-1},Y^+_{-1}]=-Y^+_{-2}, && [Z^+_1,Y^+_{-1}]=-\frac{1}{4}m_1^2Y^+_0, nonumber && (3.111)\\
&[Z^+_{-1},Y^+_{-2}]=Y^+_{-3}, && [Z^+_1,Y^+_{-2}]=\frac{1}{2}m_1^2Y^+_{-1}, nonumber && (3.112)\\
&[Z^-_{-1},Y^-_2]=-\frac{1}{2}m_2^2Y^-_1, && [Z^-_1,Y^-_2]=Y^-_3,\\
&[Z^-_{-1},Y^-_1]=\frac{1}{4}m_2^2Y^-_0, && [Z^-_1,Y^-_1]=Y^-_2,\\
&[Z^-_{-1},Y^-_0]=0, && [Z^-_1,Y^-_0]=0,\\
&[Z^-_{-1},Y^-_{-1}]=-Y^-_{-2}, && [Z^-_1,Y^-_{-1}]=-\frac{1}{4}m_2^2Y^-_0, nonumber && (3.113)\\
&[Z^-_{-1},Y^-_{-2}]=Y^-_{-3}, && [Z^-_1,Y^-_{-2}]=\frac{1}{2}m_2^2Y^-_{-1}, && (3.114)
\end{aligned}
$$

while

$$
[Z^+_{-1},Z^+_1]=-\frac{1}{2}m_1^2Z^+_0,\qquad [Z^-_{-1},Z^-_1]=-\frac{1}{2}m_2^2Z^-_0. \tag{3.115}
$$

All other commutators are zero. The vector field Y_3^+ is given by

$$
\begin{aligned}
Y_3^{+,u_1} &= \frac{m_1}{4}\Big(- 4v_{2xx} - 4\lambda R_{34}u_{2x} + 2m_1u_{1x} - 4\lambda u_2(R_{34})_{(1)} \\
&+ m_1^2 v_2 - \lambda m_1 R_{34} v_1 + \lambda^2 R_{34}^2 v_2\Big), \\
Y_3^{+,v_1} &= \frac{m_1}{4}\Big(+ 4u_{2xx} - 4\lambda R_{34}v_{2x} + 2m_1v_{1x} - 4\lambda v_2(R_{34})_{(1)} \\
&- m_1^2 u_2 + \lambda m_1 R_{34} u_1 - \lambda^2 R_{34}^2 u_2\Big), \\
Y_3^{+,u_2} &= \frac{1}{4}\Big(- 8u_{2xxx} - 4m_1v_{1xx} + 12\lambda R_{34}v_{2xx} + 8\lambda v_2(R_{34})_{(2)} \\
&+ 24\lambda v_{2x}(R_{34})_{(1)} + 8\lambda v_2(R_{34})_{(1,1)} + u_{2x}(4m_1^2 + 6\lambda^2 R_{34}^2) \\
&- 4\lambda m_1 R_{34}u_{1x} + 12\lambda^2 u_2 R_{34}(R_{34})_{(1)} - 4\lambda m_1 u_1 (R_{34})_{(1)} \\
&+ m_1^3 v_1 - 2\lambda m_1^2 R_{34} v_2 + \lambda^2 m_1 R_{34}^2 v_1 - \lambda^3 R_{34}^3 v_2\Big), \\
Y_3^{+,v_2} &= \frac{1}{4}\Big(- 8v_{2xxx} + 4m_1u_{1xx} - 12\lambda R_{34}u_{2xx} + 8\lambda u_2(R_{34})_{(2)} \\
&- 24\lambda u_{2x}(R_{34})_{(1)} - 8\lambda u_2(R_{34})_{(1,1)} + v_{2x}(4m_1^2 + 6\lambda^2 R_{34}^2) \\
&- 4\lambda m_1 R_{34}v_{1x} + 12\lambda^2 v_2 R_{34}(R_{34})_{(1)} - 4\lambda m_1 v_1 (R_{34})_{(1)} \\
&- m_1^3 u_1 + 2\lambda m_1^2 R_{34} u_2 - \lambda^2 m_1 R_{34}^2 u_1 + \lambda^3 R_{34}^3 u_2\Big), \\
Y_3^{+,u_3} &= \frac{\lambda}{4}v_3 L, \quad Y_3^{+,v_3} = -\frac{\lambda}{4}u_3 L, \\
Y_3^{+,u_4} &= \frac{\lambda}{4}v_4 L, \quad Y_3^{+,v_4} = -\frac{\lambda}{4}u_4 L,
\end{aligned}
\tag{3.116}
$$

where in (3.116)

$$
\begin{aligned}
R_{34} &= R_3 + R_4, \\
(R_{34})_{(1)} &= u_3u_{3x} + v_3v_{3x} + u_4u_{4x} + v_4v_{4x}, \\
(R_{34})_{(2)} &= u_3u_{3xx} + v_3v_{3xx} + u_4u_{4xx} + v_4v_{4xx}, \\
(R_{34})_{(1,1)} &= u_{3x}^2 + v_{3x}^2 + u_{4x}^2 + v_{4x}^2, \\
L &= 8(u_2u_{2xx} + v_2v_{2xx}) - 4(u_{2x}^2 + v_{2x}^2) + 12\lambda R_{34}(u_{2x}v_2 - v_{2x}u_2) \\
&+ 4m_1(u_1v_{2x} - v_1u_{2x} + u_2v_{1x} - v_2u_{1x}) - m_1^2(2R_2 + R_1) \\
&+ 4m_1\lambda R_{34}(u_1u_2 + v_1v_2) - 3\lambda^2 R_2 R_{34}^2.
\end{aligned}
$$

The results for the vector fields Y_{-3}^+, Y_3^-, Y_{-3}^- are similar to (3.116) and are not given here, but are obtained from discrete symmetries σ and τ, to be described in the next section.

From the above it is clear now, why the vector fields Z_{-1}^+, Z_1^+, Z_{-1}^-, Z_1^- are called *creating* and *annihilating* operators.

We thus have four infinite hierarchies of symmetries of the Federbush model, i.e., Y^+_{-n}, Y^+_n, Y^-_{-n}, Y^-_n, $n \in \mathbb{N}$. A formal proof of the infiniteness of the hierarchies is given in Subsection 7.5.3.

7.4. Discrete symmetries. In deriving the specific results for the symmetry structure of the Federbush model, we realised that there are discrete transformations which transform the Federbush model into itself and by consequence transform symmetries into symmetries. Existence of these disrete symmetries allow us to restrict to just one part of the Lie algebra of symmetries, the discrete symmetries generating the remaining parts. These discrete symmetries σ, τ are given by

$$\begin{aligned}
&\sigma : u_1 \leftrightarrow u_3, v_1 \leftrightarrow v_3, u_2 \leftrightarrow u_4, v_2 \leftrightarrow v_4, m_1 \leftrightarrow m_2, \lambda \leftrightarrow -\lambda, t \leftrightarrow t;\\
&\tau : u_1 \leftrightarrow u_2, v_1 \leftrightarrow v_2, u_3 \leftrightarrow u_4, v_3 \leftrightarrow v_4, \lambda \leftrightarrow -\lambda, x \leftrightarrow -x, t \leftrightarrow t. \qquad (3.117)
\end{aligned}$$

The transformations satisfy the following rules:

$$\begin{aligned}
\sigma^2 &= \mathrm{id},\\
\tau^2 &= \mathrm{id},\\
\sigma \circ \tau &= \tau \circ \sigma.
\end{aligned}$$

Physically, the transformation σ denotes the exchange of two particles.

The action of the discrete smmetries on the Lie algebra of symmetries is as follows:

$$\begin{aligned}
\sigma(Y^+_i) &= Y^-_i,\\
\tau(Y^+_i) &= Y^+_{-i},\\
\tau(Y^-_i) &= Y^-_{-i},
\end{aligned}$$

where $i = 0, 1, 2$,

$$\begin{aligned}
\sigma(Z^+_1) &= Z^-_1,\\
\tau(Z^+_1) &= Z^+_{-1},\\
\tau(Z^-_1) &= Z^-_{-1}, \qquad (3.118)
\end{aligned}$$

while Y^+_{-3}, Y^-_3, Y^-_{-3}, arising in the previous section, are defined by

$$Y^+_{-3} = \tau(Y^+_3), \quad Y^-_3 = \sigma(Y^+_3), \quad Y^-_{-3} = \tau\sigma(Y^+_3). \qquad (3.119)$$

7.5. Towards infinite number of hierarchies of symmetries. In this subsection, we demonstrate the existence of an infinite number of hiearchies of higher symmetries of the Federbush model. We shall do this by the construction of two (x, t)-dependent symmetries of degree 0 which are polynomial with respect to x, t and of degree 2. This will be done in Subsection 7.5.1.

Then, after writing the Federbush model as a Hamiltonian system, we show that all higher symmetries obtained thusfar are Hamitonian vector fields; this will be done in Subsection 7.5.2. Finally in Subsection 7.5.3

we give a proof of a lemma from which the existence of infinite number of hierarchies of Hamiltonians becomes evident, and from this we then obtain the obvious result for the symmetry structure of the Federbush model.

7.5.1. *Construction of* $Y^+(2,0)$ *and* $Y^+(2,0)$. First, we start from the presentation of these vector fields, which is assumed to be of the following structure

$$\begin{aligned} Y^+(2,0) &= x^2(\alpha_1 Y_2^+ + \alpha_2 m_1 Y_1^+ + \alpha_3 m_1^2 Y_0^+ + \alpha_4 m_1 Y_{-1}^+ + \alpha_5 Y_{-2}^+) \\ &+ 2xt(\beta_1 Y_2^+ + \beta_2 m_1 Y_1^+ + \beta_3 m_1^2 Y_0^+ + \beta_4 m_1 Y_{-1}^+ + \beta_5 Y_{-2}^+) \\ &+ t^2(\gamma_1 Y_2^+ + \gamma_2 m_1 Y_1^+ + \gamma_3 m_1^2 Y_0^+ + \gamma_4 m_1 Y_{-1}^+ + \gamma_5 Y_{-2}^+) \\ &+ xC_1^+ + tC_2^+ + C_0^+, \end{aligned} \tag{3.120}$$

In (3.120), the fields Y_i^+, $i = -2, \ldots, 2$, are given in previous sections, α_1, β_i, γ_i, $i = 1, \ldots, 5$, are constant, while C_1^+, C_2^+, C_0^+, which are of degree 2, 2 and 1 respectively, have to be determined.

From the symmetry condition (2.29) on p. 72 we obtained the following result: There does exist a symmetry of presentation (3.120), which is given by

$$\begin{aligned} Y^+(2,0) &= x^2(Y_2^+ - \frac{1}{2}m_1^2 Y_0^+ + Y_{-2}^+) + 2xt(Y_2^+ - Y_{-2}^+) \\ &+ t^2(Y_2^+ + \frac{1}{2}m_1^2 Y_0^+ + Y_{-2}^+) + xC_1^+ + tC_2^+, \end{aligned} \tag{3.121}$$

whereas in (3.120) and (3.121),

$$\begin{aligned} C_1^+ &= (-2v_{1x} - m_1 u_2 - \lambda R_{34} u_1)\frac{\partial}{\partial u_1} + (2u_{1x} - m_1 v_2 - \lambda R_{34} v_1)\frac{\partial}{\partial v_1} \\ &+ (-2v_{2x} + m_1 u_1 - \lambda R_{34} u_2)\frac{\partial}{\partial u_2} + (2u_{2x} + m_1 v_1 - \lambda R_{34} v_2)\frac{\partial}{\partial v_2}, \\ C_2^+ &= (2v_{1x} + m_1 u_2 + \lambda R_{34} u_1)\frac{\partial}{\partial u_1} + (-2u_{1x} + m_1 v_2 + \lambda R_{34} v_1)\frac{\partial}{\partial v_1} \\ &+ (-2v_{2x} + m_1 u_1 - \lambda R_{34} u_2)\frac{\partial}{\partial u_2} + (2u_{2x} + m_1 v_1 - \lambda R_{34} v_2)\frac{\partial}{\partial v_2}, \\ C_0^+ &= 0. \end{aligned} \tag{3.122}$$

In a similar way, motivated by the structure of the Lie algebra obtained thusfar, we get another higher symmetry of a similar structure, i.e.,

$$\begin{aligned} Y^-(2,0) &= x^2(Y_2^- - \frac{1}{2}m_2^2 Y_0^- + Y_{-2}^-) + 2xt(Y_2^- - Y_{-2}^-) \\ &+ t^2(Y_2^- - \frac{1}{2}m_2^2 Y_0^- + Y_{-2}^-) + xC_1^- + tC_2^-, \end{aligned} \tag{3.123}$$

whereas in (3.123),

$$C_1^- = (-2v_{3x} - m_2 u_4 + \lambda R_{12} u_3)\frac{\partial}{\partial u_3} + (2u_{3x} - m_2 v_4 + \lambda R_{12} v_3)\frac{\partial}{\partial v_3}$$

$$+(-2v_{4x}+m_2u_3+\lambda R_{12}u_4)\frac{\partial}{\partial u_4}+(2u_{4x}+m_2v_3+\lambda R_{12}v_4)\frac{\partial}{\partial v_4},$$

$$C_2^- = (2v_{3x}+m_2u_4-\lambda R_{12}u_3)\frac{\partial}{\partial u_3}+(-2u_{3x}+m_2v_4-\lambda R_{12}v_3)\frac{\partial}{\partial v_3}$$
$$+(-2v_{4x}+m_2u_3+\lambda R_{12}u_4)\frac{\partial}{\partial u_4}+(2u_{4x}+m_2v_3+\lambda R_{12}v_4)\frac{\partial}{\partial v_4},$$
$$C_0^- = 0. \tag{3.124}$$

To give an idea of the action of the vector fields $Y^+(2,0)$, $Y^-(2,0)$, we compute their Lie brackets with the vector fields Y_1^+, Y_0^+, Y_{-1}^+, Y_1^-, Y_0^-, Y_{-1}^-, yielding the following results

$$\begin{aligned}
&[Y^+(2,0),Y_1^+]=2Z_1^+, && [Y^-(2,0),Y_1^-]=2Z_1^-,\\
&[Y^+(2,0),Y_0^+]=0, && [Y^-(2,0),Y_0^-]=0,\\
&[Y^+(2,0),Y_{-1}^+]=2Z_{-1}^+, && [Y^-(2,0),Y_{-1}^-]=2Z_{-1}^-,\\
&[Y^+(2,0),Y_i^-]=0, && [Y^-(2,0),Y_i^+]=0,
\end{aligned}\tag{3.125}$$

where $i=-1,0,1$. These results suggest to set

$$Y^\pm(1,i)=Z_i^\pm,\quad Y^\pm(0,i)=Y_i^\pm,\qquad i\in\mathbb{Z}. \tag{3.126}$$

The complete Lie algebra structure is obtained in Subection 7.5.3.

7.5.2. *Hamiltonian structures.* We shall now discuss Hamiltonians (or conserved functionals) for the Federbush model described by (3.84),

$$\begin{aligned}
u_{1,t}+u_{1,x}-m_1v_2&=\lambda(u_4^2+v_4^2)v_1,\\
-v_{1,t}-v_{1,x}-m_1u_2&=\lambda(u_4^2+v_4^2)u_1,\\
u_{2,t}-u_{2,x}-m_1v_1&=-\lambda(u_3^2+v_3^2)v_2,\\
-v_{2,t}+v_{2,x}-m_1u_1&=-\lambda(u_3^2+v_3^2)u_2,\\
u_{3,t}+u_{3,x}-m_2v_4&=-\lambda(u_2^2+v_2^2)v_3,\\
-v_{3,t}-v_{3,x}-m_2u_4&=-\lambda(u_2^2+v_2^2)u_3,\\
u_{4,t}-u_{4,x}-m_2v_3&=\lambda(u_2^2+v_2^2)v_4,\\
-v_{4,t}+v_{4,x}-m_2u_3&=\lambda(u_2^2+v_2^2)u_4.
\end{aligned}\tag{3.127}$$

We introduce functions $R_1,\dots,R_4$ by

$$\begin{aligned}
R_1&=u_1^2+v_1^2, & R_2&=u_2^2+v_2^2,\\
R_3&=u_3^2+v_3^2, & R_4&=u_4^2+v_4^2.
\end{aligned}$$

We first rewite the Federbush model as a Hamiltonian system, i.e.,

$$\frac{du}{dt}=\Omega^{-1}\delta H, \tag{3.128}$$

where Ω is a symplectic operator, H is the Hamiltonian and δH is the Fréchet derivative[1] of H, $u = (u_1, v_1, \ldots, u_4, v_4)$. In (3.128) we have

$$\Omega = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & J \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$H = \int_{-\infty}^{\infty} \frac{1}{2}\big(u_{1x}v_1 - u_1 v_{1x} - u_{2x}v_2 + u_2 v_{2x} + u_{3x}v_3 - u_3 v_{3x} - u_{4x}v_4 + u_4 v_{4x}\big)\, dx$$
$$- m_1(u_1u_2 + v_1v_2) - m_2(u_3u_4 + v_3v_4) - \frac{\lambda}{2}R_1R_4 + \frac{\lambda}{2}R_2R_3.$$

By definition, to each Hamiltonian symmetry Y (also called canonical symmetry) there corresponds a Hamiltonian $F(Y)$, where

$$F(Y) = \int_{-\infty}^{\infty} \mathcal{F}(Y)\, dx, \tag{3.129}$$

$\mathcal{F}(Y)$ being the Hamiltonian density, such that

$$Y = \Omega^{-1}\delta F(Y), \tag{3.130}$$

and the Poisson bracket of $F(Y)$ and H vanishes.

Suppose that Y_1, Y_2 are two Hamiltonian symmetries. Then $[Y_1, Y_2]$ is a Hamiltonian symmetry and

$$F([Y_1, Y_2]) = \{F(Y_1), F(Y_2)\}, \tag{3.131}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined by

$$\{F(Y_1), F(Y_2)\} = \langle \delta F(Y_1), Y_2\rangle, \tag{3.132}$$

$\langle\cdot,\cdot\rangle$ denoting the contraction of a 1-form and a vector field:

$$\frac{d}{d\epsilon}H(x + \epsilon y)|_{\epsilon=0} = \langle \delta H, y\rangle. \tag{3.133}$$

The Hamiltonians $F(X)$ associated to the Hamiltonian densities $\mathcal{F}(X)$ are defined by (3.134):

$$F(X) = \int_{-\infty}^{\infty} \mathcal{F}(X)\, dx. \tag{3.134}$$

From these definitions it is a straightforward computation that the symmetries Y_0^+, Y_1^+, Y_{-1}^+, Y_0^-, Y_1^-, Y_{-1}^- obtained sofar are all Hamiltonian, where the Hamiltonian densities are given by

$$\mathcal{F}(Y_0^+) = \frac{1}{2}(R_1 + R_2),$$
$$\mathcal{F}(Y_1^+) = -\frac{1}{2}(u_{2x}v_2 - u_2v_{2x}) + \frac{\lambda}{4}R_{34}R_2 - \frac{1}{2}m_1(u_1u_2 + v_1v_2),$$

[1] By the Fréchet derivative the components of the Euler–Lagrange operator are understood.

$$\begin{aligned}
\mathcal{F}(Y^+_{-1}) &= -\frac{1}{2}(u_{1x}v_1 - u_1v_{1x}) + \frac{\lambda}{4}R_{34}R_1 + \frac{1}{2}m_1(u_1u_2 + v_1v_2),\\
\mathcal{F}(Y^-_0) &= \frac{1}{2}(R_3 + R_4),\\
\mathcal{F}(Y^-_1) &= -\frac{1}{2}(u_{4x}v_4 - u_4v_{4x}) - \frac{\lambda}{4}R_{12}R_4 - \frac{1}{2}m_2(u_3u_4 + v_3v_4),\\
\mathcal{F}(Y^-_{-1}) &= -\frac{1}{2}(u_{3x}v_3 - u_3v_{3x}) - \frac{\lambda}{4}R_{12}R_3 + \frac{1}{2}m_2(u_3u_4 + v_3v_4), \qquad (3.135)
\end{aligned}$$

whereas the densities $\mathcal{F}(Y^\pm_i)$, $i = -2, 2$, are given by

$$\begin{aligned}
\mathcal{F}(Y^+_2) &= -\frac{1}{2}(u_{2x}^2 + v_{2x}^2) + \frac{\lambda}{2}R_{34}(u_{2x}v_2 - u_2v_{2x}) - \frac{1}{2}m_1(u_{2x}v_1 - u_1v_{2x})\\
&\quad - \frac{1}{8}\lambda^2R_{34}^2R_2 + \frac{1}{4}m_1\lambda R_{34}(u_1u_2 + v_1v_2) - \frac{1}{8}m_1^2R_{12},\\
\mathcal{F}(Y^+_{-2}) &= -\frac{1}{2}(u_{1x}^2 + v_{1x}^2) + \frac{\lambda}{2}R_{34}(u_{1x}v_1 - u_1v_{1x}) + \frac{1}{2}m_1(u_{1x}v_2 - u_2v_{1x})\\
&\quad - \frac{1}{8}\lambda^2R_{34}^2R_1 - \frac{1}{4}m_1\lambda R_{34}(u_1u_2 + v_1v_2) - \frac{1}{8}m_1^2R_{12},\\
\mathcal{F}(Y^-_2) &= -\frac{1}{2}(u_{4x}^2 + v_{4x}^2) - \frac{\lambda}{2}R_{12}(u_{4x}v_4 - u_4v_{4x}) - \frac{1}{2}m_2(u_{4x}v_3 - u_3v_{4x})\\
&\quad - \frac{1}{8}\lambda^2R_{12}^2R_4 - \frac{1}{4}m_2\lambda R_{12}(u_3u_4 + v_3v_4) - \frac{1}{8}m_2^2R_{34},\\
\mathcal{F}(Y^-_{-2}) &= -\frac{1}{2}(u_{3x}^2 + v_{3x}^2) - \frac{\lambda}{2}R_{12}(u_{3x}v_3 - u_3v_{3x}) + \frac{1}{2}m_2(u_{3x}v_4 - u_4v_{3x})\\
&\quad - \frac{1}{8}\lambda^2R_{12}^2R_3 + \frac{1}{4}m_2\lambda R_{12}(u_3u_4 + v_3v_4) - \frac{1}{8}m_2^2R_{34}, \qquad (3.136)
\end{aligned}$$

and the densities associated to Y^+_3, Y^+_{-3} are given by

$$\begin{aligned}
\mathcal{F}(Y^+_3) &= -(u_{2xx}v_{2x} - v_{2xx}u_{2x}) - \lambda R_{34}(u_{2xx}u_2 + v_{2xx}v_2)\\
&\quad + \frac{\lambda}{2}R_{34}(u_{2x}^2 + v_{2x}^2)\\
&\quad - m_1(u_{1x}u_{2x} + v_{1x}v_{2x}) - \frac{3}{4}\lambda^2R_{34}^2(u_{2x}v_2 - u_2v_{2x})\\
&\quad + \frac{1}{2}m_1\lambda R_{34}(u_{1x}v_2 - u_1v_{2x} + u_{2x}v_1 - u_2v_{1x})\\
&\quad - \frac{1}{4}m_1^2(u_{1x}v_1 - u_1v_{1x}) - \frac{1}{2}m_1^2(u_{2x}v_2 - u_2v_{2x}) - \frac{1}{4}m_1^3(u_1u_2 + v_1v_2)\\
&\quad + \frac{1}{8}\lambda^3R_{34}^3R_2 - \frac{1}{4}m_1\lambda^2R_{34}^2(u_1u_2 + v_1v_2) + \frac{1}{8}m_1^2\lambda R_{34}(R_1 + 2R_2),\\
\mathcal{F}(Y^+_{-3}) &= u_{1xx}v_{1x} - v_{1xx}u_{1x} + \lambda R_{34}(u_{1xx}u_1 + v_{1xx}v_1) + \frac{\lambda}{2}R_{34}(u_{1x}^2 + v_{1x}^2)\\
&\quad - m_1(u_{1x}u_{2x} + v_{1x}v_{2x}) + \frac{3}{4}\lambda^2R_{34}^2(u_{1x}v_1 - u_1v_{1x})\\
&\quad + \frac{1}{2}m_1\lambda R_{34}(u_{1x}v_2 - u_1v_{2x} + u_{2x}v_1 - u_2v_{1x})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}m_1^2(u_{1x}v_1-u_1v_{1x})+\frac{1}{4}m_1^2(u_{2x}v_2-u_2v_{2x})-\frac{1}{4}m_1^3(u_1u_2+v_1v_2)\\
&-\frac{1}{8}\lambda^3R_{34}^3R_1-\frac{1}{4}m_1\lambda^2R_{34}^2(u_1u_2+v_1v_2)-\frac{1}{8}m_1^2\lambda R_{34}(2R_1+R_2).
\end{aligned}
$$

The vector fields Z_{-1}^+, Z_1^+, Z_{-1}^-, Z_1^- are Hamiltonian vector fields too, and the associated densities are given by

$$
\begin{aligned}
\mathcal{F}(Z_0^+)&=x\big(\mathcal{F}(Y_1^+)-\mathcal{F}(Y_{-1}^+)\big)+t\big(\mathcal{F}(Y_1^+)+\mathcal{F}(Y_{-1}^+)\big),\\
\mathcal{F}(Z_1^+)&=x\big(\mathcal{F}(Y_2^+)-\frac{1}{4}m_1^2\mathcal{F}(Y_0^+)\big)+t\big(\mathcal{F}(Y_2^+)+\frac{1}{4}m_1^2\mathcal{F}(Y_0^+)\big),\\
\mathcal{F}(Z_{-1}^+)&=x\big(-\mathcal{F}(Y_{-2}^+)+\frac{1}{4}m_1^2\mathcal{F}(Y_0^+)\big)+t\big(\mathcal{F}(Y_{-2}^+)+\frac{1}{4}m_1^2\mathcal{F}(Y_0^+)\big),\\
\mathcal{F}(Z_0^-)&=x\big(\mathcal{F}(Y_1^-)-\mathcal{F}(Y_{-1}^-)\big)+t\big(\mathcal{F}(Y_1^-)+\mathcal{F}(Y_{-1}^-)\big),\\
\mathcal{F}(Z_1^-)&=x\big(\mathcal{F}(Y_2^-)-\frac{1}{4}m_2^2\mathcal{F}(Y_0^-)\big)+t\big(\mathcal{F}(Y_2^-)+\frac{1}{4}m_2^2\mathcal{F}(Y_0^-)\big),\\
\mathcal{F}(Z_{-1}^-)&=x(-\mathcal{F}(Y_{-2}^-)+\frac{1}{4}m_2^2\mathcal{F}(Y_0^-)\big)+t\big(\mathcal{F}(Y_{-2}^-)+\frac{1}{4}m_2^2\mathcal{F}(Y_0^-)\big).
\end{aligned}
$$

We now arrive at the following remarkable fact: The vector fields $Y^+(2,0)$ and $Y^-(2,0)$ are again Hamiltonian vector fields, the corresponding Hamiltonian densities being given by

$$
\begin{aligned}
\mathcal{F}(Y^-(2,0))&=x^2\big(\mathcal{F}(Y_2^-)-\frac{1}{2}m_2^2\mathcal{F}(Y_0^-)+\mathcal{F}(Y_{-2}^-)\big)\\
&\quad+2xt\big(\mathcal{F}(Y_2^-)-\mathcal{F}(Y_{-2}^-)\big)\\
&\quad+t^2\big(\mathcal{F}(Y_2^-)+\frac{1}{2}m_2^2\mathcal{F}(Y_0^-)+\mathcal{F}(Y_{-2}^-)\big)\\
&=(x+t)^2\mathcal{F}(Y_2^-)-\frac{1}{2}m_2^2(x+t)(x-t)\mathcal{F}(Y_0^-)\\
&\quad+(x-t)^2\mathcal{F}(Y_{-2}^-),
\end{aligned}
\tag{3.137}
$$

and similarly

$$
\begin{aligned}
\mathcal{F}(Y^+(2,0))&=(x+t)^2\mathcal{F}(Y_2^+)-\frac{1}{2}m_1^2(x+t)(x-t)\mathcal{F}(Y_0^+)\\
&\quad+(x-t)^2\mathcal{F}(Y_{-2}^+),
\end{aligned}
\tag{3.138}
$$

Now the Hamiltonians $F(Z_1^+)$, $F(Z_{-1}^+)$, $F(Z_1^-)$, $F(Z_{-1}^-)$ act as creating and annihilating operators on the (x,t)-independent Hamiltonians $F(Y_{-3}^+),\ldots,$ $F(Y_3^+)$ and $F(Y_{-3}^-),\ldots,F(Y_3^-)$, by the action of the Poisson bracket: for example

$$
\begin{aligned}
&\{F(Z_1^+),F(Y_0^+)\}=0,\\
&\{F(Z_1^+),F(Y_{-1}^+)\}=\frac{1}{4}m_1^2\int_{-\infty}^{\infty}(R_1+R_2)=\frac{1}{4}m_1^2F(Y_0^+),\\
&\{F(Z_1^+),F(Y_1^+)\}=-F(Y_2^+).
\end{aligned}
$$

In the next subsection we give a formal proof for the existence of infinite number of hierarchies of higher symmetries by proving existence of infinite number of hierarchies of Hamiltonians, thus leading to those for the symmetries.

7.5.3. *The infinity of the hierarchies.* We shall prove here a lemma concerning the infiniteness of the hierarchies of Hamiltonians for the Federbush model. From this we obtain a similar result for the associated hierarchies of Hamiltonian vector fields.

LEMMA 3.12. *Let $H_n^r(u,v)$ and $K_n^r(u,v)$ be defined by*

$$\begin{aligned} H_n^r(u,v) &= \int_{-\infty}^{\infty} x^r(u_n^2+v_n^2), \\ K_n^r(u,v) &= \int_{-\infty}^{\infty} x^r(u_{n+1}v_n - v_{n+1}u_n), \end{aligned} \tag{3.139}$$

whereas in (3.139) $r,n=0,1,\dots,$ *and* r, n *are such that the degrees of* $H_n^r(u,v)$ *and* $K_n^r(u,v)$ *are positive.*

Let the Poisson bracket of F *and* L, *denoted by* $\{F,L\}$, *be defined as*

$$\{F,L\} = \int_{-\infty}^{\infty} \Big(\frac{\delta F}{\delta v}\frac{\delta L}{\delta u} - \frac{\delta F}{\delta u}\frac{\delta L}{\delta v}\Big). \tag{3.140}$$

Then the following results hold

$$\begin{aligned} &\{H_1^1,H_n^r\} = 4(n-r)K_n^r, \\ &\{H_1^1,K_n^r\} = (4(n-r)+2)H_{n+1}^r + r(r-1)(r-n-1)H_n^{r-2}, \\ &\{H_1^2,H_n^r\} = 4(2n-r)K_n^{r+1}, \\ &\{H_1^2,K_n^r\} = (2n+1-r)(4H_{n+1}^{r+1} - r^2H_n^{r-1}), \end{aligned} \tag{3.141}$$

$r,n=0,1,\dots$

PROOF. We shall now prove the third and fourth relation in (3.141), the proofs of the other two statements running along similar lines.

Calculation of the Fréchet derivatives of H_n^r, K_n^r yields

$$\begin{aligned} \frac{\delta H_n^r}{\delta u} &= (-D_x)^n(2x^r u_n), \\ \frac{\delta H_n^r}{\delta v} &= (-D_x)^n(2x^r v_n), \\ \frac{\delta K_n^r}{\delta u} &= (-D_x)^{n+1}(x^r v_n) - (-D_x)^n(x^r v_{n+1}), \\ \frac{\delta K_n^r}{\delta v} &= -(-D_x)^{n+1}(x^r u_n) + (-D_x)^n(x^r u_{n+1}). \end{aligned} \tag{3.142}$$

Substitution of (3.142) into the third relation of (3.141) yields

$$\begin{aligned} \{H_1^2,H_n^r\} = \int_{-\infty}^{\infty} &-D_x(2x^2v_1)\cdot(-1)^nD_x^n(2x^ru_n) \\ &+ D_x(2x^2u_1)\cdot(-1)^nD_x^n(2x^rv_n) \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{2n-1} \int_{-\infty}^{\infty} D_x^n(2x^2u_1)D_x(2x^r v_n) - D_x^n(2x^2v_1)D_x(2x^r u_n) \\
&= -4 \int_{-\infty}^{\infty} (x^2u_{n+1} + 2nxu_n + n(n-1)u_{n-1})(x^r v_{n+1} + rx^{r-1}v_n) \\
&\qquad - (x^2v_{n+1} + 2nxv_n + n(n-1)v_{n-1})(x^r u_{n+1} + rx^{r-1}u_n) \\
&= -4 \int_{-\infty}^{\infty} rx^{r+1}(u_{n+1}v_n - v_{n+1}u_n) - 2nx^{r+1}(u_{n+1}v_n - v_{n+1}u_n) \\
&\quad + n(n-1)x^r(v_{n+1}u_{n-1} - u_{n+1}v_{n-1}) + n(n-1)rx^{r-1}(v_nu_{n-1} - u_nv_{n-1}) \\
&= 4(2n-r)K_n^{r+1}, \qquad (3.143)
\end{aligned}
$$

which proves the third relation in (3.141).

The last equality in (3.143) results from the fact that the last two terms are just constituting a total derivative of

$$n(n-1)x^r(v_nu_{n-1} - u_nv_{n-1}). \qquad (3.144)$$

In order to prove the fourth relation in (3.141), we substitute (3.142), which leads to

$$
\begin{aligned}
\{H_1^2, H_n^r\} &= \int_{-\infty}^{\infty} -D_x(2x^2v_1)\cdot\Big((-1)^{n+1}D_x^{n+1}(x^rv_n) - (-1)^nD_x^n(x^rv_{n+1})\Big) \\
&\quad + D_x(2x^2u_1)\cdot\Big((-1)^{n+1}D_x^{n+1}(x^ru_n) - (-1)^nD_x^n(x^ru_{n+1})\Big). \qquad (3.145)
\end{aligned}
$$

Integration, n times, of the terms in brackets leads to

$$
\begin{aligned}
\{H_1^2, H_n^r\} &= 2\int_{-\infty}^{\infty} D_x^{n+1}(x^2v_1)\cdot(D_x(x^rv_n) + x^rv_{n+1}) \\
&\qquad + D_x^{n+1}(x^2u_1)\cdot(D_x(x^ru_n) + x^ru_{n+1}) \\
&= 2\int_{-\infty}^{\infty} (x^2v_{n+2} + 2(n+1)xv_{n+1} + n(n+1)v_n)(2x^rv_{n+1} + rx^{r-1}v_n) \\
&\quad + (x^2u_{n+2} + 2(n+1)xu_{n+1} + n(n+1)u_n)(2x^ru_{n+1} + rx^{r-1}u_n). \qquad (3.146)
\end{aligned}
$$

By expanding the expressions in (3.146), we arrive, after a short calculation, at

$$\{H_1^2, K_n^r\} = (2n+1-r)(4H_{n+1}^{r+1} - r^2H_n^{r-1}), \qquad (3.147)$$

which proves the fourth relation in (3.141). □

We are now in a position to formulate and prove the main theorem of this subsection.

THEOREM 3.13. *The conserved functionals $F(Y^{\pm}(2,0))$ associated to the symmetries $Y^{\pm}(2,0)$ generate infinite number of hierarchies of Hamiltonians, starting at the hierarchies $F(Y_i^+)$, $F(Y_i^-)$, where $i \in \mathbb{Z}$, by repeated action of the Poisson bracket (3.140). The hierarchies $F(Z_j^+)$, $F(Z_j^-)$, $j \in \mathbb{Z}$, are obtained by the first step of this procedure.*

Moreover, the hierarchies $F(Y_j^+)$, $F(Y_j^-)$, $j \in \mathbb{Z}$, *are obtained from* $F(Y_{\pm 1}^{\pm})$ *by repeated action of the conserved functionals* $F(Z_{\pm 1}^{\pm})$

$$F(Z_{\pm 1}^{\pm}) = \pm\frac{1}{2}F\big([Y^{\pm}(2,0), Y_{\pm 1}^{\pm}]\big). \tag{3.148}$$

PROOF. The proof of this theorem is a straightforward application of the previous lemma, and the observation that the (λ, m_1, m_2)-independent parts of the conserved densities $Y_{\pm 1}^{\pm}$, $Y^+(2,0)$, $Y^-(2,0)$ are just given by

$$\begin{aligned}
\mathcal{F}(Y_1^+) &\longrightarrow -\frac{1}{2}(u_{2x}v_2 - v_{2x}u_2),\\
\mathcal{F}(Y_{-1}^+) &\longrightarrow -\frac{1}{2}(u_{1x}v_1 - v_{1x}u_1),\\
\mathcal{F}(Y_1^-) &\longrightarrow -\frac{1}{2}(u_{4x}v_4 - v_{4x}u_4),\\
\mathcal{F}(Y_{-1}^-) &\longrightarrow -\frac{1}{2}(u_{3x}v_3 - v_{3x}u_3),
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(Y^+(2,0)) &\longrightarrow -\frac{1}{2}(x+t)^2(u_{2x}^2 + v_{2x}^2) - \frac{1}{2}(x-t)^2(u_{1x}^2 + v_{1x}^2),\\
\mathcal{F}(Y^-(2,0)) &\longrightarrow -\frac{1}{2}(x+t)^2(u_{4x}^2 + v_{4x}^2) - \frac{1}{2}(x-t)^2(u_{3x}^2 + v_{3x}^2).
\end{aligned}$$

Note that in applying the lemma we have to choose $(u, v) = (u_1, v_1)$, etc. □

7.6. Nonlocal symmetries. In this last subsection concerning the Federbush model, we discuss existence of nonlocal symmetries. We start from the conservation laws, conserved quantities and the associated nonlocal variables p_1, p_2:

$$\begin{aligned}
p_{1x} &= R_1 + R_2, & p_{1t} &= -R_1 + R_2,\\
p_{2x} &= R_3 + R_4, & p_{2t} &= -R_3 + R_4.
\end{aligned} \tag{3.149}$$

Including these two nonlocal variables, we find two new nonlocal symmetries

$$\begin{aligned}
Z^+(0,0) &= u_1\frac{\partial}{\partial u_1} + v_1\frac{\partial}{\partial v_1} + u_2\frac{\partial}{\partial u_2} + v_2\frac{\partial}{\partial v_2} - \lambda p_1\Big(v_3\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial v_3}\\
&\quad + v_4\frac{\partial}{\partial u_4} - u_4\frac{\partial}{\partial v_4}\Big) + 2p_1\frac{\partial}{\partial p_1},\\
Z^-(0,0) &= u_3\frac{\partial}{\partial u_3} + v_3\frac{\partial}{\partial v_3} + u_4\frac{\partial}{\partial u_4} + v_4\frac{\partial}{\partial v_4} + \lambda p_2\Big(v_1\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial v_1}\\
&\quad + v_2\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial v_2}\Big) + 2p_2\frac{\partial}{\partial p_2}.
\end{aligned} \tag{3.150}$$

Analogously to the construction of conservation laws and nonlocal variables in previous sections, we obtained nonlocal variables p_3, p_4, p_5, p_6 defined by

$$\begin{aligned}
p_{3x} &= \frac{1}{2}\lambda(R_1 + R_2)R_4 + m_2(u_3u_4 + v_3v_4) - u_4v_{4x} + v_4u_{4x},\\
p_{3t} &= \frac{1}{2}\lambda(R_1 + R_2)R_4 - u_4v_{4x} + v_4u_{4x},
\end{aligned}$$

$$\begin{aligned}
p_{4x} &= \frac{1}{2}\lambda(R_1+R_2)R_3 + m_2(u_3u_4+v_3v_4) + u_3v_{3x} - v_3u_{3x},\\
p_{4t} &= \frac{1}{2}\lambda(R_1+R_2)R_3 - u_3v_{3x} + v_3u_{3x},\\
p_{5x} &= \frac{1}{2}\lambda(R_3+R_4)R_2 - m_1(u_1u_2+v_1v_1) + u_2v_{2x} - v_2u_{2x},\\
p_{5t} &= \frac{1}{2}\lambda(R_3+R_4)R_2 + u_2v_{2x} - v_2u_{2x},\\
p_{6x} &= \frac{1}{2}\lambda(R_3+R_4)R_1 + m_1(u_1u_2+v_1v_1) + u_1v_{1x} - v_1u_{1x},\\
p_{6t} &= -\frac{1}{2}\lambda(R_3+R_4)R_1 - u_1v_{1x} + v_1u_{1x}.
\end{aligned} \tag{3.151}$$

Using these nonlocal variables we find four additional nonlocal symmetries $Z^+(0,-1)$, $Z^+(0,+1)$, $Z^-(0,-1)$, $Z^-(0,+1)$:

$$\begin{aligned}
Z^+(0,-1) &= \frac{1}{2}\Big(-\lambda u_1(R_3+R_4) - m_1u_2 - 2v_{1x}\Big)\frac{\partial}{\partial u_1}\\
&\quad + \frac{1}{2}\Big(-\lambda v_1(R_3+R_4) - m_1v_2 + 2u_{1x}\Big)\frac{\partial}{\partial v_1}\\
&\quad - \frac{1}{2}m_1u_1\frac{\partial}{\partial u_2} - \frac{1}{2}m_1v_1\frac{\partial}{\partial v_2}\\
&\quad + \lambda p_6\Big(v_3\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial v_3} + v_4\frac{\partial}{\partial u_4} - u_4\frac{\partial}{\partial v_4}\Big),\\
Z^+(0,+1) &= \frac{1}{2}m_1u_2\frac{\partial}{\partial u_1} + \frac{1}{2}m_1v_2\frac{\partial}{\partial v_1}\\
&\quad + \frac{1}{2}\Big(-\lambda u_2(R_3+R_4) + m_1u_1 - 2v_{2x}\Big)\frac{\partial}{\partial u_2}\\
&\quad + \frac{1}{2}\Big(-\lambda v_2(R_3+R_4) + m_1v_1 + 2u_{2x}\Big)\frac{\partial}{\partial v_2}\\
&\quad + \lambda p_5\Big(v_3\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial v_3} + v_4\frac{\partial}{\partial u_4} - u_4\frac{\partial}{\partial v_4}\Big),\\
Z^-(0,-1) &= -\lambda p_4(v_1\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial v_2})\\
&\quad + \frac{1}{2}\Big(-\lambda u_3(R_1+R_2) - m_2u_4 - 2v_{3x}\Big)\frac{\partial}{\partial u_3}\\
&\quad + \frac{1}{2}\Big(+\lambda v_3(R_1+R_2) - m_2v_4 + 2u_{3x}\Big)\frac{\partial}{\partial v_3}\\
&\quad - \frac{1}{2}m_2u_3\frac{\partial}{\partial u_4} - \frac{1}{2}m_2v_3\frac{\partial}{\partial v_4},\\
Z^-(0,+1) &= \lambda p_3\Big(v_1\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial v_2}\Big)\\
&\quad + \frac{1}{2}m_2u_4\frac{\partial}{\partial u_3} + \frac{1}{2}m_2v_4\frac{\partial}{\partial v_3}
\end{aligned}$$

$$+\frac{1}{2}\Big(\lambda u_4(R_1+R_2)+m_2u_3-2v_{4x}\Big)\frac{\partial}{\partial u_4}$$
$$+\frac{1}{2}\Big(\lambda v_4(R_1+R_2)+m_2v_3+2u_{4x}\Big)\frac{\partial}{\partial v_4}.$$

According to standard lines of computations, including prolongation towards nonlocal variables as explained in previous sections, we arrive at the following commutators:

$$
\begin{aligned}
[Y^{\pm}(1,\pm1),Z^{\pm}(0,0)]&=0,\\
[Y^{+}(1,-1),Z^{+}(0,-1)]&=Z^{+}(0,-2),\\
[Y^{+}(1,-1),Z^{+}(0,+1)]&=-\frac{1}{4}m_1^2Z^{+}(0,0),\\
[Y^{+}(1,+1),Z^{+}(0,-1)]&=\frac{1}{4}m_1^2Z^{+}(0,0),\\
[Y^{+}(1,+1),Z^{+}(0,+1)]&=Z^{+}(0,+2)
\end{aligned}
$$

and

$$
\begin{aligned}
[Y^{-}(1,-1),Z^{-}(0,-1)]&=Z^{-}(0,-2),\\
[Y^{-}(1,-1),Z^{-}(0,+1)]&=-\frac{1}{4}m_2^2Z^{-}(0,0),\\
[Y^{-}(1,+1),Z^{-}(0,-1)]&=\frac{1}{4}m_2^2Z^{-}(0,0),\\
[Y^{-}(1,+1),Z^{-}(0,+1)]&=Z^{-}(0,+2),
\end{aligned}
\tag{3.152}
$$

the vector fields $Z^{+}(0,-2)$, $Z^{+}(0,+2)$, $Z^{-}(0,-2)$, $Z^{-}(0,+2)$ just being new nonlocal symmetries.

Summarising these results, we conclude that the action of the symmetries $Y^{\pm}(1,\pm1)$ on $Z^{\pm}(0,\pm1)$ constitute hierarchies of nonlocal symmetries.

Finally we compute the Lie brackets of $Y^{+}(2,0)$, (3.121), and $Z^{+}(0,\pm1)$ which results in

$$
\begin{aligned}
[Y^{+}(2,0),Z^{+}(0,-1)]&=Z^{+}(1,-1),\\
[Y^{+}(2,0),Z^{+}(0,+1)]&=Z^{+}(1,+1),
\end{aligned}
\tag{3.153}
$$

whereas in (3.153) $Z^{+}(0,\pm1)$ are defined by

$$
\begin{aligned}
Z^{+}(1,-1)&=2(-x+t)Z^{+}(0,-2)+\frac{1}{2}m_1^2(x+t)Z^{+}(0,0)\\
&\quad+\Big(\lambda v_1R_{34}+m_1v_2-2u_{1x}\Big)\frac{\partial}{\partial u_1}-\Big(\lambda u_1R_{34}+m_1u_2+2v_{1x}\Big)\frac{\partial}{\partial v_1}\\
&\quad-\frac{\lambda}{2}\Big(v_3\frac{\partial}{\partial u_3}-u_3\frac{\partial}{\partial v_3}+v_4\frac{\partial}{\partial u_4}-u_4\frac{\partial}{\partial v_4}\Big)K^{+}_{-1},\\
Z^{+}(1,+1)&=2(x+t)Z^{+}(0,-2)+\frac{1}{2}m_1^2(x-t)Z^{+}(0,0)\\
&\quad+\Big(\lambda v_2R_{34}-m_1v_1-2u_{2x}\Big)\frac{\partial}{\partial u_2}
\end{aligned}
$$

$$
\begin{aligned}
&- \Big(\lambda u_2 R_{34} + m_1 u_1 + 2v_{2x}\Big)\frac{\partial}{\partial v_2} \\
&- \frac{\lambda}{2}\Big(v_3 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial v_3} + v_4 \frac{\partial}{\partial u_4} - u_4 \frac{\partial}{\partial v_4}\Big) K^+_{+1},
\end{aligned} \tag{3.154}
$$

while $K^+_{\pm 1}$ are given by

$$
\begin{aligned}
K^+_{-1} &= 8 \int_{-\infty}^{x} \int_{-\infty}^{x} \mathcal{F}(Y^+(0,-2)) - m_1^2 \int_{-\infty}^{x} \int_{-\infty}^{x} \mathcal{F}(Y^+(0,0)), \\
K^+_{+1} &= 8 \int_{-\infty}^{x} \int_{-\infty}^{x} \mathcal{F}(Y^+(0,+2)) - m_1^2 \int_{-\infty}^{x} \int_{-\infty}^{x} \mathcal{F}(Y^+(0,0)).
\end{aligned} \tag{3.155}
$$

The previous formulas reflect the fact that $Y^+(2,0)$ constructs an (x,t)-dependent hierarchy $Z^+(1,*)$ from $Z^+(0,*)$ by action of the Lie bracket. We expect similar results for the action of $Y^+(2,0)$ on the hierarchy $Z^+(1,*)$.

Results conserning the action of $Y^-(2,0)$ on $Z^-(0,*)$ and from this, on $Z^-(1,*)$ will be similar.

8. Bäcklund transformations and recursion operators

In this section, we mainly follow the results by M. Marvan exposed in [**73**]. Our aim here is to show that recursion opeartors for higher symmetries may be unberstood as Bäcklund transformations of a special type.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two differential equations in unknown functions u^1 and u^2 respectively. Informally speaking, a Bäcklund transformation between $\mathcal{E}_1$ and $\mathcal{E}_2$ is a third equation $\mathcal{E}$ containing both independent variables u^1 and u^2 and possessing the following property:

1. If u^1_0 is a solution of $\mathcal{E}_1$, then solving the equation $\mathcal{E}[u^1_0]$ with respect to u^2, we obtain a family of solutions to $\mathcal{E}_2$.
2. Vice versa, if u^2_0 is a solution of $\mathcal{E}_2$, then solving the equation $\mathcal{E}[u^2_0]$ with respect to u^1, we obtain a family of solutions to $\mathcal{E}_1$.

Geometrically this construction is expressed in a quite simple manner.

DEFINITION 3.10. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be objects of the category $\mathcal{DM}^\infty$. A *Bäklund transformation* between $\mathcal{N}_1$ and $\mathcal{N}_2$ is a pair of coverings

$$
\begin{tikzcd}
 & \mathcal{N} \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\
\mathcal{N}_1 & & \mathcal{N}_2
\end{tikzcd}
$$

where $\mathcal{N}$ is a third object of $\mathcal{DM}^\infty$. A Bäcklund transformation is called a *Bäcklund auto-transformation*, if $\mathcal{N}_1 = \mathcal{N}_2$.

In fact, let $\mathcal{N}_i = \mathcal{E}_i^\infty$, $i = 1, 2$, and $s \subset \mathcal{E}_1^\infty$ be a solution. Then the set $\varphi_1^{-1} s \subset \mathcal{N}$ is fibered by solutions of $\mathcal{N}$ and they are projected by φ_2 (at nonsingular points) to a family of solutions of $\mathcal{E}_2^\infty$.

We are now interested in Bäcklund auto-transformations of the total space of the Cartan covering $\tau^v : V\mathcal{E}^\infty \to \mathcal{E}^\infty$ (see Example 3.2). The reason to this is the following

PROPOSITION 3.14. *A section $X : \mathcal{E}^\infty \to V\mathcal{E}^\infty$ of the projection τ^v is a symmetry of the equation $\mathcal{E}$ if and only if it is a morphism in the category $\mathcal{DM}^\infty$, i.e., if it preserves Cartan distributions.*

The proof is straightforward and is based on the definition of the Cartan distribution on $V\mathcal{E}^\infty$. The result is in full agreement with equalities (3.2) on p. 101: the equations for $V\mathcal{E}^\infty$ are just linearization of $\mathcal{E}$ and symmetries are solutions of the linearized equation.

Thus, we can hope that Bäcklund auto-transformations of $V\mathcal{E}^\infty$ will relate symmetries of $\mathcal{E}$ to each other. This motivates the following

DEFINITION 3.11. Let $\mathcal{E}^\infty$ be an infinitely prolonged equation. A *recursion operator* for symmetries of $\mathcal{E}$ is a pair of coverings $K, L : \mathcal{R} \to V\mathcal{E}^\infty$ such that the diagram

$$\begin{tikzcd} & \mathcal{R} \arrow[dl, "K"'] \arrow[dr, "L"] & \\ V\mathcal{E}^\infty \arrow[dr, "\tau^v"'] & & V\mathcal{E}^\infty \arrow[dl, "\tau^v"] \\ & \mathcal{E}^\infty & \end{tikzcd}$$

is commutative. A recursion operator is called *linear*, if both K and L are linear coverings.

EXAMPLE 3.4. Consider the KdV equation $\mathcal{E} = \{u_t = uu_x + u_{xxx}\}$. Then $V\mathcal{E}^\infty$ is described by additional equation

$$v_t = uv_x + u_x v + v_{xxx}.$$

Let us take for $\mathcal{R}$ the system of equations

$$\begin{aligned} w_x &= v, \\ w_t &= v_{xx} + uv, \\ v_t &= v_{xxx} uv_x + u_x v, \\ u_t &= u_{xxx} + uu_x, \end{aligned}$$

while the mappings K and L are given by

$$\begin{aligned} K &: v = w_x, \\ L &: v = v_{xx} + \frac{2}{3}uv + \frac{1}{3}u_x w. \end{aligned}$$

Obviously, K and L determine covering structures over $V\mathcal{E}^\infty$ (the first being one-dimensional and the second three-dimensional) while the triple $(\mathcal{R}, K, L)$ corresponds to the classical Lenard operator $D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$.

Let us now study action of recursion operators on symmetries in more details. Let X be a symmetry of an equation $\mathcal{E}$. Then, due to Proposition 3.14, it can be considered as a section $X\colon \mathcal{E}^\infty \to V\mathcal{E}^\infty$ which is a morphism in $\mathcal{DM}^\infty$. Thus we obtain the following commutative diagram

$$\begin{tikzcd}
\mathcal{R}^* \arrow[r, "X^*"] \arrow[d, "P = X^*(K)"'] & \mathcal{R} \arrow[r, "L"] \arrow[d, "K"'] & V\mathcal{E}^\infty \arrow[d, "\tau^v"'] \\
\mathcal{E}^\infty \arrow[r, "X"] & V\mathcal{E}^\infty \arrow[r, "\tau^v"] & \mathcal{E}^\infty
\end{tikzcd}$$

where the composition of the arrows below is the identity while $P = X^*(K)$ is the pull-back. As a consequence, we obtain the following morphism of coverings

$$\begin{tikzcd}
\mathcal{R}^* \arrow[rr, "L \circ X^*"] \arrow[dr, "P"'] & & V\mathcal{E}^\infty \arrow[dl, "\tau^v"] \\
& \mathcal{E}^\infty &
\end{tikzcd}$$

But a morphism of this type, as it can be easily checked, is exactly a *shadow* of a nonlocal symmetry in the covering P (cf. Section 2). And as we know, action of the Lenard operator on the scaling symmetry of the KdV equation results in a shadow which can be reconstructed using the methods of Section 3.

We conclude this section with discussing the problem of inversion of recursion operators. This nontrivial, from analytical point of view, procedure, becomes quite trivial in the geometrical setting.

In fact, to invert a recursion operator $(\mathcal{R}, K, L)$ just amounts to changing arrows in the corresponding diagram:

$$\begin{tikzcd}
& \mathcal{R} \arrow[dl, "K' = L"'] \arrow[dr, "L' = K"] & \\
V\mathcal{E}^\infty \arrow[dr, "\tau^v"'] & & V\mathcal{E}^\infty \arrow[dl, "\tau^v"] \\
& \mathcal{E}^\infty &
\end{tikzcd}$$

We shall illustrate the procedure using the example of the modified KdV equation (mKdV).

EXAMPLE 3.5 (see also [**28, 27, 29**]). Consider the mKdV eqiation written in the form

$$u_t = u_{xxx} - u^2 u_x.$$

Then the corresponding Cartan covering is given by the pair of equations

$$\begin{aligned} u_t &= u_{xxx} - u^2 u_x, \\ v_t &= v_{xxx} - u^2 v_x - 2uu_x v, \end{aligned}$$

while the recursion operator for the mKdV equation comes out of the covering $\mathcal{R}$ of the form

$$\begin{aligned} w_x &= uv, \\ w_t &= uv_x x - u_x v_x + u_{xx} v - u^3 v \end{aligned}$$

and is of the form $L\colon z = v_{xx} - \frac{2}{3}u^2 v - \frac{2}{3}u_x w$, where z stands for the nonlocal coordinate in the second copy of $V\mathcal{E}^\infty$.

To invert L, it needs to reconstruct the covering over the second copy of $V\mathcal{E}^\infty$ using the above information. From the form of L we obtain $v_{xx} = z + \frac{2}{3}u^2 v + \frac{2}{3}u_x w$, from where it follows that the needed nonlocal variables are v, w, and s satisfying the relations

$$\begin{aligned} w_x &= uv, \\ v_x &= s, \\ s_x &= \frac{2}{3}u_x w + \frac{2}{3}u^2 v + z \end{aligned}$$

and

$$\begin{aligned} w_t &= \frac{2}{3}uu_x w + \left(u_{xx} - \frac{1}{3}u^3\right) z - u_x s + uz, \\ v_t &= \frac{2}{3}u_{xx} w - \frac{1}{3}u^2 s + z_x, \\ s_t &= \left(\frac{2}{3}u_{xxx} - \frac{2}{9}u^2 u_x\right) w + \left(\frac{2}{3}uu_{xx} - \frac{2}{9}u^4\right) v - \frac{2}{3}uu_x s + z_x - \frac{1}{3}u^2 z. \end{aligned}$$

Consequently, we got the covering $L'\colon \mathcal{R}' \to V\mathcal{E}^\infty$ with $(w, v, s) \mapsto v$, and it is natural to identify the triple $(\mathcal{R}' = \mathcal{R}, L' = K, K' = L)$ with the inverted recursion operator.

It should be noted that the covering $\mathcal{R}'$ can be simplified in the following way: set

$$p^+ = w + \sqrt{\frac{3}{2}}, \quad p^- = w - \sqrt{\frac{3}{2}}, \quad q = -\frac{2}{3}uw + s.$$

Then we get

$$
\begin{aligned}
p_x^{\pm} &= \pm\sqrt{\frac{2}{3}}up^{\pm} \pm \sqrt{\frac{3}{2}}q, \\
q_x &= z, \\
p_t^{\pm} &= \pm\sqrt{\frac{2}{3}}\left(u_{xx} - \frac{1}{3}u^3\right)p^{\pm} - \left(u_x \pm \frac{\sqrt{6}}{6}u^2\right)q \pm \sqrt{\frac{3}{2}}z_x + uz, \\
q_t &= z_{xx} - u^2 z,
\end{aligned}
$$

while K acquires the form $v = p^+ - p^-$.

CHAPTER 4

Brackets

This chapter is of a purely algebraic nature. Following [**99**] (see also [**60**, Ch. 1]), we construct differential calculus in the category of modules over a unitary commutative K-algebra A, K being a commutative ring with unit (in the corresponding geometrical setting K is usually the field $\mathbb{R}$ and $A = C^\infty(M)$ for a smooth manifold M). Properly understood, this calculus is a system of special functors, together with their natural transformations and representative objects.

In the framework of the calculus constructed, we study form-valued derivations and deduce, in particular, two types of brackets: the Richardson–Nijenhuis and Frölicher–Nijenhuis ones. If a derivation is *integrable* in the sense of the second one, a cohomology theory can be related to it. A source of integrable elements are algebras with flat connections.

These algebras serve as an adequate model for infinitely prolonged differential equations, and we shall also show that all basic conceptual constructions introduced on $\mathcal{E}^\infty$ in previous chapters are also valid for algebras with flat connections, becoming much more transparent. In particular, the notions of a symmetry and a recursion operator for an algebra with flat connection are introduced in cohomological terms and the structure of symmetry Lie algebras is analyzed. Later (in Chapter 5) we specify all these results for the case of the bundle $\mathcal{E}^\infty \to M$.

1. Differential calculus over commutative algebras

Throughout this section, K is a commutative ring with unit, A is a commutative K-algebra, $P, Q, \dots$ are modules over A. We introduce linear differential operators $\Delta\colon P \to Q$, modules of jets $\mathcal{J}^k(P)$, derivations, and differential forms $\Lambda^i(A)$.

1.1. Linear differential operators. Consider two A-modules P and Q and the K-module $\hom_K(P, Q)$. Then there exist two A-module structures in $\hom_K(P, Q)$: the left one

$$(\mathrm{l}_a f)(p) = a f(p), \qquad a \in A, \quad f \in \hom_K(P, Q), \quad p \in P,$$

and the right one

$$(\mathrm{r}_a f)(p) = f(ap), \qquad a \in A, \quad f \in \hom_K(P, Q), \quad p \in P.$$

Let us introduce the notation $\delta_a = \mathrm{l}_a - \mathrm{r}_a$.

DEFINITION 4.1. A *linear differential operator* of order $\leq k$ acting from an A-module P to an A-module Q is a mapping $\Delta \in \hom_K(P, Q)$ satisfying the identity

$$(\delta_{a_0} \circ \cdots \circ \delta_{a_k})\Delta = 0 \tag{4.1}$$

for all $a_0, \dots a_k \in A$.

For any $a, b \in A$, one has

$$\mathrm{l}_a \circ \mathrm{r}_b = \mathrm{r}_b \circ \mathrm{l}_a$$

and consequently the set of all differential operators of order $\leq k$

(i) is stable under both left and right multiplication and
(ii) forms an A-bimodule.

This bimodule is denoted by $\mathrm{Diff}_k^{(+)}(P, Q)$, while the left and the right multiplications in it are denoted by $a\Delta$ and $a^+\Delta$ respectively, $a \in A$, $\Delta \in \mathrm{Diff}_k^{(+)}(P, Q)$. When $P = A$, we use the notation $\mathrm{Diff}_k^{(+)}(Q)$.

Obviously, one has embeddings of A-bimodules

$$\mathrm{Diff}_k^{(+)}(P, Q) \hookrightarrow \mathrm{Diff}_{k'}^{(+)}(P, Q)$$

for any $k \leq k'$ and we can define the module

$$\mathrm{Diff}_*^{(+)}(P, Q) \stackrel{\mathrm{def}}{=} \bigcup_{k \geq 0} \mathrm{Diff}_k^{(+)}(P, Q).$$

Note also that for $k = 0$ we have $\mathrm{Diff}_0^{(+)}(P, Q) = \hom_A(P, Q)$.

Let P, Q, R be A-modules and $\Delta\colon P \to Q$, $\Delta'\colon Q \to R$ be differential operators of orders k and k' respectively. Then the composition $\Delta' \circ \Delta\colon P \to R$ is defined.

PROPOSITION 4.1. *The composition $\Delta' \circ \Delta$ is a differential operator of order $\leq k + k'$.*

PROOF. In fact, by definition we have

$$\delta_a(\Delta' \circ \Delta) = \delta_a(\Delta') \circ \Delta + \Delta' \circ \delta_a(\Delta). \tag{4.2}$$

for any $a \in A$. Let $\mathfrak{a} = \{a_0, \dots, a_s\}$ be a set of elements of the algebra A. Say that two subsets $\mathfrak{a}_r = \{a_{i_1}, \dots, a_{i_r}\}$ and $\mathfrak{a}_{s-r+1} = \{a_{j_1}, \dots, a_{j_{s-r+1}}\}$ form an *unshuffle* of $\mathfrak{a}$, if $i_1 < \cdots < i_r$, $j_1 < \cdots < j_{s-r+1}$. Denote the set of all unshuffles of $\mathfrak{a}$ by $\mathrm{unshuffle}(\mathfrak{a})$ and set $\delta_{\mathfrak{a}} \stackrel{\mathrm{def}}{=} \delta_{a_0} \circ \cdots \circ \delta_{a_s}$. Then from (4.2) it follows that

$$\delta_{\mathfrak{a}}(\Delta \circ \Delta') = \sum_{(\mathfrak{a}_r, \mathfrak{a}_{s-r+1}) \in \mathrm{unshuffle}(\mathfrak{a})} \delta_{\mathfrak{a}_r}(\Delta) \circ \delta_{\mathfrak{a}_{s-r+1}}(\Delta') \tag{4.3}$$

for any Δ, Δ'. Hence, if $s \geq k + k' + 1$, both summands in (4.3) vanish which finishes the proof. $\square$

REMARK 4.1. Let M be a smooth manifold, π, ξ be vector bundles over M and $P = \Gamma(\pi)$, $Q = \Gamma(\xi)$. Then Δ is a differential operator in the sense of Definition 4.1 if and only if it is a linear differential operator acting from sections of π to those of ξ.

First note that it suffices to consider the case $M = \mathbb{R}^n$, π and ξ being trivial one-dimensional bundles over M. Obviously, any linear differential operator in a usual analytical sense satisfies Definition 4.1. Conversely, let $\Delta\colon C^\infty(M) \to C^\infty(M)$ satisfy Definition 4.1 and be an operator of order k. Consider a function $f \in C^\infty(M)$ and a point $x^0 \in M$. Then in a neighborhood of x^0 the function f is represented in the form

$$f(x) = \sum_{|\sigma|\le k} \frac{(x-x^0)^\sigma}{\sigma!} \left.\frac{\partial^{|\sigma|} f}{\partial x^{|\sigma|}}\right|_{x=x^0} + \sum_{|\sigma|=k+1} (x-x^0)^\sigma g_\sigma(x),$$

where $(x-x^0)^\sigma = (x_1 - x_1^0)^{i_1} \dots (x_n - x_n^0)^{i_n}$, $\sigma! = i_1! \dots i_n!$, and g_σ are some smooth functions. Introduce the notation

$$\Delta_\sigma = \Delta\left(\frac{(x-x^0)^\sigma}{\sigma!}\right);$$

then

$$\Delta(f) = \sum_{|\sigma|\le k} \Delta_\sigma \left.\frac{\partial^{|\sigma|} f}{\partial x^{|\sigma|}}\right|_{x=x^0} + \Delta\left(\sum_{|\sigma|=k+1} (x-x^0)^\sigma g_\sigma(x)\right). \tag{4.4}$$

Due to the fact that Δ is a k-th order operator, from equality (4.3) it follows that the last summand in (4.4) vanishes. Hence, Δf is completely determined by the values of partial derivatives of f up to order k and depends on these derivatives linearly.

Consider a differential operator $\Delta\colon P \to Q$ and A-module homomorphisms $f\colon Q \to R$ and $f'\colon R' \to P$. Then from Definition 4.1 it follows that both $f \circ \Delta\colon P \to R$ and $\Delta \circ f'\colon R' \to Q$ are differential operators of order $\operatorname{ord}\Delta$. Thus the correspondence $(P, Q) \to \operatorname{Diff}_k^{(+)}(P, Q)$, $k = 0, 1, \dots, *$, is a bifunctor from the category of A-modules to the category of A-bimodules.

PROPOSITION 4.2. *Let us fix a module Q. Then the functor* $\operatorname{Diff}_k^+(\bullet, Q)$ *is representable in the category of A-modules. Moreover, for any differential operator $\Delta\colon P \to Q$ of order k there exists a unique homomorphism* $f_\Delta\colon P \to \operatorname{Diff}_k^+(Q)$ *such that the diagram*

$$\begin{tikzcd}
P \arrow[rr, "\Delta"] \arrow[dr, "f_\Delta"'] & & Q \arrow[dl, "\mathcal{D}_k"] \\
& \operatorname{Diff}_k^+(Q) &
\end{tikzcd} \tag{4.5}$$

is commutative, where the operator $\mathcal{Д}_k$ *is defined by* $\mathcal{Д}_k(\square) \stackrel{\text{def}}{=} \square(1)$, $\square \in \mathrm{Diff}_k^+(Q)$.

PROOF. Let $p \in P, a \in A$ and set $(f_\Delta(p))(a) \stackrel{\text{def}}{=} \Delta(ap)$. It is easily seen that it is the mapping we are looking for. $\square$

DEFINITION 4.2. Let $\Delta\colon P \to Q$ be a k-th order differential operator. The composition $\Delta_{(l)} \stackrel{\text{def}}{=} \mathcal{Д}_l \circ \Delta\colon P \to \mathrm{Diff}_l^+(Q)$ is called the *l-th* Diff-*prolongation* of Δ.

Consider, in particular, the l-th prolongation of the operator $\mathcal{Д}_k$. By definition, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Diff}_{l,k}^+(P) \arrow[r, "\mathcal{Д}_l"] \arrow[d, "c_{l,k}"'] \arrow[dr, "(\mathcal{Д}_k)_{(l)}"] & \mathrm{Diff}_k^+(P) \arrow[d, "\mathcal{Д}_k"] \\
\mathrm{Diff}_{l+k}^+(P) \arrow[r, "\mathcal{Д}_{k+l}"] & P
\end{tikzcd}$$

where $\mathrm{Diff}_{i_1,\dots,i_n}^+ \stackrel{\text{def}}{=} \mathrm{Diff}_{i_1}^+ \circ \cdots \circ \mathrm{Diff}_{i_n}^+$ and $c_{l,k} \stackrel{\text{def}}{=} f_{\mathcal{Д}_k \circ \mathcal{Д}_l}$. The mapping $c_{l,k} = c_{l,k}(P)\colon \mathrm{Diff}_{l,k}(P) \to \mathrm{Diff}_{l+k}(P)$ is called the *gluing homomorphism* while the correspondence $P \Rightarrow c_{l,k}(P)$ is a natural transformation of functors called the *gluing transformation.*

Let $\Delta\colon P \to Q$, $\square\colon Q \to R$ be differential operators of orders k and l respectively. The A-module homomorphisms

$$f\Delta\colon P \to \mathrm{Diff}_k^+(Q), \quad f_{\square\circ\Delta}\colon P \to \mathrm{Diff}_{k+l}^+(R), \quad f_\square\colon Q \to \mathrm{Diff}_l^+(R)$$

are defined. On the other hand, since $\mathrm{Diff}_k^+(\bullet)$ is a functor, we have the homomorphism $\mathrm{Diff}_k^+(f_\square)\colon \mathrm{Diff}_k^+(Q) \to \mathrm{Diff}_k^+(\mathrm{Diff}_l^+(R))$.

PROPOSITION 4.3. *The diagram*

$$\begin{tikzcd}
P \arrow[r, "f_{\square\circ\Delta}"] \arrow[d, "f_\Delta"] & \mathrm{Diff}_{k+l}^+(R) \\
\mathrm{Diff}_k^+(Q) \arrow[r, "\mathrm{Diff}_k^+(f_\square)"] & \mathrm{Diff}_{k,l}^+(R) \arrow[u, "c_{k,l}"']
\end{tikzcd} \tag{4.6}$$

is commutative.

By this reason, the transformation $c_{k,l}$ is also called the *universal composition transformation.*

1.2. Jets. Let us now study representability of the functors $\mathrm{Diff}_k(P, \bullet)$.

Consider an A-module P and the tensor product $A \otimes_K P$ endowed with two A-module structures

$$\mathrm{l}^a(b \otimes p) = (ab) \otimes p, \quad \mathrm{r}^a(b \otimes p) = b \otimes (ap), \quad a, b \in A, \quad p \in P.$$

We also set $\delta^a = \mathrm{l}^a - \mathrm{r}^a$ and denote by μ_k the submodule[1] in $A \otimes_K P$ spanned by all elements of the form

$$(\delta^{a_0} \circ \cdots \circ \delta^{a_s})(a \otimes p), \quad a_0, \dots, a_s \in A, \quad s \geq k.$$

DEFINITION 4.3. The module $\mathcal{J}^k(P) \overset{\text{def}}{=} (A \otimes_K P)/\mu_k$ is called the *module of k-jets* for the module P. The correspondence

$$j_k \colon P \to \mathcal{J}^k(P), \qquad p \mapsto (1 \otimes p) \bmod \mu_k,$$

is called the *k-jet operator*.

PROPOSITION 4.4. *The mapping j_k is a linear differential operator of order $\leq k$. Moreover, for any linear differential operator $\Delta \colon P \to Q$ there exists a uniquely defined homomorphism $f^\Delta \colon \mathcal{J}^k(P) \to Q$ such that the diagram*

$$\begin{tikzcd} P \arrow[rr, "j_k"] \arrow[dr, "\Delta"'] & & \mathcal{J}^k(P) \arrow[dl, "f^\Delta"] \\ & Q & \end{tikzcd}$$

is commutative.

Hence, $\mathrm{Diff}_k(P, \bullet)$ is a representable functor. Note also that $\mathcal{J}^k(P)$ carries two structures of an A-module (with respect to l^a and r^a) and the correspondence $P \Rightarrow \mathcal{J}^k(P)$ is a functor from the category of A-modules to the category of A-bimodules.

Note that by definition we have short exact sequences of A-modules

$$0 \to \mu_{k+1}/\mu_k \to \mathcal{J}^{k+1}(P) \xrightarrow{\nu_{k+1,k}} \mathcal{J}^k(P) \to 0$$

and thus we are able to define the A-module

$$\mathcal{J}^\infty(P) \overset{\text{def}}{=} \operatorname*{proj\,lim}_{\{\nu_{k+1,k}\}} \mathcal{J}^k(P)$$

which is called the *module of infinite jets* for P. Denote by $\nu_{\infty,k} \colon \mathcal{J}^\infty(P) \to \mathcal{J}^k(P)$ the corresponding projections. Since $\nu_{k+1,k} \circ j_k = j_{k+1}$ for any $k \geq 0$, the system of operators j_k induces the mapping $j_\infty \colon P \to \mathcal{J}^\infty(P)$ satisfying the condition $\nu_{\infty,k} \circ j_\infty = j_k$. Obviously, $\mathcal{J}^\infty(P)$ is the representative object for the functor $\mathrm{Diff}_*(P, \bullet)$ while the mapping j_∞ possesses the universal property similar to that of j_k: for any $\Delta \in \mathrm{Diff}_*(P, Q)$ there exists a unique

[1] It makes no difference whether we span μ_k by the left or the right multiplication due to the identity $\mathrm{l}^{a'} \delta^a(b \otimes p) = \mathrm{r}^{a'} \delta^a(b \otimes p) + \delta^{a'} \delta^a(b \otimes p)$.

homomorphism $f^{\Delta}\colon \mathcal{J}^{\infty}(P) \to Q$ such that $\Delta = f^{\Delta} \circ j_{\infty}$. Note that j_{∞} is not a differential operator in the sense of Definition 4.1.[2]

The functors $\mathcal{J}^k(\bullet)$ possess the properties dual to those of $\mathrm{Diff}_k^+(\bullet)$. Namely, we can define the *l-th* Jet-*prolongation* of $\Delta \in \mathrm{Diff}_k(P,Q)$ by setting

$$\Delta^{(l)} \stackrel{\text{def}}{=} j_l \circ \Delta\colon P \to \mathcal{J}^l(Q)$$

and consider the commutative diagram

$$\begin{tikzcd} P \arrow[r, "j_k"] \arrow[d, "j_{k+l}"'] \arrow[dr, "j_k^{(l)}"] & \mathcal{J}^k(P) \arrow[d, "j_l"] \\ \mathcal{J}^{k+l}(P) \arrow[r, "c^{l,k}"] & \mathcal{J}^l\mathcal{J}^k(P) \end{tikzcd}$$

where $c^{l,k} = f^{j_k^{(l)}}$ is called the *cogluing transformation.* Similar to Diagram (4.6), for any operators $\Delta\colon P \to Q$, $\Box\colon Q \to R$ of orders k and l respectively, we have the commutative diagram

$$\begin{tikzcd} \mathcal{J}^{k+l}(P) \arrow[r, "f^{\Box\circ\Delta}"] \arrow[d, "c^{l,k}"'] & R \\ \mathcal{J}^l\mathcal{J}^k(P) \arrow[r, "\mathcal{J}^l(f^{\Delta})"] & \mathcal{J}^l(Q) \arrow[u, "f^{\Box}"'] \end{tikzcd}$$

and call $c^{l,k}$ the *universal cocompositon operation.* This operation is *coassociative*, i.e., the diagram

$$\begin{tikzcd} \mathcal{J}^{k+l+s}(P) \arrow[r, "c^{k+l,s}"] \arrow[d, "c^{k,l+s}"'] & \mathcal{J}^{k+l}\mathcal{J}^s(P) \arrow[d, "c^{k,l}"] \\ \mathcal{J}^k\mathcal{J}^{l+s}(P) \arrow[r, "\mathcal{J}^k(c^{l,s})"] & \mathcal{J}^k\mathcal{J}^l\mathcal{J}^s(P) \end{tikzcd}$$

is commutative for all $k, l, s \geq 0$.

1.3. Derivations. We shall now deal with special differential operators of order 1.

DEFINITION 4.4. Let P be an A-module. A *P-valued derivation* is a first order operator $\Delta\colon A \to P$ satisfying $\Delta(1) = 0$.

[2]One might say that j_{∞} is a differential operator of "infinite order", but this concept needs to be more clarified. Some remarks concerning a concept of infinite order differential operators were made in Chapter 1, see also [**51**] for more details.

The set of such derivations will be denoted by $D(P)$. From the above definition and from Definition 4.1 it follows that $\Delta \in D(P)$ if and only if

$$\Delta(ab) = a\Delta(b) + b\Delta(a), \qquad a, b \in A. \tag{4.7}$$

It should be noted that the set $D(P)$ is a submodule in $\mathrm{Diff}_1(P)$ but not in $\mathrm{Diff}_1^+(P)$.

REMARK 4.2. In the case $A = C^\infty(M)$, M being a smooth manifold, and $P = A$ the module $D(A)$ coincides with the module $D(M)$ of vector fields on the manifold M.

For any A-homomorphism $f\colon P \to Q$ and a derivation $\Delta \in D(P)$, the composition $D(f) \stackrel{\text{def}}{=} f \circ \Delta$ lies in $D(Q)$ and thus $P \Rightarrow D(P)$ is a functor from the category of A-modules into itself. This functor can be generalized as follows.

Let P be an A-module and $N \subset P$ be a subset in P. Let us define

$$D(N) \stackrel{\text{def}}{=} \{\Delta \in D(P) \mid \Delta(A) \subset N\}.$$

Let us also set $(\mathrm{Diff}_1^+)^i \stackrel{\text{def}}{=} \mathrm{Diff}_1^+ \circ \cdots \circ \mathrm{Diff}_1^+$, where the composition is taken i times. We now define a series of functors D_i, $i \geq 0$, together with natural embeddings $D_i(P) \hookrightarrow (\mathrm{Diff}_1^+)^i(P)$ by setting $D_0(P) = P$, $D_1(P) = D(P)$ and, assuming that all $D_j(P)$, $j < i$, were defined,

$$D_i(P) = D(D_{i-1}(P) \subset (\mathrm{Diff}_1^+)^{i-1}(P)).$$

Since

$$D(D_{i-1}(P) \subset (\mathrm{Diff}_1^+)^{i-1}(P)) \subset D((\mathrm{Diff}_1^+)^{i-1}(P)) \subset (\mathrm{Diff}_1^+)^i(P), \tag{4.8}$$

the modules $D_i(P)$ are well defined.

Let us show now that the correspondences $P \Rightarrow D_i(P)$ are functors for all $i \geq 0$. In fact, the case $i = 0$ is obvious while $i = 1$ was considered above. We use induction on i and assume that $i > 1$ and that for $j < i$ all D_j are functors. We shall also assume that the embeddings $\alpha_P^j\colon D_j(P) \hookrightarrow (\mathrm{Diff}_1^+)^i(P)$ are natural, i.e., the diagrams

$$\begin{array}{ccc} D_j(P) & \xrightarrow{\alpha_P^j} & (\mathrm{Diff}_1^+)^j(P) \\ {\scriptstyle D_j(f)}\downarrow & & \downarrow{\scriptstyle (\mathrm{Diff}_1^+)^j(f)} \\ D_j(Q) & \xrightarrow{\alpha_Q^j} & (\mathrm{Diff}_1^+)^j(Q) \end{array} \tag{4.9}$$

are commutative for any homomorphism $f\colon P \to Q$ (in the cases $j = 0, 1$, this is obvious). Then, if $\Delta \in D_i(P)$ and $a \in A$, we set $(D_i(f))(\Delta) \stackrel{\text{def}}{=} D_{i-1}(\Delta(a))$. Then from commutativity of diagram (4.9) it follows that $D_i(f)$ takes $D_i(P)$ to $D_i(Q)$ while (4.8) implies that $\alpha_P^i\colon D_i(P) \hookrightarrow (\mathrm{Diff}_1^+)^i(P)$ is a natural embedding.

Note now that, by definition, elements of $D_i(P)$ may be understood as K-linear mappings $A \to D_{i-1}(P)$ possessing "special properties". Given an element $a \in A$ and an operator $\Delta \in D_i(P)$, we have $\Delta(a) \in D_{i-1}(P)$, i.e., $\Delta\colon A \to D_{i-1}(P)$, etc. Thus Δ is a polylinear mapping

$$\Delta\colon \underbrace{A \otimes_K \cdots \otimes_K A}_{i \text{ times}} \to P. \tag{4.10}$$

Let us describe the module $D_i(P)$ in these terms.

Proposition 4.5. *A polylinear mapping of the form* (4.10) *is an element of $D_i(P)$ if and only if*

$$\begin{aligned}\Delta(a_1, \dots, a_{\alpha-1}, ab, a_{\alpha+1}, \dots, a_i)&\\ = a\Delta(\dots, a_{\alpha-1}, b, a_{\alpha+1}, \dots) + b\Delta(\dots, a_{\alpha-1}, a, a_{\alpha+1}, \dots)&\end{aligned} \tag{4.11}$$

and

$$\Delta(\dots, a_\alpha, \dots, a_\beta, \dots) = (-1)^{\alpha\beta}\Delta(\dots, a_\beta, \dots, a_\alpha, \dots) \tag{4.12}$$

for all $a, b, a_1, \dots, a_i \in A$, $1 \le \alpha < \beta \le i$. In other words, $D_i(P)$ consists of skew-symmetric polyderivations (*of degree i*) *of the algebra A with the values in P.*

Proof. Note first that to prove the result it suffices to consider the case $i = 2$. In fact, the general case is proved by induction on i whose step literally repeats the proof for $i = 2$.

Let now $\Delta \in D_2(P)$. Then, since Δ is a derivation with the values in $\mathrm{Diff}_1^+(P)$, one has

$$\Delta(ab) = a^+\Delta(b) + b^+\Delta(a), \qquad a, b \in A.$$

Consequently,

$$\Delta(ab, c) = \Delta(b, ac) + \Delta(a, bc) \tag{4.13}$$

for any $c \in A$. But $\Delta(ab) \in D(P)$ and thus $\Delta(ab, 1) = 0$. Therefore, (4.13) implies $\Delta(a, b) + \Delta(b, a) = 0$ which proves (4.12). On the other hand, from the result proved we obtain that $\Delta(ab, c) = -\Delta(c, ab)$ while, by definition, one has $\Delta(c) \in D(P)$ for any $c \in A$. Hence,

$$\Delta(ab, c) = -\Delta(c, ab) = -a\Delta(c, b) - b\Delta(c, a) = a\Delta(b, c) + b\Delta(a, c)$$

which finishes the proof. $\square$

To finish this subsection, we establish an additional algebraic structure in the modules $D_i(P)$. Namely, we define by induction the *wedge product* $\wedge\colon D_i(A) \otimes_K D_j(P) \to D_{i+j}(P)$ by setting

$$a \wedge p \overset{\text{def}}{=} ap, \qquad a \in D_0(A) = A, \quad p \in D_0(P) = P, \tag{4.14}$$

and

$$(\Delta \wedge \Box)(a) \overset{\text{def}}{=} \Delta \wedge \Box(a) + (-1)^j \Delta(a) \wedge \Box \tag{4.15}$$

for any $\Delta \in D_i(A)$, $\Box \in D_j(P)$, $i + j > 0$.

PROPOSITION 4.6. *The wedge product of polyderivations is a well-defined operation.*

PROOF. It needs to prove that $\Delta \wedge \Box$ defined by (4.14) and (4.15) lies in $D_{i+j}(P)$. To do this, we shall use Proposition 4.5 and induction on $i+j$. The case $i+j<2$ is trivial.

Let now $i+j \geq 2$ and assume that the result was proved for all $k<i+j$. Then from (4.15) it follows that $(\Delta \wedge \Box)(a) \in D_{i+j-1}(P)$. Let us prove that $\Delta \wedge \Box$ satisfies identities (4.11) and (4.12) of Proposition 4.5. In fact, we have

$$\begin{aligned}(\Delta \wedge \Box)(a,b) &= (\Delta \wedge \Box(a))(b) + (-1)^j(\Delta(a) \wedge \Box)(b) \\ &= \Delta \wedge \Box(a,b) + (-1)^{j-1}\Delta(b) \wedge \Box(a) + (-1)^j(\Delta(a) \wedge \Box(b) \\ &+ (-1)^j\Delta(a,b) \wedge \Box) = -\big(\Delta \wedge \Box(b,a) + (-1)^{j-1}\Delta(a) \wedge \Box(b) \\ &+ (-1)^j\Delta(b) \wedge \Box(a) + \Delta(b,a) \wedge \Box\big) = -(\Delta \wedge \Box)(b,a),\end{aligned}$$

where a and b are arbitrary elements of A.

On the other hand,

$$\begin{aligned}(\Delta \wedge \Box)(ab) &= \Delta \wedge \Box(ab) + (-1)^j\Delta(ab) \wedge \Box \\ &= \Delta \wedge \big(a\Box(b) + b\Box(a)\big) + (-1)^j\big(a\Delta(b) + b\Delta(a)\big) \wedge \Box \\ &= a\big(\Delta \wedge \Box(b) + (-1)^j\Delta(b) \wedge \Box\big) + b\big(\Delta \wedge \Box(a) + (-1)^j\Delta(a) \wedge \Box\big) \\ &= a\big(\Delta \wedge \Box)(b) + b(\Delta \wedge \Box\big)(a).\end{aligned}$$

We used here the fact that $\Delta \wedge (a\Box) = a(\Delta \wedge \Box)$ which is proved by trivial induction. $\Box$

PROPOSITION 4.7. *For any derivations* Δ, Δ_1, $\Delta_2 \in D_*(A)$ *and* $\Box$, $\Box_1$, $\Box_2 \in D_*(P)$, *one has*

(i) $(\Delta_1 + \Delta_2) \wedge \Box = \Delta_1 \wedge \Box + \Delta_2 \wedge \Box$,
(ii) $\Delta \wedge (\Box_1 + \Box_2) = \Delta \wedge \Box_1 + \Delta \wedge \Box_2$,
(iii) $\Delta_1 \wedge (\Delta_2 \wedge \Box) = (\Delta_1 \wedge \Delta_2) \wedge \Box$,
(iv) $\Delta_1 \wedge \Delta_2 = (-1)^{i_1 i_2}\Delta_2 \wedge \Delta_1$,

where $\Delta_1 \in D_{i_1}(A)$, $\Delta_2 \in D_{i_2}(A)$.

PROOF. All statements are proved in a similar way. As an example, let us prove equality (iv). We use induction on $i_1 + i_2$. The case $i_1 + i_2 = 0$ is obvious (see (4.14)). Let now $i_1 + i_2 > 0$ and assume that (iv) is valid for all $k < i_1 + i_2$. Then

$$\begin{aligned}(\Delta_1 \wedge \Delta_2)(a) &= \Delta_1 \wedge \Delta_2(a) + (-1)^{i_2}\Delta_1(a) \wedge \Delta_2 \\ &= (-1)^{i_1(i_2-1)}\Delta_2(a) \wedge \Delta_1 + (-1)^{i_2}(-1)^{(i_1-1)i_2}\Delta_2 \wedge \Delta_1(a) \\ &= (-1)^{i_1 i_2}(\Delta_2 \wedge \Delta_1(a) + (-1)^{i_1}\Delta_2(a) \wedge \Delta_1) = (-1)^{i_1 i_2}(\Delta_2 \wedge \Delta_1)(a)\end{aligned}$$

for any $a \in A$. $\Box$

COROLLARY 4.8. *The correspondence $P \Rightarrow D_*(P)$ is a functor from the category of A-modules to the category of graded modules over the graded commutative algebra $D_*(A)$.*

1.4. Forms. Consider the module $\mathcal{J}^1(A)$ and the submodule in it generated by $j_1(1)$, i.e., by the class of the element $1 \otimes 1 \in A \otimes_K A$. Denote by $\nu\colon \mathcal{J}^1(A) \to \mathcal{J}^1(A)/(A \cdot j_1(1))$ the natural projection of modules.

DEFINITION 4.5. The quotient module $\Lambda^1(A) \stackrel{\text{def}}{=} \mathcal{J}^1(A)/(A \cdot j_1(1))$ is called the *module of differential* 1*-forms* of the algebra A. The composition $d = d_1 \stackrel{\text{def}}{=} \nu \circ j_1 \colon A \to \Lambda^1(A)$ is called the (*first*) *de Rham differential* of A.

PROPOSITION 4.9. *For any derivation $\Delta\colon A \to P$, a uniquely defined A-homomorphism $\varphi_\Delta\colon \Lambda^1(A) \to P$ exists such that the diagram*

$$\begin{tikzcd} A \arrow[rr, "d"] \arrow[dr, "\Delta"'] & & \Lambda^1(A) \arrow[dl, "\varphi_\Delta"] \\ & P & \end{tikzcd}$$

is commutative. In particular, $\Lambda^1(A)$ is the representative object for the functor $D(\bullet)$.

PROOF. The mapping d, being the composition of j_1 with a homomorphism, is a first order differential operator and it is a tautology that f^d (see Proposition 4.4) coincides with the projection $\nu\colon \mathcal{J}^1(A) \to \Lambda^1(A)$. On the other hand, consider the diagram

$$\begin{tikzcd} \mathcal{J}^1(A) \arrow[rr, "f^\Delta"] \arrow[dr, "\nu"] & & P \\ & \Lambda^1(A) \arrow[ur, "\varphi_\Delta"] & \\ & A \arrow[uul, "j_1"] \arrow[u, "d"] \arrow[uur, "\Delta"'] & \end{tikzcd}$$

Since Δ is a first order differential operator, there exists a homomorphism $f^\Delta\colon \mathcal{J}^1(A) \to P$ satisfying the equality $\Delta = f^\Delta \circ j_1$. But Δ is a derivation, i.e., $\Delta(1) = 0$, which means that $\ker(f^\Delta)$ contains $A \cdot j_1(1)$. Hence, there exists a unique mapping φ_Δ such that the above diagram is commutative. $\square$

REMARK 4.3. From the definition it follows that $\Lambda^1(A)$, as an A-module, is generated by the elements da, $a \in A$, with the relations

$$d(\alpha a + \beta b) = \alpha da + \beta db, \quad d(ab) = adb + bda,$$

$\alpha, \beta \in K$, $a, b \in A$, while the de Rham differential takes a to the coset $a \bmod (A \cdot j_1(1))$.

Let us set now

$$\Lambda^i(A) = \underbrace{\Lambda^1(A) \wedge \cdots \wedge \Lambda^1(A)}_{i \text{ times}}. \tag{4.16}$$

The elements of $\Lambda^i(A)$ are called *differential i-forms* of the algebra A. We also formally set $\Lambda^0(A) \stackrel{\text{def}}{=} A$.

PROPOSITION 4.10. *The modules $\Lambda^i(A)$, $i \geq 0$, are representative objects for the functors $D_i(\bullet)$.*

PROOF. The case $i = 0$ is trivial while the case $i = 1$ was proved already (see Proposition 4.9). Let now $i > 1$ and $a \in A$. Define the mappings

$$\lambda_a \colon \hom_A(\Lambda^i(A), P) \to \hom_A(\Lambda^{i-1}(A), P), \quad \mathrm{i}_a \colon D_i(P) \to D_{i-1}(P)$$

by setting

$$(\lambda_a \varphi)(\omega) \stackrel{\text{def}}{=} \varphi(da \wedge \omega), \quad \mathrm{i}_a \Delta \stackrel{\text{def}}{=} \Delta(a),$$

where $\omega \in \Lambda^{i-1}(A)$, $\varphi \in \hom_A(\Lambda^i(A), P)$, and $\Delta \in D_i(P)$.

Using induction on i, let us construct isomorphisms

$$\psi_i \colon \hom_A(\Lambda^i(A), P) \to D_i(P)$$

in such a way that the diagrams

$$\begin{tikzcd}
\hom_A(\Lambda^i(A), P) \arrow[r, "\psi_i"] \arrow[d, "\lambda_a"'] & D_i(P) \arrow[d, "\mathrm{i}_a"] \\
\hom_A(\Lambda^{i-1}(A), P) \arrow[r, "\psi_{i-1}"] & D_{i-1}(P)
\end{tikzcd} \tag{4.17}$$

are commutative for all $a \in A$.

The case $i = 1$ reduces to Proposition 4.9. Let now $i > 1$ and assume that for $i - 1$ the statement is valid. Then from (4.17) we should have

$$(\psi_i(\varphi))(a) = \psi_{i-1}(\lambda_a(\varphi)), \quad \varphi \in \hom_A(\Lambda^i(A), P),$$

which completely determines ψ_i. From the definition of the mapping λ_a it follows that

$$\lambda_{ab} = a\lambda_b + b\lambda_a, \quad \lambda_a \circ \lambda_b = -\lambda_b \circ \lambda_a, \qquad a, b \in A,$$

i.e., $\operatorname{im} \psi_i \in D_i(P)$ (see Proposition 4.5).

Let us now show that ψ_i constructed in such a way is an isomorphism. Take $\Delta \in D_i(P)$, $a_1, \dots, a_i$ and set

$$\bar{\psi}_i(da_1 \wedge \dots da_i) \stackrel{\text{def}}{=} \left(\psi_{i-1}^{-1}(X(a_1))\right)(da_2 \wedge \cdots \wedge da_i).$$

It may be done since ψ_{i-1}^{-1} exists by the induction assumption. Directly from definitions one obtains that $\psi_i \circ \bar{\psi}_i = \mathrm{id}$, $\bar{\psi}_i \circ \psi_i = \mathrm{id}$. It is also obvious that the isomorphisms ψ_i are natural, i.e., the diagrams

$$\begin{tikzcd} \hom_A(\Lambda^i(A), P) \arrow[r, "\psi_i"] \arrow[d, "\hom_A(\Lambda^i(A), f)"'] & D_i(P) \arrow[d, "D_i(f)"] \\ \hom_A(\Lambda^i(A), Q) \arrow[r, "\psi_i"] & D_i(Q) \end{tikzcd}$$

are commutative for all homomorphisms $f \in \hom_A(P, Q)$. $\square$

From the result proved we obtain the pairing

$$\langle \cdot, \cdot \rangle \colon D_i(P) \otimes_A \Lambda^i(A) \to P \tag{4.18}$$

defined by

$$\langle \Delta, \omega \rangle \stackrel{\text{def}}{=} \left(\psi_i^{-1}(\Delta)\right)(\omega), \quad \omega \in \Lambda^i(A), \quad \Delta \in D_i(P).$$

A direct consequence of the proof of Proposition 4.10 is the following

COROLLARY 4.11. *The identity*

$$\langle \Delta, da \wedge \omega \rangle = \langle \Delta(a), \omega \rangle \tag{4.19}$$

holds for any $\omega \in \Lambda^i(A)$, $\Delta \in D_{i+1}(A)$, $a \in A$.

Let us define the mappings $d = d_i \colon \Lambda^{i-1}(A) \to \Lambda^i(A)$ by taking the first de Rham differential for d_1 and setting

$$d_i(a_0\, da_1 \wedge \cdots \wedge da_i) \stackrel{\text{def}}{=} da_0 \wedge da_1 \wedge \cdots \wedge da_i$$

for $i > 1$. From (4.16) and Remark 4.3 it follows that the mappings d_i are well defined.

PROPOSITION 4.12. *The mappings d_i possess the following properties*:

(i) *d_i is a first order differential operator acting from $\Lambda^{i-1}(A)$ to $\Lambda^i(A)$;*
(ii) *$d(\omega \wedge \theta) = d(\omega) \wedge \theta + (-1)^i \omega \wedge d(\theta)$ for any $\omega \in \Lambda^i(A)$, $\theta \in \Lambda^j(A)$;*
(iii) *$d_i \circ d_{i-1} = 0$.*

The proof is trivial.

In particular, (iii) means that the sequence of mappings

$$0 \to A \xrightarrow{d_1} \Lambda^1(A) \to \cdots \to \Lambda^{i-1}(A) \xrightarrow{d_i} \Lambda^i(A) \to \cdots \tag{4.20}$$

is a complex.

DEFINITION 4.6. The mapping d_i is called the (i-th) *de Rham differential.* The sequence (4.20) is called the *de Rham complex* of the algebra A.

REMARK 4.4. Before proceeding with further exposition, let us make some important comments on the relation between algebraic and geometrical settings. As we saw above, the algebraic definition of a linear differential operator is in full accordance with the analytical one. The same is true if we compare algebraic "vector fields" (i.e., elements of the module $D(A)$) with vector fields on a smooth manifold M: derivations of the algebra $C^\infty(M)$ are identical to vector fields on M.

This situation changes, when we pass to representative objects. A simple example illustrates this effect. Let $M = \mathbb{R}$ and $A = C^\infty(M)$. Consider the differential one-form $\omega = de^x - e^x\,dx \in \Lambda^1(A)$. This form is nontrivial as an element of the module $\Lambda^1(A)$. On the other hand, for any A-module P let us define the value of an element $p \in P$ at point $x \in M$ as follows. Denote by μ_x the ideal

$$\mu_x \stackrel{\text{def}}{=} \{f \in C^\infty(M) \mid f(x) = 0\} \subset C^\infty(M)$$

and set $p_x \stackrel{\text{def}}{=} p \bmod \mu_x$. In particular, if $P = A$, thus defined value coincides with the value of a function f at a point. One can easily see that $\omega_x = 0$ for any $x \in M$. Thus, ω is a kind of a "ghost", not observable at any point of the manifold. The reader will easily construct similar examples for the modules $\mathcal{J}^k(A)$. In other words, we can state that

$$\Lambda^i(M) \neq \Lambda^i(C^\infty(M)), \quad \Gamma(\pi_k) \neq \mathcal{J}^k(\Gamma(\pi))$$

for an arbitrary smooth manifold M and a vector bundle $\pi\colon E \to M$.

Let us say that $C^\infty(M)$-module P is *geometrical*, if

$$\bigcap_{x \in M} \mu_x \cdot P = 0.$$

Obviously, all modules of the form $\Gamma(\pi)$ are geometrical. We can introduce the *geometrization functor* by setting

$$\mathfrak{G}(P) \stackrel{\text{def}}{=} P/\bigcap_{x \in M} \mu_x \cdot P.$$

Then the following result is valid:

PROPOSITION 4.13. *Let M be a smooth manifold and $\pi\colon E \to M$ be a smooth vector bundle. Denote by A the algebra $C^\infty(M)$ and by P the module $\Gamma(\pi)$. then:*

(ii) *The functor $D_i(\bullet)$ is representable in the category of geometrical A-modules and one has*

$$D_i(Q) = \hom_A(\mathfrak{G}(\Lambda^i(A)), Q)$$

for any geometrical module Q.

(i) *The functor* $\mathrm{Diff}(P, \bullet)$ *is representable in the category of geometrical A-modules and one has*

$$\mathrm{Diff}_k(P, Q) = \hom_A(\mathfrak{G}(\mathcal{J}^k(P)), Q)$$

for any geometrical module Q.

In particular,

$$\Lambda^i(M) = \mathfrak{G}(\Lambda^i(C^\infty(M))), \quad \Gamma(\pi_k) = \mathfrak{G}(\mathcal{J}^k(\Gamma(\pi))).$$

1.5. Smooth algebras. Let us introduce a class of algebras which plays an important role in geometrical theory.

DEFINITION 4.7. A commutative algebra A is called *smooth*, if $\Lambda^1(A)$ is a projective A-module of finite type while A itself is an algebra over the field of rational numbers $\mathbb{Q}$.

Denote by $S^i(P)$ the i-th symmetric power of an A-module P.

LEMMA 4.14. *Let A be a smooth algebra. Then both $S^i(\Lambda^1(A))$ and $\Lambda^i(A)$ are projective modules of finite type.*

PROOF. Denote by $T^i \overset{\text{def}}{=} T^i(\Lambda^1(A))$ the i-th tensor power of $\Lambda^1(A)$. Since the module $\Lambda^1(A)$ is projective, then it can be represented as a direct summand in a free module, say P. Consequently, T^i is a direct summand in the free module $T^i(P)$ and thus is projective with finite number of generators.

On the other hand, since A is a $\mathbb{Q}$-algebra, both $S^i(\Lambda^i(A))$ and $\Lambda^i(A)$ are direct summands in T^i which finishes the proof. □

PROPOSITION 4.15. *If A is a smooth algebra, then the following isomorphisms are valid*:

(i) $D_i(A) \simeq \underbrace{D_1(A) \wedge \cdots \wedge D_1(A)}_{i \text{ times}}$,

(ii) $D_i(P) \simeq D_i(A) \otimes_A P$,

where P is an arbitrary A-module.

PROOF. The result follows from Lemma 4.14 combined with Proposition 4.10 □

For smooth algebras, one can also efficiently describe the modules $\mathcal{J}^k(A)$. Namely, the following statement is valid:

PROPOSITION 4.16. *If A is a smooth algebra, then all the modules $\mathcal{J}^k(A)$ are projective of finite type and the isomorphisms*

$$\mathcal{J}^k(A) \simeq \bigoplus_{i \le k} S^i(\Lambda^1(A))$$

take place.

PROOF. We shall use induction on k. First note that the mapping $a \mapsto aj_1(1)$ splits the exact sequence

$$0 \to \ker(\nu_{1,0}) \to \mathcal{J}^1(A) \xrightarrow{\nu_{1,0}} \mathcal{J}^0(A) = A \to 0.$$

But by definition, $\ker(\nu_{1,0}) = \Lambda^1(A)$ and thus $\mathcal{J}^1(A) = A \oplus \Lambda^1(A)$.

Let now $k > 1$ and assume that for $k-1$ the statement is true. By definition, $\ker(\nu_{k,k-1}) = \mu_{k-1}/\mu_k$, where $\mu_i \subset A \otimes_K A$ are the submodules introduced in Subsection 1.2. Note that the identity $a \otimes b = a(b \otimes 1) - a\delta^b(1 \otimes 1)$ implies the direct sum decomposition $\mu_{k-1} = \mu_k \oplus (\mu_{k-1}/\mu_k)$ and thus the quotient module μ_{k-1}/μ_k is identified with the submodule in $A \otimes_K A$ spanned by

$$(\delta^{a_1} \circ \cdots \circ \delta^{a_k})\,(1 \otimes 1), \quad a_0, \dots, a_k \in A.$$

Consequently, any $a \in A$ determines the homomorphism

$$\delta^a \colon \mu_{k-2}/\mu_{k-1} \to \mu_{k-1}/\mu_k$$

by

$$\delta^a \colon a' \otimes a'' \mapsto aa' \otimes a'' - a' \otimes aa''.$$

But one has $\delta^{ab} = a\delta^b + b\delta^a$ and hence $\delta \colon a \mapsto \delta^a$ is an element of the module $D_1(\hom_A(\mu_{k-2}/\mu_{k-1}, \mu_{k-1}/\mu_k))$. Consider the corresponding homomorphism

$$\varphi = \varphi_\delta \in \hom_A(\Lambda^1(A), \hom_A(\mu_{k-2}/\mu_{k-1}, \mu_{k-1}/\mu_k)).$$

Due to the canonical isomorphism

$$\begin{aligned} &\hom_A(\Lambda^1(A), \hom_A(\mu_{k-2}/\mu_{k-1}, \mu_{k-1}/\mu_k)) \simeq \\ &\qquad\qquad \simeq \hom_A(\Lambda^1(A) \otimes_A \mu_{k-2}/\mu_{k-1}, \mu_{k-1}/\mu_k), \end{aligned}$$

we obtain the mapping

$$\varphi \colon \Lambda^1(A) \otimes_A \bigl(\mu_{k-2}/\mu_{k-1}\bigr) \to \mu_{k-1}/\mu_k,$$

and repeating the procedure, get eventually the mapping $\varphi \colon T^k \to \mu_{k-1}/\mu_k$. Due to the identity $\delta_a \circ \delta_b = \delta_b \circ \delta_a$, this mapping induces the homomorphism $\varphi_S \colon S^k(\Lambda^1(A)) \to \mu_{k-1}/\mu_k$ which, in terms of generators, acts as

$$\varphi_S(da_1 \cdot \dots \cdot da_k) = (\delta^{a_1} \circ \dots \circ \delta^{a_k})\,(1 \otimes 1)$$

and thus is epimorphic.

Consider the dual monomorphism

$$\varphi_S^* \colon \mu_{k-1}/\mu_k = \operatorname{Diff}_k(A)/\operatorname{Diff}_{k-1}(A) \to (S^k(\Lambda^1(A)))^* = S^k(D_1(A)).$$

Let $\sigma \in \operatorname{Diff}_k(A)/\operatorname{Diff}_{k-1}(A)$ and $\Delta \in \operatorname{Diff}_k(A)$ be a representative of the class σ. Then

$$(\varphi_S^*(\sigma))(da_1 \cdot \dots \cdot da_k) = (\delta_{a_1} \circ \cdots \circ \delta_{a_k})\,(\Delta).$$

But, on the other hand, it is not difficult to see that the mapping

$$\bar{\varphi}_S^* \colon X_1 \cdot \ldots X_k \mapsto \frac{1}{k!}[X_1 \circ \ldots X_k],$$

$\bar{\varphi}_S^* \colon S^k(D_1(A)) \to \operatorname{Diff}_k(A)/\operatorname{Diff}_{k-1}(A)$, where $[\Delta]$ denotes the coset of the operator $\Delta \in \operatorname{Diff}_k(A)$ in the quotient module $\operatorname{Diff}_k(A)/\operatorname{Diff}_{k-1}(A)$, is inverse to φ_S^*. Thus, φ_S^* is an isomorphism. Then the mapping

$$\mu_{k-1}/\mu_k \to (\mu_{k-1}/\mu_k)^{**} \simeq S^k(\Lambda^1(A)),$$

where the first arrow is the natural homomorphism, is the inverse to φ_S.

From the above said it follows that $\mu_{k-1}/\mu_k \simeq S^k(\Lambda^1(A))$ and we have the exact sequence

$$0 \to S^k(\Lambda^1(A)) \to \mathcal{J}^k(A) \to \mathcal{J}^{k-1}(A) \to 0.$$

But, by the induction assumption, $\mathcal{J}^{k-1}(A)$ is a projective module isomorphic to $\bigoplus_{i\le k-1} S^i(\Lambda^1(A))$. Hence,

$$\mathcal{J}^k(A) \simeq S^k(\Lambda^1(A)) \oplus \mathcal{J}^{k-1}(A) \simeq \bigoplus_{i\le k} S^i(\Lambda^1(A))$$

which finishes the proof. □

DEFINITION 4.8. Let P be an A-module. The module $\operatorname{Smbl}_*(P) \stackrel{\text{def}}{=} \sum_{k\ge 0} \operatorname{Smbl}_k(P)$, where

$$\operatorname{Smbl}_k(P) \stackrel{\text{def}}{=} \operatorname{Diff}_k(P)/\operatorname{Diff}_{k-1}(P),$$

is called the *module of symbols* for P. The coset of $\Delta \in \operatorname{Diff}_k(P)$ in $\operatorname{Smbl}_k(P)$ is called the *symbol of the operator* Δ.

Let $\sigma \in \operatorname{Smbl}_i(A)$ and $\sigma' \in \operatorname{Smbl}_j(A)$ and assume that $\Delta \in \operatorname{Diff}_i(A)$ and $\Delta' \in \operatorname{Diff}_j(A)$ are representatives of σ, σ' respectively. Define the product $\sigma\sigma'$ as the coset of $\Delta \circ \Delta'$ in $\operatorname{Diff}_{i+j}(A)$. It is easily checked that $\operatorname{Smbl}_*(A)$ forms a commutative A-algebra with respect to thus defined multiplication.

As a direct consequence of the last proposition and of Proposition 4.4, we obtain

COROLLARY 4.17. *If A is a smooth algebra, then the following statements are valid*:

(i) $\operatorname{Diff}_k(P) \simeq \operatorname{Diff}_k(A) \otimes_A P$,
(ii) $\operatorname{Diff}_*(A)$, *as an associative algebra, is generated by* $A = \operatorname{Diff}_0(A)$ *and* $D_1(A) \subset \operatorname{Diff}_1(A)$,
(iii) $\operatorname{Smbl}_k(P) \simeq \operatorname{Smbl}_k(A) \otimes_A P$,
(iv) $\operatorname{Smbl}_*(A)$, *as a commutative algebra, is isomorphic to the symmetric tensor algebra of* $D_1(A)$.

REMARK 4.5. It should be noted that $\operatorname{Smbl}_* A$ is more than just a commutative algebra. In fact, in the case $A = C^\infty(M)$, as it can be easily seen, elements of $\operatorname{Smbl}_* A$ can be naturally identified with smooth functions on T^*M polynomial along the fibers of the natural projection $T^*M \to M$. The manifold T^*M is symplectic and, in particular, the algebra $C^\infty(T^*M)$ possesses a Poisson bracket which induces a bracket in $\operatorname{Smbl}_* A \subset C^\infty(T^*M)$. This bracket, as it happens, is of a purely algebraic nature.

Let us consider two symbols $\sigma_1 \in \operatorname{Smbl}_{i_1} A$, $\sigma_2 \in \operatorname{Smbl}_{i_2} A$ such that $\sigma_r = \Delta_r \bmod \operatorname{Diff}_{i_r-1} A$, $r = 1, 2$, and set

$$\{\sigma_1, \sigma_2\} \stackrel{\text{def}}{=} [\Delta_1, \Delta_2] \bmod \operatorname{Diff}_{i_1+i_2-2}. \tag{4.21}$$

The operation $\{\cdot,\cdot\}$ defined by (4.21) is called the *Poisson bracket* in the algebra of symbols and in the case $A = C^\infty(M)$ coincides with the classical

Poisson bracket on the cotangent space. It possesses the usual properties, i.e.,

$$\{\sigma_1, \sigma_2\} + \{\sigma_2, \sigma_1\} = 0,$$
$$\{\sigma_1, \{\sigma_2, \sigma_3\}\} + \{\sigma_2, \{\sigma_3, \sigma_1\}\} + \{\sigma_3, \{\sigma_1, \sigma_2\}\} = 0,$$
$$\{\sigma_1, \sigma_2\sigma_3\} = \{\sigma_1, \sigma_2\}\sigma_3 + \sigma_2\{\sigma_1, \sigma_3\}$$

and, in particular, $\mathrm{Smbl}_* A$ becomes a Lie K-algebra with respect to this bracket. This is a starting point to construct Hamiltonian formalism in a general algebraic setting. For details and generalizations see [**104, 53, 54**].

2. Frölicher–Nijenhuis bracket

We still consider the general algebraic setting of the previous section and extend standard constructions of calculus to form-valued derivations. It allows us to define *Frölicher–Nijenhuis brackets* and introduce a cohomology theory (∇-cohomologies) associated to commutative algebras with *flat connections*. In the next chapter, applying this theory to infinitely prolonged partial differential equations, we obtain an algebraic and analytical description of recursion operators for symmetries and describe efficient tools to compute these operators. These and related results, together with their generalizations, were first published in the papers [**55, 56, 57**] and [**59, 58, 40**].

2.1. Calculus in form-valued derivations. Let $\Bbbk$ be a field of characteristic zero and A be a commutative unitary $\Bbbk$-algebra. Let us recall the basic notations:

- $D(P)$ is the module of P-valued derivations $A \to P$, where P is an A-module;
- $D_i(P)$ is the module of P-valued skew-symmetric i-derivations. In particular, $D_1(P) = D(P)$;
- $\Lambda^i(A)$ is the module of differential i-forms of the algebra A;
- $d\colon \Lambda^i(A) \to \Lambda^{i+1}(A)$ is the de Rham differential.

Recall also that the modules $\Lambda^i(A)$ are representative objects for the functors $D_i\colon P \Rightarrow D_i(P)$, i.e., $D_i(P) = \mathrm{Hom}_A(\Lambda^i(A), P)$. The isomorphism $D(P) = \mathrm{Hom}_A(\Lambda^1(A), P)$ can be expressed in more exact terms: for any derivation $X\colon A \to P$, there exists a uniquely defined A-module homomorphism $\varphi_X\colon \Lambda^1(A) \to P$ satisfying the equality $X = \varphi_X \circ d$. Denote by $\langle Z, \omega\rangle \in P$ the value of the derivation $Z \in D_i(P)$ at $\omega \in \Lambda^i(A)$.

Both $\Lambda^*(A) = \bigoplus_{i \geq 0} \Lambda^i(A)$ and $D_*(A) = \bigoplus_{i \geq 0} D_i(A)$ are endowed with the structures of superalgebras with respect to the wedge product operations

$$\wedge\colon \Lambda^i(A) \otimes \Lambda^j(A) \to \Lambda^{i+j}(A),$$
$$\wedge\colon D_i(A) \otimes D_j(A) \to D_{i+j}(A),$$

the de Rham differential $d\colon \Lambda^*(A) \to \Lambda^*(A)$ becoming a derivation of $\Lambda^*(A)$. Note also that $D_*(P) = \bigoplus_{i \geq 0} D_i(P)$ is a $D_*(A)$-module.

Using the paring $\langle\cdot,\cdot\rangle$ and the wedge product, we define the *inner product* (or *contraction*) $\mathrm{i}_X\omega \in \Lambda^{j-i}(A)$ of $X \in D_i(A)$ and $\omega \in \Lambda^j(A)$, $i \le j$, by setting

$$\langle Y, \mathrm{i}_X\omega\rangle = (-1)^{i(j-i)}\langle X \wedge Y, \omega\rangle, \tag{4.22}$$

where Y is an arbitrary element of $D_{j-i}(P)$, P being an A-module. We formally set $\mathrm{i}_X\omega = 0$ for $i > j$. When $i = 1$, this definition coincides with the one given in Section 1. Recall that the following duality is valid:

$$\langle X, da \wedge \omega\rangle = \langle X(a), \omega\rangle, \tag{4.23}$$

where $\omega \in \Lambda^i(A)$, $X \in D_{i+1}(P)$, and $a \in A$ (see Corollary 4.11). Using the property (4.23), one can show that

$$\mathrm{i}_X(\omega \wedge \theta) = \mathrm{i}_X(\omega) \wedge \theta + (-1)^{X\omega}\omega \wedge \mathrm{i}_X(\omega)$$

for any $\omega, \theta \in \Lambda^*(A)$, where (as everywhere below) the symbol of a graded object used as the exponent of (-1) denotes the degree of that object.

We now define the *Lie derivative* of $\omega \in \Lambda^*(A)$ along $X \in D_*(A)$ as

$$\mathrm{L}_X\omega = \bigl(\mathrm{i}_X \circ d - (-1)^X d \circ \mathrm{i}_X\bigr)\omega = [\mathrm{i}_X, d]\omega, \tag{4.24}$$

where $[\cdot,\cdot]$ denotes the graded (or super) commutator: if $\Delta, \Delta'\colon \Lambda^*(A) \to \Lambda^*(A)$ are graded derivations, then

$$[\Delta, \Delta'] = \Delta \circ \Delta' - (-1)^{\Delta\Delta'}\Delta' \circ \Delta.$$

For $X \in D(A)$ this definition coincides with the ordinary commutator of derivations.

Consider now the graded module $D(\Lambda^*(A))$ of $\Lambda^*(A)$-valued derivations $A \to \Lambda^*(A)$ (corresponding to form-valued vector fields — or, which is the same — vector-valued differential forms on a smooth manifold). Note that the graded structure in $D(\Lambda^*(A))$ is determined by the splitting $D(\Lambda^*(A)) = \bigoplus_{i\ge 0} D(\Lambda^i(A))$ and thus elements of grading i are derivations X such that $\operatorname{im} X \subset \Lambda^i(A)$. We shall need three algebraic structures associated to $D(\Lambda^*(A))$.

First note that $D(\Lambda^*(A))$ is a graded $\Lambda^*(A)$-module: for any $X \in D(\Lambda^*(A))$, $\omega \in \Lambda^*(A)$ and $a \in A$ we set $(\omega \wedge X)a = \omega \wedge X(a)$. Second, we can define the inner product $\mathrm{i}_X\omega \in \Lambda^{i+j-1}(A)$ of $X \in D(\Lambda^i(A))$ and $\omega \in \Lambda^j(A)$ in the following way. If $j = 0$, we set $\mathrm{i}_X\omega = 0$. Then, by induction on j and using the fact that $\Lambda^*(A)$ as a graded A-algebra is generated by the elements of the form da, $a \in A$, we set

$$\mathrm{i}_X(da \wedge \omega) = X(a) \wedge \omega - (-1)^X da \wedge \mathrm{i}_X(\omega), \quad a \in A. \tag{4.25}$$

Finally, we can contract elements of $D(\Lambda^*(A))$ with each other in the following way:

$$(\mathrm{i}_X Y)a = \mathrm{i}_X(Ya), \quad X, Y \in D(\Lambda^*(A)), \quad a \in A. \tag{4.26}$$

Three properties of contractions are essential in the sequel.

PROPOSITION 4.18. *Let* $X, Y \in D(\Lambda^*(A))$ *and* $\omega, \theta \in \Lambda^*(A)$. *Then*

$$\mathrm{i}_X(\omega \wedge \theta) = \mathrm{i}_X(\omega) \wedge \theta + (-1)^{\omega(X-1)} \omega \wedge \mathrm{i}_X(\theta), \tag{4.27}$$

$$\mathrm{i}_X(\omega \wedge Y) = \mathrm{i}_X(\omega) \wedge Y + (-1)^{\omega(X-1)} \omega \wedge \mathrm{i}_X(Y), \tag{4.28}$$

$$[\mathrm{i}_X, \mathrm{i}_Y] = \mathrm{i}_{[\![X,Y]\!]^{\mathrm{rn}}}, \tag{4.29}$$

where

$$[\![X, Y]\!]^{\mathrm{rn}} = \mathrm{i}_X(Y) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(X). \tag{4.30}$$

PROOF. Equality (4.27) is a direct consequence of (4.25). To prove (4.28), it suffices to use the definition and expressions (4.26) and (4.27).

Let us prove (4.29) now. To do this, note first that due to (4.26), the equality is sufficient to be checked for elements $\omega \in \Lambda^j(A)$. Let us use induction on j. For $j = 0$ it holds in a trivial way. Let $a \in A$; then one has

$$\begin{aligned}[\mathrm{i}_X, \mathrm{i}_Y](da \wedge \omega) &= \big(\mathrm{i}_X \circ \mathrm{i}_Y - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y \circ \mathrm{i}_X\big)(da \wedge \omega) \\ &= \mathrm{i}_X(\mathrm{i}_Y(da \wedge \omega)) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(\mathrm{i}_X(da \wedge \omega)).\end{aligned}$$

But

$$\begin{aligned}\mathrm{i}_X(\mathrm{i}_Y(da \wedge \omega)) &= \mathrm{i}_X(Y(a) \wedge \omega - (-1)^Y da \wedge \mathrm{i}_Y \omega) \\ &= \mathrm{i}_X(Y(a)) \wedge \omega + (-1)^{(X-1)Y} Y(a) \wedge \mathrm{i}_X \omega - (-1)^Y (X(a) \wedge \mathrm{i}_Y \omega \\ &\qquad - (-1)^X da \wedge \mathrm{i}_X(\mathrm{i}_Y \omega)),\end{aligned}$$

while

$$\begin{aligned}\mathrm{i}_Y(\mathrm{i}_X(da \wedge \omega) &= \mathrm{i}_Y(X(a) \wedge \omega - (-1)^X da \wedge \mathrm{i}_X \omega) \\ &= \mathrm{i}_Y(X(a)) \wedge \omega + (-1)^{X(Y-1)} X(a) \wedge \mathrm{i}_Y \omega - (-1)^X (Y(a) \wedge \mathrm{i}_X \omega \\ &\qquad - (-1)^Y da \wedge \mathrm{i}_Y(\mathrm{i}_X \omega)).\end{aligned}$$

Hence,

$$\begin{aligned}[\mathrm{i}_X, \mathrm{i}_Y](da \wedge \omega) &= \big(\mathrm{i}_X(Y(a)) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(X(a))\big) \wedge \omega \\ &\quad + (-1)^{X+Y} da \wedge \big(\mathrm{i}_X(\mathrm{i}_Y \omega) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(\mathrm{i}_X \omega)\big).\end{aligned}$$

But, by definition,

$$\begin{aligned}&\mathrm{i}_X(Y(a)) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(X(a)) \\ &\qquad = (\mathrm{i}_X Y - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y X)(a) = [\![X, Y]\!]^{\mathrm{rn}}(a),\end{aligned}$$

whereas

$$\mathrm{i}_X(\mathrm{i}_Y \omega) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(\mathrm{i}_X \omega) = \mathrm{i}_{[\![X,Y]\!]^{\mathrm{rn}}}(\omega)$$

by induction hypothesis. □

Note also that the following identity is valid for any $X, Y, Z \in D(\Lambda^*(A))$:

$$X \lrcorner (Y \lrcorner Z) = (X \lrcorner Y) \lrcorner Z + (-1)^X (X \wedge Y) \lrcorner Z. \tag{4.31}$$

DEFINITION 4.9. The element $[\![X,Y]\!]^{\mathrm{rn}}$ defined by (4.30) is called the *Richardson–Nijenhuis bracket* of elements X and Y.

Directly from Proposition 4.18 we obtain the following

PROPOSITION 4.19. *For any derivations $X, Y, Z \in D(\Lambda^*(A))$ and a form $\omega \in \Lambda^*(A)$ one has*

$$[\![X,Y]\!]^{\mathrm{rn}} + (-1)^{(X+1)(Y+1)}[\![Y,X]\!]^{\mathrm{rn}} = 0, \tag{4.32}$$

$$\oint (-1)^{(Y+1)(X+Z)}[\![[\![X,Y]\!]^{\mathrm{rn}}, Z]\!]^{\mathrm{rn}} = 0, \tag{4.33}$$

$$[\![X, \omega \wedge Y]\!]^{\mathrm{rn}} = \mathrm{i}_X(\omega) \wedge Y + (-1)^{(X+1)\omega}\omega \wedge [\![X,Y]\!]^{\mathrm{rn}}. \tag{4.34}$$

Here and below the symbol $\oint$ denotes the sum of cyclic permutations.

REMARK 4.6. Note that Proposition 4.19 means that $D(\Lambda^*(A))^{\downarrow}$ is a Gerstenhaber algebra with respect to the Richardson–Nijenhuis bracket [**48**]. Here the superscript $^{\downarrow}$ denotes the shift of grading by 1.

Similarly to (4.24), let us define the Lie derivative of $\omega \in \Lambda^*(A)$ along $X \in D(\Lambda^*(A))$ by

$$\mathrm{L}_X\omega = (\mathrm{i}_X \circ d - (-1)^{X-1} d \circ \mathrm{i}_X)\omega = [\mathrm{i}_X, d]\omega \tag{4.35}$$

REMARK 4.7. Let us clarify the change of sign in (4.35) with respect to formula (4.24). If A is a commutative algebra, then the module $D_*(\Lambda^*(A))$ is a *bigraded* module: if $\Delta \in D_i(\Lambda^j(A))$, then bigrading of this element is (i,j). We can also consider the total grading by setting $\deg\Delta \stackrel{\mathrm{def}}{=} i+j$. In this sense, if $X \in D_i(A)$, then $\deg X = i$, and for $X \in D_1(\Lambda^j(A))$, then $\deg X = j+1$. This also explains shift of grading in Remark 4.6.

From the properties of i_X and d we obtain

PROPOSITION 4.20. *For any $X \in D(\Lambda^*(A))$ and $\omega, \theta \in \Lambda^*(A)$, one has the following identities*:

$$\mathrm{L}_X(\omega \wedge \theta) = \mathrm{L}_X(\omega) \wedge \theta + (-1)^{X\omega}\omega \wedge \mathrm{L}_X(\theta), \tag{4.36}$$

$$\mathrm{L}_{\omega\wedge X} = \omega \wedge \mathrm{L}_X + (-1)^{\omega+X} d(\omega) \wedge \mathrm{i}_X, \tag{4.37}$$

$$[\mathrm{L}_X, d] = 0. \tag{4.38}$$

Our main concern now is to analyze the commutator $[\mathrm{L}_X, \mathrm{L}_Y]$ of two Lie derivatives. It may be done efficiently for smooth algebras (see Definition 4.7).

PROPOSITION 4.21. *Let A be a smooth algebra. Then for any derivations $X, Y \in D(\Lambda^*(A))$ there exists a uniquely determined element $[\![X,Y]\!]^{\mathrm{fn}} \in D(\Lambda^*(A))$ such that*

$$[L_X, L_Y] = L_{[\![X,Y]\!]^{\mathrm{fn}}}. \tag{4.39}$$

PROOF. To prove existence, recall that for smooth algebras one has

$$D_i(P) = \mathrm{Hom}_A(\Lambda^i(A), P) = P \otimes_A \mathrm{Hom}_A(\Lambda^i(A), A) = P \otimes_A D_i(A)$$

for any A-module P and integer $i \geq 0$. Using this identification, let us represent elements $X, Y \in D(\Lambda^*(A))$ in the form

$$X = \omega \otimes X' \text{ and } Y = \theta \otimes Y' \text{ for } \omega, \theta \in \Lambda^*(A), X', Y' \in D(A).$$

Then it is easily checked that the element

$$\begin{aligned} Z = \omega \wedge \theta \otimes [X', Y'] &+ \omega \wedge \mathrm{L}_{X'}\theta \otimes Y + (-1)^{\omega} d\omega \wedge \mathrm{i}_{X'}\theta \otimes Y' \\ &- (-1)^{\omega\theta}\theta \wedge \mathrm{L}_{Y'}\omega \otimes X' - (-1)^{(\omega+1)\theta} d\theta \wedge \mathrm{i}_{Y'}\omega \otimes X' \\ &= \omega \wedge \theta \otimes [X', Y'] + \mathrm{L}_X\theta \otimes Y' - (-1)^{\omega\theta}\mathrm{L}_Y\omega \otimes X' \end{aligned} \tag{4.40}$$

satisfies (4.39).

Uniqueness follows from the fact that $\mathrm{L}_X(a) = X(a)$ for any $a \in A$. □

DEFINITION 4.10. The element $[\![X, Y]\!]^{\mathrm{fn}} \in D^{i+j}(\Lambda^*(A))$ defined by formula (4.39) (or by (4.40)) is called the *Frölicher–Nijenhuis bracket* of form-valued derivations $X \in D^i(\Lambda^*(A))$ and $Y \in D^j(\Lambda^*(A))$.

The basic properties of this bracket are summarized in the following

PROPOSITION 4.22. *Let A be a smooth algebra, $X, Y, Z \in D(\Lambda^*(A))$ be derivations and $\omega \in \Lambda^*(A)$ be a differential form. Then the following identities are valid*:

$$[\![X, Y]\!]^{\mathrm{fn}} + (-1)^{XY}[\![Y, X]\!]^{\mathrm{fn}} = 0, \tag{4.41}$$

$$\oint (-1)^{Y(X+Z)}[\![X, [\![Y, Z]\!]^{\mathrm{fn}}]\!]^{\mathrm{fn}} = 0, \tag{4.42}$$

$$\mathrm{i}_{[\![X,Y]\!]^{\mathrm{fn}}} = [\mathrm{L}_X, \mathrm{i}_Y] + (-1)^{X(Y+1)}\mathrm{L}_{\mathrm{i}_Y X}, \tag{4.43}$$

$$\begin{aligned} \mathrm{i}_Z[\![X, Y]\!]^{\mathrm{fn}} = [\![\mathrm{i}_Z X, Y]\!]^{\mathrm{fn}} &+ (-1)^{X(Z+1)}[\![X, \mathrm{i}_Z Y]\!]^{\mathrm{fn}} \\ &+ (-1)^X \mathrm{i}_{[\![Z,X]\!]^{\mathrm{fn}}} Y - (-1)^{(X+1)Y}\mathrm{i}_{[\![Z,Y]\!]^{\mathrm{fn}}} X, \end{aligned} \tag{4.44}$$

$$\begin{aligned} [\![X, \omega \wedge Y]\!]^{\mathrm{fn}} = \mathrm{L}_X\omega \wedge Y &- (-1)^{(X+1)(Y+\omega)} d\omega \wedge \mathrm{i}_Y X \\ &+ (-1)^{X\omega}\omega \wedge [\![X, Y]\!]^{\mathrm{fn}}. \end{aligned} \tag{4.45}$$

Note that the first two equalities in the previous proposition mean that the module $D(\Lambda^*(A))$ is a Lie superalgebra with respect to the Frölicher–Nijenhuis bracket.

REMARK 4.8. The above exposed algebraic scheme has a geometrical realization, if one takes $A = C^\infty(M)$, M being a smooth finite-dimensional manifold. The algebra $A = C^\infty(M)$ is smooth in this case. However, in the geometrical theory of differential equations we have to work with infinite-dimensional manifolds[3] of the form $N = \operatorname{proj\,lim}_{\{\pi_{k+1,k}\}} N_k$, where

[3]Infinite jets, infinite prolongations of differential equations, total spaces of coverings, etc.

all the mappings $\pi_{k+1,k}\colon N_{k+1} \to N_k$ are surjections of finite-dimensional smooth manifolds. The corresponding algebraic object is a filtered algebra $A = \bigcup_{k\in\mathbb{Z}} A_k$, $A_k \subset A_{k+1}$, where all A_k are subalgebras in A. As it was already noted, self-contained differential calculus over A is constructed, if one considers the category of all filtered A-modules with filtered homomorphisms for morphisms between them. Then all functors of differential calculus in this category become filtered, as well as their representative objects.

In particular, the A-modules $\Lambda^i(A)$ are filtered by A_k-modules $\Lambda^i(A_k)$. We say that the algebra A is *finitely smooth*, if $\Lambda^1(A_k)$ is a projective A_k-module of finite type for any $k \in \mathbb{Z}$. For finitely smooth algebras, elements of $D(P)$ may be represented as formal infinite sums $\sum_k p_k \otimes X_k$, such that any finite sum $S_n = \sum_{k\le n} p_k \otimes X_k$ is a derivation $A_n \to P_{n+s}$ for some fixed $s \in \mathbb{Z}$. Any derivation X is completely determined by the system $\{S_n\}$ and Proposition 4.22 obviously remains valid.

REMARK 4.9. In fact, the Frölicher–Nijenhuis bracket can be defined in a completely general situation, with no additional assumption on the algebra A. To do this, it suffices to define $[\![X,Y]\!]^{\mathrm{fn}} = [X,Y]$, when $X, Y \in D_1(A)$ and then use equality (4.44) as inductive definition. Gaining in generality, we then loose of course in simplicity of proofs.

2.2. Algebras with flat connections and cohomology. We now introduce the second object of our interest. Let A be an $\Bbbk$-algebra, $\Bbbk$ being a field of zero characteristic, and B be an algebra over A. We shall assume that the corresponding homomorphism $\varphi\colon A \to B$ is an embedding. Let P be a B-module; then it is an A-module as well and we can consider the B-module $D(A,P)$ of P-valued derivations $A \to P$.

DEFINITION 4.11. Let $\nabla^\bullet\colon D(A,\bullet) \Rightarrow D(\bullet)$ be a natural transformations of the functors $D(A,\bullet)\colon A \Rightarrow D(A,P)$ and $D(\bullet)\colon P \Rightarrow D(P)$ in the category of B-modules, i.e., a system of homomorphisms $\nabla^P\colon D(A,P) \to D(P)$ such that the diagram

$$\begin{array}{ccc} D(A,P) & \xrightarrow{\nabla^P} & D(P) \\ {\scriptstyle D(A,f)}\downarrow & & \downarrow{\scriptstyle D(f)} \\ D(A,Q) & \xrightarrow{\nabla^Q} & D(Q) \end{array}$$

is commutative for any B-homomorphism $f\colon P \to Q$. We say that $\nabla^\bullet$ is a *connection* in the triad (A, B, φ), if $\nabla^P(X)\big|_A = X$ for any $X \in D(A,P)$.

Here and below we use the notation $Y|_A = Y \circ \varphi$ for any derivation $Y \in D(P)$.

REMARK 4.10. When $A = C^\infty(M)$, $B = C^\infty(E)$, $\varphi = \pi^*$, where M and E are smooth manifolds and $\pi\colon E \to M$ is a smooth fiber bundle, Definition

4.11 reduces to the ordinary definition of a connection in the bundle π. In fact, if we have a connection $\nabla^\bullet$ in the sense of Definition 4.11, then the correspondence

$$D(A) \hookrightarrow D(A,B) \xrightarrow{\nabla^B} D(B)$$

allows one to lift any vector field on M up to a π-projectable field on E. Conversely, if ∇ is such a correspondence, then we can construct a natural transformation $\nabla^\bullet$ of the functors $D(A,\bullet)$ and $D(\bullet)$ due to the fact that for smooth finite-dimensional manifolds one has $D(A,P) = P \otimes_A D(A)$ and $D(P) = P \otimes_B D(P)$ for an arbitrary B-module P. We use the notation $\nabla = \nabla^B$ in the sequel.

Definition 4.12. Let $\nabla^\bullet$ be a connection in (A,B,φ) and consider two derivations $X, Y \in D(A,B)$. The *curvature form* of the connection $\nabla^\bullet$ on the pair X, Y is defined by

$$R_\nabla(X,Y) = [\nabla(X), \nabla(Y)] - \nabla(\nabla(X)\circ Y - \nabla(Y)\circ X). \tag{4.46}$$

Note that (4.46) makes sense, since $\nabla(X)\circ Y - \nabla(Y)\circ X$ is a B-valued derivation of A.

Consider now the de Rham differential $d = d_B\colon B \to \Lambda^1(B)$. Then the composition $d_B \circ \varphi\colon A \to B$ is a derivation. Consequently, we may consider the derivation $\nabla(d_B\circ\varphi) \in D(\Lambda^1(B))$.

Definition 4.13. The element $U_\nabla \in D(\Lambda^1(B))$ defined by

$$U_\nabla = \nabla(d_B\circ\varphi) - d_B \tag{4.47}$$

is called the *connection form* of ∇.

Directly from the definition we obtain the following

Lemma 4.23. *The equality*

$$\mathrm{i}_X(U_\nabla) = X - \nabla(X|_A) \tag{4.48}$$

holds for any $X \in D(B)$.

Using this result, we now prove

Proposition 4.24. *If B is a smooth algebra, then*

$$\mathrm{i}_Y\mathrm{i}_X[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2R_\nabla(X|_A, Y|_A) \tag{4.49}$$

for any $X, Y \in D(B)$.

Proof. First note that $\deg U_\nabla = 1$. Then using (4.44) and (4.41) we obtain

$$\begin{aligned}\mathrm{i}_X[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = [\![\mathrm{i}_X U_\nabla, U_\nabla]\!]^{\mathrm{fn}} + [\![U_\nabla, \mathrm{i}_X U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X,U_\nabla]\!]^{\mathrm{fn}}}U_\nabla - \mathrm{i}_{[\![X,U_\nabla]\!]^{\mathrm{fn}}}U_\nabla\\ = 2\big([\![\mathrm{i}_X U_\nabla, U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X,U_\nabla]\!]^{\mathrm{fn}}}U_\nabla\big).\end{aligned}$$

Applying i_Y to the last expression and using (4.42) and (4.44), we get now

$$\mathrm{i}_Y \mathrm{i}_X [\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2\big([\![\mathrm{i}_X U_\nabla, \mathrm{i}_Y U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X,Y]\!]^{\mathrm{fn}}} U_\nabla\big).$$

But $[\![V,W]\!]^{\mathrm{fn}} = [V,W]$ for any $V, W \in D(\Lambda^0(A)) = D(A)$. Hence, by (4.48), we have

$$\mathrm{i}_Y \mathrm{i}_X [\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2\big([X - \nabla(X|_A), Y - \nabla(Y|_A)] - ([X,Y] - \nabla([X,Y]|_A))\big).$$

It only remains to note now that $\nabla(X|_A)|_A = X|_A$ and $[X,Y]|_A = X \circ Y|_A - Y \circ X|_A$. $\square$

DEFINITION 4.14. A connection ∇ in (A, B, φ) is called *flat*, if $R_\nabla = 0$.

Fix an algebra A and let us introduce the category $\mathcal{FC}(A)$, whose *objects* are triples (A, B, φ) endowed with a connection $\nabla^\bullet$ while morphisms are defined as follows. Let $\mathcal{O} = (A, B, \varphi, \nabla^\bullet)$ and $\widetilde{\mathcal{O}} = (A, \widetilde{B}, \widetilde{\varphi}, \widetilde{\nabla}^\bullet)$ be two objects of $\mathcal{FC}(A)$. Then a *morphism* from $\mathcal{O}$ to $\widetilde{\mathcal{O}}$ is a mapping $f\colon B \to \widetilde{B}$ such that:

(i) f is an A-algebra homomorphism, i.e., the diagram

$$\begin{tikzcd} B \arrow[rr, "f"] & & \widetilde{B} \\ & A \arrow[ul, "\varphi"] \arrow[ur, "\widetilde{\varphi}"'] & \end{tikzcd}$$

is commutative, and

(ii) for any $\widetilde{B}$-module P (which can be considered as a B-module as well due to the homomorphism f the diagram

$$\begin{tikzcd} D(\widetilde{B}, P) \arrow[rr, "D(\widetilde{B}, f)"] & & D(B, P) \\ & D(A, P) \arrow[ul, "\widetilde{\nabla}^P"] \arrow[ur, "\nabla^P"'] & \end{tikzcd}$$

is commutative, where $D(\widetilde{B}, f)(X) = X \circ f$ for any derivation $X\colon \widetilde{B} \to P$.

Due to Proposition 4.24, for flat connections we have

$$[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 0. \tag{4.50}$$

Let $U \in D(\Lambda^1(B))$ be an element satisfying equation (4.50). Then from the graded Jacobi identity (4.42) we obtain

$$2[\![U, [\![U, X]\!]^{\mathrm{fn}}]\!]^{\mathrm{fn}} = [\![[\![U, U]\!]^{\mathrm{fn}}, X]\!]^{\mathrm{fn}} = 0$$

for any $X \in D(\Lambda^*(A))$. Consequently, the operator

$$\partial_U = [\![U, \cdot]\!]^{\mathrm{fn}} \colon D(\Lambda^i(B)) \to D(\Lambda^{i+1}(B))$$

defined by the equality $\partial_U(X) = [\![U, X]\!]^{\mathrm{fn}}$ satisfies the identity $\partial_U \circ \partial_U = 0$.

Consider now the case $U = U_\nabla$, where ∇ is a flat connection.

DEFINITION 4.15. An element $X \in D(\Lambda^*(B))$ is called *vertical*, if $X(a) = 0$ for any $a \in A$. Denote the B-submodule of such elements by $D^v(\Lambda^*(B))$.

LEMMA 4.25. *Let ∇ be a connection in (A, B, φ). Then*

(1) *an element $X \in D(\Lambda^*(B))$ is vertical if and only if $\mathrm{i}_X U_\nabla = X$;*
(2) *the connection form U_∇ is vertical, $U_\nabla \in D^v(\Lambda^1(B))$;*
(3) *the mapping ∂_{U_∇} preserves verticality, i.e., for all i one has the embeddings $\partial_{U_\nabla}(D^v(\Lambda^i(B))) \subset D^v(\Lambda^{i+1}(B))$.*

PROOF. To prove (1), use Lemma 4.23: from (4.48) it follows that $\mathrm{i}_X U_\nabla = X$ if and only if $\nabla(X|_A) = 0$. But $\nabla(X|_A)|_A = X|_A$. The second statements follows from the same lemma and from the first one:

$$\mathrm{i}_{U_\nabla} U_\nabla = U_\nabla - \nabla(U_\nabla|_A) = U_\nabla - \nabla\big((U_\nabla - \nabla(U_\nabla|_A))|_A \big) = U_\nabla.$$

Finally, (3) is a consequence of (4.44). □

DEFINITION 4.16. Denote the restriction $\partial_{U_\nabla}|_{D^v(\Lambda^*(A))}$ by ∂_∇ and call the complex

$$0 \to D^v(B) \xrightarrow{\partial_\nabla} D^v(\Lambda^1(B)) \to \cdots \to D^v(\Lambda^i(B)) \xrightarrow{\partial_\nabla} D^v(\Lambda^{i+1}(B)) \to \cdots \tag{4.51}$$

the ∇-*complex* of the triple (A, B, φ). The corresponding cohomology is denoted by $H^*_\nabla(B; A, \varphi) = \bigoplus_{i \geq 0} H^i_\nabla(B; A, \varphi)$ and is called the ∇-*cohomology* of the triple (A, B, φ).

Introduce the notation

$$d^v_\nabla = \mathrm{L}_{U_\nabla} \colon \Lambda^i(B) \to \Lambda^{i+1}(B). \tag{4.52}$$

PROPOSITION 4.26. *Let ∇ be a flat connection in a triple (A, B, φ) and B be a smooth (or finitely smooth) algebra. Then for any $X, Y \in D^v(\Lambda^*(A))$ and $\omega \in \Lambda^*(A)$ one has*

$$\partial_\nabla [\![X, Y]\!]^{\mathrm{fn}} = [\![\partial_\nabla X, Y]\!]^{\mathrm{fn}} + (-1)^X [\![X, \partial_\nabla Y]\!]^{\mathrm{fn}}, \tag{4.53}$$

$$[\mathrm{i}_X, \partial_\nabla] = (-1)^X \mathrm{i}_{\partial_\nabla X}, \tag{4.54}$$

$$\partial_\nabla(\omega \wedge X) = (d^v_\nabla - d)(\omega) \wedge X + (-1)^\omega \omega \wedge \partial_\nabla X, \tag{4.55}$$

$$[d^v_\nabla, \mathrm{i}_X] = \mathrm{i}_{\partial_\nabla X} + (-1)^X \mathrm{L}_X. \tag{4.56}$$

PROOF. Equality (4.53) is a direct consequence of (4.42). Equality (4.54) follows from (4.44). Equality (4.55) follows from (4.45) and (4.48). Finally, (4.56) is obtained from (4.43). □

COROLLARY 4.27. *The module $H^*_\nabla(B;A,\varphi)$ inherits the graded Lie algebra structure with respect to the Frölicher–Nijenhuis bracket $[\![\cdot,\cdot]\!]^{\mathrm{fn}}$, as well as the contraction operation.*

PROOF. Note that $D^v(\Lambda^*(A))$ is closed with respect to the Frölicher–Nijenhuis bracket: to prove this fact, it suffices to apply (4.44). Then the first statement follows from (4.53). The second one is a consequence of (4.54). $\square$

REMARK 4.11. We preserve the same notations for the inherited structures. Note, in particular, that $H^0_\nabla(B;A,\varphi)$ is a Lie algebra with respect to the Frölicher–Nijenhuis bracket (which reduces to the ordinary Lie bracket in this case). Moreover, $H^1_\nabla(B;A,\varphi)$ is an associative algebra with respect to the inherited contraction, while the action

$$\mathcal{R}_\Omega\colon X \mapsto \mathrm{i}_X\Omega, \quad X \in H^0_\nabla(B;A,\varphi), \quad \Omega \in H^1_\nabla(B;A,\varphi)$$

is a representation of this algebra as endomorphisms of $H^0_\nabla(B;A,\varphi)$.

Consider now the mapping $d^v_\nabla\colon \Lambda^*(B) \to \Lambda^*(B)$ defined by (4.52) and define $d^h_\nabla = d_B - d^v_\nabla$.

PROPOSITION 4.28. *Let B be a (finitely) smooth algebra and ∇ be a flat connection in the triple $(B;A,\varphi)$. Then*

(1) *The pair (d^h_∇, d^v_∇) forms a bicomplex, i.e.,*

$$d^v_\nabla \circ d^v_\nabla = 0, \quad d^h_\nabla \circ d^h_\nabla = 0, \quad d^h_\nabla \circ d^v_\nabla + d^v_\nabla \circ d^h_\nabla = 0. \tag{4.57}$$

(2) *The differential d^h_∇ possesses the following properties*

$$[d^h_\nabla, \mathrm{i}_X] = -\mathrm{i}_{\partial_\nabla X}, \tag{4.58}$$

$$\partial_\nabla(\omega \wedge X) = -d^h_\nabla(\omega) \wedge X + (-1)^\omega \omega \wedge \partial_\nabla X, \tag{4.59}$$

where $\omega \in \Lambda^*(B)$, $X \in D^v(\Lambda^*(B))$.

PROOF. (1) Since $\deg d^v_\nabla = 1$, we have

$$2d^v_\nabla \circ d^v_\nabla = [d^v_\nabla, d^v_\nabla] = [\mathrm{L}_{U_\nabla}, \mathrm{L}_{U_\nabla}] = \mathrm{L}_{[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}}} = 0.$$

Since $d^v_\nabla = \mathrm{L}_{U_\nabla}$, the identity $[d_B, d^v_\nabla] = 0$ holds (see (4.38)), and it concludes the proof of the first part.

(2) To prove (4.58), note that

$$[d^h_\nabla, \mathrm{i}_X] = [d_B - d^h_\nabla, \mathrm{i}_X] = (-1)^X \mathrm{L}_X - [d^v_\nabla, \mathrm{i}_X],$$

and (4.58) holds due to (4.56). Finally, (4.59) is just the other form of (4.55). $\square$

DEFINITION 4.17. Let ∇ be a connection in (A,B,φ).

(1) The bicomplex $(B, d^h_\nabla, d^v_\nabla)$ is called the *variational bicomplex* associated to the connection ∇.
(2) The corresponding spectral sequence is called the ∇-*spectral sequence* of the triple (A,B,φ).

Obviously, the ∇-spectral sequence converges to the de Rham cohomology of B.

To finish this section, note the following. Since the module $\Lambda^1(B)$ is generated by the image of the operator $d_B\colon B \to \Lambda^1(B)$ while the graded algebra $\Lambda^*(B)$ is generated by $\Lambda^1(B)$, we have the direct sum decomposition

$$\Lambda^*(B) = \bigoplus_{i\geq 0} \bigoplus_{p+q=i} \Lambda_v^p(B) \otimes \Lambda_h^q(B),$$

where

$$\Lambda_v^p(B) = \underbrace{\Lambda_v^1(B) \wedge \cdots \wedge \Lambda_v^1(B)}_{p \text{ times}}, \quad \Lambda_h^q(B) = \underbrace{\Lambda_h^1(B) \wedge \cdots \wedge \Lambda_h^1(B)}_{q \text{ times}},$$

while the submodules $\Lambda_v^1(B) \subset \Lambda^1(B)$, $\Lambda_h^1(B) \subset \Lambda^1(B)$ are spanned in $\Lambda^1(B)$ by the images of the differentials d_∇^v and d_∇^h respectively. Obviously, we have the following embeddings:

$$d_\nabla^h \left(\Lambda_v^p(B) \otimes \Lambda_h^q(B)\right) \subset \Lambda_v^p(B) \otimes \Lambda_h^{q+1}(B),$$
$$d_\nabla^v \left(\Lambda_v^p(B) \otimes \Lambda_h^q(B)\right) \subset \Lambda_v^{p+1}(B) \otimes \Lambda_h^q(B).$$

Denote by $D^{p,q}(B)$ the module $D^v(\Lambda_v^p(B) \otimes \Lambda_h^q(B))$. Then, obviously, $D^v(B) = \bigoplus_{i\geq 0} \bigoplus_{p+q=i} D^{p,q}(B)$, while from equalities (4.58) and (4.59) we obtain

$$\partial_\nabla\left(D^{p,q}(B)\right) \subset D^{p,q+1}(B).$$

Consequently, the module $H_\nabla^*(B; A, \varphi)$ is split as

$$H_\nabla^*(B; A, \varphi) = \bigoplus_{i\geq 0} \bigoplus_{p+q=i} H_\nabla^{p,q}(B; A, \varphi) \tag{4.60}$$

with the obvious meaning of the notation $H_\nabla^{p,q}(B; A, \varphi)$.

PROPOSITION 4.29. *If $\mathcal{O} = (B, \nabla)$ is an object of the category $\mathcal{FC}(A)$, then*

$$H_\nabla^{p,0}(B) = \ker\left(\partial_\nabla \Big|_{D_1^v(\mathcal{C}^p\Lambda(B))}\right).$$

3. Structure of symmetry algebras

Here we expose the theory of symmetries and recursion operators in the categories $\mathcal{FC}(A)$. Detailed motivations for the definition can be found in previous chapters as well as in Chapter 5. A brief discussion concerning relations of this algebraic scheme to further applications to differential equations the reader will find in concluding remarks below.

3.1. Recursion operators and structure of symmetry algebras. We start with the following

DEFINITION 4.18. Let $\mathcal{O} = (B, \nabla)$ be an object of the category $\mathcal{FC}(A)$.
(i) The elements of $H^{0,0}_\nabla(B) = H^0_\nabla(B)$ are called *symmetries* of $\mathcal{O}$.
(ii) The elements of $H^{1,0}_\nabla(B)$ are called *recursion operators* of $\mathcal{O}$.

We use the notations

$$\mathrm{Sym} \overset{\text{def}}{=} H^{0,0}_\nabla(B)$$

and

$$\mathrm{Rec} \overset{\text{def}}{=} H^{1,0}_\nabla(B).$$

From Corollary 4.27 and Proposition 4.29 one obtains

THEOREM 4.30. *For any object $\mathcal{O} = (B, \nabla)$ of the category $\mathcal{FC}(A)$ the following facts take place*:
(i) Sym *is a Lie algebra with respect to commutator of derivations.*
(ii) Rec *is an associative algebra with respect to contraction*, U_∇ *being the unit of this algebra.*
(iii) *The mapping* $\mathcal{R}\colon \mathrm{Rec} \to \mathrm{End}_\Bbbk(\mathrm{Sym})$, *where*

$$\mathcal{R}_\Omega(X) = \mathrm{i}_X(\Omega), \quad \Omega \in \mathrm{Rec}, X \in \mathrm{Sym},$$

is a representation of this algebra and hence
(iv) $i_{(\mathrm{Sym})}(\mathrm{Rec}) \subset \mathrm{Sym}$.

In what follows we shall need a simple consequence of basic definitions:

PROPOSITION 4.31. *For any object $\mathcal{O} = (B, \nabla)$ of $\mathcal{FC}(A)$*

$$[\![\, \mathrm{Sym}, \mathrm{Rec}]\!] \subset \mathrm{Rec}$$

and

$$[\![\mathrm{Rec}, \mathrm{Rec}]\!] \subset H^{2,0}_\nabla(B).$$

COROLLARY 4.32. *If $H^{2,0}_\nabla(B) = 0$, then all recursion operators of the object $\mathcal{O} = (B, \nabla)$ commute with each other with respect to the Frölicher–Nijenhuis bracket.*

We call the objects satisfying the conditions of the previous corollary 2-*trivial.* To simplify notations we denote

$$\mathcal{R}_\Omega(X) = \Omega(X), \quad \Omega \in \mathrm{Rec}, \quad X \in \mathrm{Sym}.$$

From Proposition 4.31 and equality (4.42) one gets

PROPOSITION 4.33. *Consider an object $\mathcal{O} = (B, \nabla)$ of $\mathcal{FC}(A)$ and let $X, Y \in \mathrm{Sym}$, $\Omega, \theta \in \mathrm{Rec}$. Then*

$$\begin{aligned}[\![\Omega, \theta]\!](X, Y) = [\Omega(X), \theta(Y)] + [\theta(X), \Omega(Y)] - \Omega([\theta(X), Y] \\ + [X, \theta(Y)]) - \theta([\Omega(X), Y] + [X, \Omega(Y)]) + (\Omega \circ \theta + \theta \circ \Omega)\, [X, Y].\end{aligned}$$

In particular, for $\Omega = \theta$ *one has*

$$\frac{1}{2}[\![\Omega, \Omega]\!](X, Y) = [\Omega(X), \Omega(Y)] \\ - \Omega([\Omega(X), Y]) - \Omega([X, \Omega(Y)]) + \Omega(\Omega([X, Y])). \quad (4.61)$$

The proof of this statement is similar to that of Proposition 4.24. The right-hand side of (4.61) is called the *Nijenhuis torsion* of Ω (cf. [**49**]).

COROLLARY 4.34. *If* $\mathcal{O}$ *is a* 2*-trivial object, then*

$$[\Omega(X), \Omega(Y)] = \Omega\left([\Omega(X), Y] + [X, \Omega(Y)] - \Omega[X, Y]\right). \quad (4.62)$$

Choose a recursion operator $\Omega \in \mathrm{Rec}$ and for any symmetry $X \in \mathrm{Sym}$ denote $\Omega^i(X) = \mathcal{R}^i_\Omega(X)$ by X_i. Then (4.62) can be rewritten as

$$[X_1, Y_1] = [X_1, Y]_1 + [X, Y_1]_1 - [X, Y]_2. \quad (4.63)$$

Using (4.63) as the induction base, one can prove the following

PROPOSITION 4.35. *For any* 2*-trivial object* $\mathcal{O}$ *and* $m, n \geq 1$ *one has*

$$[X_m, Y_n] = [X_m, Y]_n + [X, Y_n]_m - [X, Y]_{m+n}.$$

Let, as before, X be a symmetry and Ω be a recursion operator. Then $\Omega_X \stackrel{\text{def}}{=} [\![X, \Omega]\!]$ is a recursion operator again (Proposition 4.31). Due to (4.42), its action on $Y \in \mathrm{Sym}$ can be expressed as

$$\Omega_X(Y) = [X, \Omega(Y)] - \Omega[X, Y]. \quad (4.64)$$

From (4.64) one has

PROPOSITION 4.36. *For any* 2*-trivial object* $\mathcal{O}$*, symmetries* $X, Y \in \mathrm{Sym}$*, a recursion* $\Omega \in \mathrm{Rec}$*, and integers* $m, n \geq 1$ *one has*

$$[X, Y_n] = [X, Y]_n + \sum_{i=0}^{n-1} (\Omega_X Y_i)_{n-i-1}$$

and

$$[X_m, Y] = [X, Y]_m - \sum_{j=0}^{m-1} (\Omega_Y X_j)_{m-j-1}.$$

From the last two results one obtains

THEOREM 4.37 (the structure of a Lie algebra for Sym). *For any* 2*-trivial object* $\mathcal{O}$*, its symmetries* $X, Y \in \mathrm{Sym}$*, a recursion operator* $\Omega \in \mathrm{Rec}$*, and integers* $m, n \geq 1$ *one has*

$$[X_m, Y_n] = [X, Y]_{m+n} + \sum_{i=0}^{n-1} (\Omega_X Y_i)_{m+n-i-1} - \sum_{j=0}^{m-1} (\Omega_Y X_j)_{m+n-j-1}.$$

COROLLARY 4.38. *If* $X, Y \in \mathrm{Sym}$ *are such that* Ω_X *and* Ω_Y *commute with* $\Omega \in \mathrm{Rec}$ *with respect to the Richardson–Nijenhuis bracket, then*

$$[X_m, Y_n] = [X, Y]_{m+n} + n(\Omega_X Y)_{m+n-1} - m(\Omega_Y X)_{m+n-1}.$$

We say that a recursion operator $\Omega \in \mathrm{Rec}$ is *X-invariant*, if $\Omega_X = 0$.

COROLLARY 4.39 (on infinite series of commuting symmetries). *If $\mathcal{O}$ is a 2-trivial object and if a recursion operator $\Omega \in \mathrm{Rec}$ is X-invariant, $X \in \mathrm{Sym}$, then a hierarchy $\{X_n\}$, $n = 0, 1, \dots,$ generated by X and Ω is commutative*:

$$[X_m, X_n] = 0$$

for all m, n.

3.2. Concluding remarks. Here we briefly discuss relations of the above exposed algebraic scheme to geometry of partial differential equations exposed in the previous chapters and the theory of recursion operators discussed in Chapters 5–7.

First recall that correspondence between algebraic approach and geometrical picture is established by identifying the category of vector bundles over a smooth manifold M with the category of geometrical modules over $A = C^\infty(M)$, see [**60**]. In the case of differential equations, M plays the role of the manifold of independent variables while $B = \bigcup_\alpha B_\alpha$ is the function algebra on the infinite prolongation of the equation $\mathcal{E}$ and $B_\alpha = C^\infty(\mathcal{E}^\alpha)$, where $\mathcal{E}^\alpha$, $\alpha = 0, 1, \dots, \infty$, is the α-prolongation of $\mathcal{E}$. The mapping $\varphi\colon A \to B$ is dual to the natural projection $\pi_\infty\colon \mathcal{E}^\infty \to M$ and thus in applications to differential equations it suffices to consider the case $A = \bigcup_\alpha B_\alpha$.

If $\mathcal{E}$ is a formally integrable equation, the bundle $\pi_\infty\colon \mathcal{E}^\infty \to M$ possesses a natural connection (the Cartan connection $\mathcal{C}$) which takes a vector field X on M to corresponding total derivative on $\mathcal{E}^\infty$. Consequently, the category of differential equations [**100**] is embedded to the category of algebras with flat connections $\mathcal{FC}(C^\infty(M))$. Under this identification the spectral sequence defined in Definition 4.17 coincides with A. Vinogradov's $\mathcal{C}$-spectral sequence [**102**] (or variational bicomplex), the module Sym, where $\mathcal{O} = (C^\infty(M), C^\infty(\mathcal{E}^\infty), \mathcal{C})$, is the Lie algebra of higher symmetries for the equation $\mathcal{E}$ and, in principle, Rec consists of recursion operators for these symmetries. This last statement should be clarified.

In fact, as we shall see later, if one tries to compute the algebra Rec straightforwardly, the results will be trivial usually — even for equations which really possess recursion operators. The reason lies in nonlocal character of recursion operators for majority of interesting equations [**1, 31, 4**]. Thus extension of the algebra $C^\infty(\mathcal{E}^\infty)$ with nonlocal variables (see 3) is the way to obtain nontrivial solutions — and actual computation show that all known (as well as new ones!) recursion operators can be obtained in such a way (see examples below and in [**58, 40**]). In practice, it usually suffices to extend $C^\infty(\mathcal{E}^\infty)$ by integrals of conservation laws (of a sufficiently high order).

The algorithm of computations becomes rather simple due to the following fact. It will shown that for non-overdetermined equations all cohomology groups $H_{\mathcal{C}}^{p,q}(\mathcal{E})$ are trivial except for the cases $q = 0, 1$ while the differential $\partial_{\mathcal{C}}\colon D_1^v(\mathcal{C}^p(\mathcal{E})) \to D_1^v(\mathcal{C}^p(\mathcal{E}) \wedge \Lambda_1^h(\mathcal{E}))$ coincides with the universal linearization operator $\ell_{\mathcal{E}}$ of the equation $\mathcal{E}$ extended to the module of Cartan forms. Therefore, the modules $H_{\mathcal{C}}^{p,0}(\mathcal{E})$ coincide with $\ker(\ell_{\mathcal{E}})$ (see 4.29)

$$H_{\mathcal{C}}^{p,0}(\mathcal{E}) = \ker(\ell_{\mathcal{E}}) \tag{4.65}$$

and thus can be computed efficiently.

In particular, it will shown that for scalar evolution equations all cohomologies $H_{\mathcal{C}}^{p,0}(\mathcal{E})$, $p \geq 2$, vanish and consequently equations of this type are 2-trivial and satisfy the conditions of Theorem 4.37 which explains commutativity of some series of higher symmetries (e.g., for the KdV equation).

CHAPTER 5

Deformations and recursion operators

In this chapter, we apply the algebraic formalism of Chapter 4 to the specific case of partial differential equations. Namely, we consider a formally integrable equation $\mathcal{E} \subset J^k(\pi)$, $\pi \colon E \to M$, taking the associated triple $(C^\infty(M), \mathcal{F}(\mathcal{E}), \pi_\infty^*)$ for the algebra with flat connection, where $\mathcal{F}(\mathcal{E}) = \bigcup_k \mathcal{F}_k(\mathcal{E})$ is the algebra of smooth functions on $\mathcal{E}^\infty$, $\pi_\infty \colon \mathcal{E}^\infty \to M$ is the natural projection and the Cartan connection $\mathcal{C}$ plays the role of ∇.

We compute the corresponding cohomology groups for the case $\mathcal{E}^\infty = J^\infty(\pi)$ and deduce defining equations for a general $\mathcal{E}$. We also establish relations between infinitesimal deformations of the equation structure and recursion operators for symmetries and consider several illustrative examples.

We start with repeating some definitions and proofs of the previous chapter in the geometrical situation.

1. $\mathcal{C}$-cohomologies of partial differential equations

Here we introduce cohomological invariants of partial differential equations based on the results of Sections 1, 2 of Chapter 4. We call these invariants *$\mathcal{C}$-cohomologies* since they are determined by the Cartan connection $\mathcal{C}$ on $\mathcal{E}^\infty$. We follow the scheme from the classical paper by Nijenhuis and Richardson [**78**], especially in interpretation of the cohomology in question.

Let $\xi \colon P \to M$ be a fiber bundle with a connection ∇, which is considered as a $C^\infty(M)$-homomorphism $\nabla \colon D(M) \to D(P)$ taking a field $X \in D(M)$ to the field $\nabla(X) = \nabla_X \in D(P)$ and satisfying the condition $\nabla_X(\xi^* f) = X(f)$ for any $f \in C^\infty(M)$.

Let $y \in P$, $\xi(y) = x \in M$, and denote by $P_y = \xi^{-1}(x)$ the fiber of the projection ξ passing through y. Then ∇ determines a linear mapping $\nabla_y \colon T_x(M) \to T_y(P)$ such that $\xi_{*,y}(\nabla_y(v)) = v$ for any $v \in T_x(M)$. Thus with any point $y \in P$ a linear subspace $\nabla_y(T_x(M)) \subset T_y(P)$ is associated. It determines a distribution $\mathcal{D}_\nabla$ on P which is called the *horizontal distribution* of the connection ∇. If ∇ is flat, then $\mathcal{D}_\nabla$ is integrable.

As it is well known (see, for example, [**46, 47**]), the *connection form* $U = U_\nabla \in \Lambda^1(P) \otimes D(P)$ can be defined as follows. Let $y \in P$, $Y \in D(P)$, $Y_y \in T_y(P)$ and $v = \xi_{*,y}(Y_y)$. Then we set

$$(Y \lrcorner U_\nabla)_y = Y_y - \nabla_y(v). \tag{5.1}$$

In other words, the value of U_∇ at the vector $Y_y \in T_y(P)$ is the projection of Y_y onto the tangent plane $T_y(P)$ along the horizontal plane[1] passing through $y \in P$.

If $(x_1, \dots, x_n)$ are local coordinates in M and $(y^1, \dots, y^s)$ are coordinates along the fiber of ξ (the case $s = \infty$ is included), we can define ∇ by the following equalities

$$\nabla\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial}{\partial x_i} + \sum_{j=1}^{s} \nabla_i^j \frac{\partial}{\partial y^j} = \nabla_i. \tag{5.2}$$

Then U_∇ is of the form

$$U_\nabla = \sum_{j=1}^{s} \Big(dy^j - \sum_{i=1}^{n} \nabla_i^j \, dx_i\Big) \otimes \frac{\partial}{\partial y^j}, \tag{5.3}$$

From equality (4.40) on p. 175 it follows that

$$[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2 \sum_{i,j,k} \Big(\frac{\partial \nabla_j^k}{\partial x_i} + \sum_\alpha \nabla_i^\alpha \frac{\partial \nabla_j^k}{\partial y^\alpha}\Big) dx_i \wedge dx_j \otimes \frac{\partial}{\partial y^k}. \tag{5.4}$$

Recall that the *curvature form* R_∇ of the connection ∇ is defined by the equality

$$R_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in D(M).$$

We shall express the element $[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}}$ in terms of the form R_∇ now (cf. Proposition 4.24). Let us consider a field $X \in D(P)$ and represent it in the form

$$X = X^v + X^h, \tag{5.5}$$

where, by definition,

$$X^v = X \lrcorner U_\nabla, \quad X^h = X - X^v$$

are *vertical* and *horizontal* components of X respectively. In the same manner one can define vertical and horizontal components of any element $\Omega \in \Lambda^*(P) \otimes D(P)$.

Obviously, $X^v \in D^v(P)$, where

$$D^v(P) = \{X \in D(P) \mid X\xi^*(f) = 0, \ f \in C^\infty(M)\},$$

while the component X^h is of the form

$$X^h = \sum_i f_i \nabla_{X_i}, \quad f_i \in C^\infty(P), \ X_i \in D(M),$$

and lies in the distribution $\mathcal{D}_\nabla$.

[1]With respect to ∇.

PROPOSITION 5.1. *Let $\nabla\colon D(M) \to D(P)$ be a connection in the fiber bundle $\xi\colon P \to M$. Then for any ξ-vertical vector field X^v one has*

$$X^v \lrcorner [\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 0.$$

If $X^h = \sum_i f_i \nabla_{X_i}$, $Y^h = \sum_j g_j \nabla_{Y_j}$, $f_i, g_j \in C^\infty(P)$, $X_i, Y_j \in D(M)$, are horizontal vector fields, then

$$Y^h \lrcorner X^h \lrcorner [\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2\sum_{i,j} f_i g_j R_\nabla(X_i, Y_j), \tag{5.6}$$

or, to be short,

$$[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2R_\nabla.$$

PROOF. Let $X \in D(P)$. Then from equality (4.45) on p. 175 it follows that

$$X \lrcorner [\![U, U]\!]^{\mathrm{fn}} = 2([\![U, X]\!]^{\mathrm{fn}} \lrcorner U - [\![U, X \lrcorner U]\!]^{\mathrm{fn}}),$$

where $U = U_\nabla$. Hence, if $X = X^v$ is a vertical field, then

$$X^v \lrcorner [\![U, U]\!]^{\mathrm{fn}} = 2([\![U, X^v]\!]^{\mathrm{fn}} \lrcorner U - [\![U, X^v \lrcorner U]\!]^{\mathrm{fn}}) = -2([\![U, X^v]\!]^{\mathrm{fn}})^h.$$

But the left-hand side of this equality is vertical (see (5.4)) and thus vanishes. This proves the first part of the proposition.

Let now $X = X^h$ be a horizontal vector field. Then

$$X^h \lrcorner [\![U, U]\!]^{\mathrm{fn}} = 2[\![U, X^h]\!]^{\mathrm{fn}} \lrcorner U = 2([\![U, X^h]\!]^{\mathrm{fn}})^v.$$

Hence, if Y^h is another horizontal field, then, by (4.31) on p. 173, one has

$$Y^h \lrcorner (X^h \lrcorner [\![U, U]\!]^{\mathrm{fn}}) = 2Y^h \lrcorner ([\![U, X^h]\!]^{\mathrm{fn}} \lrcorner U) = 2(Y^h \lrcorner [\![U, X^h]\!]^{\mathrm{fn}}) \lrcorner U.$$

But from (4.45) (see p. 175) it follows that

$$Y^h \lrcorner [\![U, X^h]\!]^{\mathrm{fn}} = [\![X^h, Y^h]\!]^{\mathrm{fn}} \lrcorner U = [X^h, Y^h] \lrcorner U.$$

Therefore,

$$\begin{aligned} Y^h \lrcorner X^h \lrcorner [\![U, U]\!]^{\mathrm{fn}} &= 2[X^h, Y^h] \lrcorner U = 2([X^h, Y^h] - [X^h, Y^h]^h) \\ &= 2\sum_{i,j} f_i g_i ([\nabla_{X_i}, \nabla_{Y_j}] - [\nabla_{X_i}, \nabla_{Y_j}]^h) \end{aligned}$$

for $X = \sum_i f_i \nabla_{X_i}$ and $Y = \sum_j g_j \nabla_{Y_j}$. But obviously, for any $f \in C^\infty(M)$ one has

$$[\nabla_{X_i}, \nabla_{Y_j}](f) = [X_i, Y_j](f)$$

and, consequently,

$$[\nabla_{X_i}, \nabla_{Y_j}]^h = \nabla_{[X_i, Y_j]},$$

which finishes the proof. $\square$

From equality (5.6) and from the considerations in the end of Section 2 of Chapter 4 it follows that if the connection in question is flat, i.e. $R_\nabla = 0$, then the element U_∇ determines a complex

$$0 \to D(P) \xrightarrow{\partial^0_\nabla} \Lambda^1(P) \otimes D(P) \to \cdots$$
$$\to \Lambda^i(P) \otimes D(P) \xrightarrow{\partial^i_\nabla} \Lambda^{i+1}(P) \otimes D(P) \to \cdots, \quad (5.7)$$

where $\partial_\nabla = \partial^i_\nabla = [\![U_\nabla, \cdot\,]\!]^{\mathrm{fn}}$.

REMARK 5.1. Horizontal vector fields X^h are defined by the condition $X^h \lrcorner U_\nabla = 0$. Denote the module of such fields by $D^h_\nabla(P)$:

$$D^h_\nabla(P) = \{X \in D(P) \mid X \lrcorner U_\nabla = 0\}.$$

Then, by setting $\Theta = U = U_\nabla$ in (4.31) on p. 173, one can see that

$$\partial_\nabla(\Omega \lrcorner U) = \partial_\nabla(\Omega) \lrcorner U$$

for any $\Omega \in \Lambda^*(P) \otimes D(P)$. Hence,

$$\partial_\nabla(\Lambda^*(P) \otimes D^v(P)) \subset \Lambda^*(P) \otimes D^v(P)$$

and

$$\partial_\nabla(\Lambda^*(P) \otimes D^h_\nabla(P)) \subset \Lambda^*(P) \otimes D^h_\nabla(P).$$

Considering a direct sum decomposition

$$\Lambda^*(P) \otimes D(P) = \Lambda^*(P) \otimes D^v(P) \oplus \Lambda^*(P) \otimes D^h_\nabla(P)$$

one can see that

$$\partial_\nabla = \partial^v_\nabla \oplus \partial^h_\nabla,$$

where

$$\partial^v_\nabla = \partial_\nabla|_{\Lambda^*(P)\otimes D^v(P)}, \quad \partial^h_\nabla = \partial_\nabla|_{\Lambda^*(P)\otimes D^h_\nabla(P)}.$$

To proceed further let us compute 0-cohomology of the complex (5.7).

From equality (4.31) on p. 173 it follows that for any two vector fields $Y, Z \in D(P)$ the equality

$$Z \lrcorner \partial^0_\nabla Y + [Z, Y] \lrcorner U_\nabla = [Z \lrcorner U_\nabla, Y]$$

holds. Thus $Y \in \ker(\partial^0_\nabla)$ if and only if

$$[Z, Y] \lrcorner U_\nabla = [Z \lrcorner U_\nabla, Y]$$

for any $Z \in D(P)$. Using decomposition (5.5) for the fields Y and Z and substituting it into the last equation, we get that the condition $Y \in \ker(\partial^0_\nabla)$ is equivalent to the system of equations

$$[Z^v, Y^h] \lrcorner U_\nabla = [Z^v, Y^h], \quad [Z^h, Y^v] \lrcorner U_\nabla = 0. \quad (5.8)$$

Let $Y^h = \sum_i f_i \nabla_{X_i}$ (see above). Then from the first equality of (5.8) it follows that

$$\sum_i Z^v(f_i)\nabla_{X_i} = \sum_i f_i[\nabla_{X_i}, Z^v].$$

But the left-hand side of this equation is a horizontal vector field while the right-hand side is always vertical. Hence,

$$\sum_i Z^v(f_i)\nabla_{X_i} = 0$$

for any vertical field Z^v. Choosing locally independent vector fields X_i, we see that the functions f_i actually lie in $C^\infty(M)$ (or, strictly speaking, in $\xi^*(C^\infty(M)) \subset C^\infty(P)$). It means that, at least locally, Y^h is of the form

$$Y^h = \nabla_X, \quad X \in D(M).$$

But since $\nabla_X = \nabla_{X'}$ if and only if $X = X'$, the field X is well defined on the whole manifold M.

On the other hand, from the second equality of (5.8) we see that $Y^v \in \ker(\partial^0_\nabla)$ if and only if the commutator $[Z^h, Y^v]$ is a horizontal field for any horizontal Z^h. Thus we get the following result:

PROPOSITION 5.2. *A direct sum decomposition*

$$\ker(\partial^0_\nabla) = D^v_\nabla(P) \oplus \nabla(D(M))$$

takes place, where $\nabla(D(M))$ *is the image of the mapping* $\nabla\colon D(M) \to D(P)$ *and*

$$D^v_\nabla(P) = \{Y \in D^v(P) \mid [Y, D^h_\nabla(P)] \subset D^h_\nabla(P)\}.$$

One can see now that $D^v_\nabla(P)$ consists of nontrivial infinitesimal symmetries of the distribution $\mathcal{D}_\nabla$ while the elements of $\nabla(D(M))$ are trivial symmetries (in the sense that the corresponding transformations slide integral manifolds of $\mathcal{D}_\nabla$ along themselves). To skip this trivial part of $\ker(\partial^0_\nabla)$, note that

(i) $U_\nabla \in \Lambda^1(P) \otimes D^v(P)$,

and (see Remark 5.1)

(ii) $\partial^i_\nabla\big(\Lambda^i(P) \otimes D^v(P)\big) \subset \Lambda^{i+1}(P) \otimes D^v(P)$.

Thus we have a vertical complex

$$0 \to D^v(P) \xrightarrow{\partial^0_\nabla} \Lambda^1(P) \otimes D^v(P) \to \cdots$$
$$\to \Lambda^i(P) \otimes D^v(P) \xrightarrow{\partial^i_\nabla} \Lambda^{i+1}(P) \otimes D^v(P) \to \cdots,$$

the i-th cohomology of which is denoted by $H^i_\nabla(P)$. From the above said it follows that $H^0_\nabla(P)$ coincides with the Lie algebra of nontrivial infinitesimal symmetries for the distribution $\mathcal{D}_\nabla$.

Consider now an infinitely prolonged equation $\mathcal{E}^\infty \subset J^\infty(\pi)$ and the Cartan connection $\mathcal{C} = \mathcal{C}_\mathcal{E}$ in the fiber bundle $\pi_\infty\colon \mathcal{E}^\infty \to M$. The corresponding connection form U_∇, where $\nabla = \mathcal{C}$, will be denoted by $U_\mathcal{E}$ in this case. Knowing the form $U_\mathcal{E}$, one can reconstruct the Cartan distribution on $\mathcal{E}^\infty$. Since this distribution contains all essential information about solutions of $\mathcal{E}$, one can state that $U_\mathcal{E}$ *determines the equation structure* on $\mathcal{E}^\infty$ (see Definition 2.4 in Chapter 2).

By rewriting the vertical complex defined above in the case $\xi = \pi_\infty$, we get a complex

$$0 \to D^v(\mathcal{E}) \xrightarrow{\partial^0_\mathcal{C}} \Lambda^1(\mathcal{E}) \otimes D^v(\mathcal{E}) \to \cdots$$
$$\to \Lambda^i(\mathcal{E}) \otimes D^v(\mathcal{E}) \xrightarrow{\partial^i_\mathcal{C}} \Lambda^{i+1}(\mathcal{E}) \otimes D^v(\mathcal{E}) \to \cdots, \quad (5.9)$$

where, for the sake of simplicity, $\Lambda^i(\mathcal{E})$ stands for $\Lambda^i(\mathcal{E}^\infty)$. The cohomologies of (5.9) are denoted by $H^i_\mathcal{C}(\mathcal{E})$ and are called *$\mathcal{C}$-cohomologies of the equation $\mathcal{E}$.*

From the definition of the Lie algebra $\mathrm{sym}(\mathcal{E})$ and from the previous considerations we get the following

THEOREM 5.3. *For any formally integrable equation $\mathcal{E}$ one has the isomorphism*

$$H^0_\mathcal{C}(\mathcal{E}) = \mathrm{sym}(\mathcal{E}).$$

To obtain an interpretation of the group $H^1_\mathcal{C}(\mathcal{E})$, consider the element $U = U_\mathcal{E} \in \Lambda^1(\mathcal{E}) \otimes D^v(\mathcal{E})$ and its deformation $U(\varepsilon)$, $U(0) = U$, where $\varepsilon \in \mathbb{R}$ is a small parameter. It is natural to expect this deformation to satisfy the following conditions:

(i)
$$U(\varepsilon) \in \Lambda^1(\mathcal{E}) \otimes D^v(\mathcal{E}) \quad \text{(verticality)}$$
and

(ii)
$$[\![U(\varepsilon), U(\varepsilon)]\!]^{\mathrm{fn}} = 0 \qquad \text{(integrability)}. \quad (5.10)$$

Let us expand $U(\varepsilon)$ into a formal series in ε,

$$U(\varepsilon) = U_0 + U_1\varepsilon + \cdots + U_i\varepsilon^i + \cdots, \quad (5.11)$$

and substitute (5.11) into (i) and (ii). Then one can see that $U_1 \in \Lambda^1(\mathcal{E}) \otimes D^v(\mathcal{E})$ and

$$[\![U_0, U_1]\!]^{\mathrm{fn}} = 0.$$

Since $U_0 = U(0) = U$, it follows that $U_1 \in \ker(\partial^1_\mathcal{E})$. Thus $\ker(\partial^1_\mathcal{E})$ consists of all (vertical) *infinitesimal deformations* of U preserving the natural conditions (i) and (ii).

On the other hand, $\operatorname{im}(\partial^0_{\mathcal{E}})$ consists of elements of the form $\partial^0_{\mathcal{E}}(X) = [\![U, X]\!]^{\mathrm{fn}}$, $X \in D^v(\mathcal{E})$. Such elements can be viewed as infinitesimal deformations of U originating from transformations of $\mathcal{E}^\infty$ which are trivial on M (i.e., fiber-wise transformations of the bundle $\pi_\infty \colon \mathcal{E}^\infty \to M$). In fact, let P be a manifold and $A_t \colon P \to P$, $t \in \mathbb{R}$, $A_0 = \mathrm{id}$, be a one-parameter group of diffeomorphisms with

$$\left.\frac{d}{dt}\right|_{t=0} (A_t) = X \in D(P).$$

Then for any $\Theta \in \Lambda^*(P) \otimes D(P)$ one can consider the element $A_{t,*}(\mathrm{L}_\Theta)$ defined by means of the commutative diagram

$$\begin{array}{ccc} \Lambda^*(P) & \xrightarrow{\mathrm{L}_\Theta} & \Lambda^*(P) \\ {\scriptstyle A_t^*}\Big\downarrow & & \Big\downarrow{\scriptstyle A_t^*} \\ \Lambda^*(P) & \xrightarrow{A_{t,*}(\mathrm{L}_\Theta)} & \Lambda^*(P) \end{array} \tag{5.12}$$

Then, obviously, for any homogeneous element $\Theta = \theta \otimes Y \in \Lambda^*(P) \otimes D(P)$ and a form $\omega \in \Lambda^*(P)$ we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} & A_{t,*}(\mathrm{L}_\Theta)(\omega) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left(A_t^*(\theta) \wedge A_t^* Y A_{-t}^* \omega + (-1)^\theta d A_t^* \theta \wedge A_t^*(Y \mathbin{\lrcorner} A_{-t}^* \omega)\right) \\ &= X(\theta) \wedge Y(\omega) + \theta \wedge [X, Y](\omega) + (-1)^\theta dX(\theta) \wedge (Y \mathbin{\lrcorner} \omega) \\ &\qquad\qquad + (-1)^\theta d\theta \wedge [X, Y] \mathbin{\lrcorner} \omega = \mathrm{L}_{([\![X, \theta \otimes Y]\!]^{\mathrm{fn}})}(\omega). \end{aligned}$$

Thus, if one takes $\Theta = \sum_i \theta_i \otimes Y_i \in \Lambda^*(P) \otimes D(P)$ and sets

$$A_t^*(\Theta) = \Theta(t) = \sum_i A_t^*(\theta_i) \otimes A_t^* Y_i A_{-t}^*, \tag{5.13}$$

then

$$\Theta(t) = \Theta + [\![X, \Theta]\!]^{\mathrm{fn}} t + o(t).$$

In other words, $[\![X, \Theta]\!]^{\mathrm{fn}}$ is the velocity of the transformation of Θ with respect to A_t. Taking $P = \mathcal{E}^\infty$ and $\Theta = U$, one can see that the elements $V = [\![U, X]\!]^{\mathrm{fn}}$ are infinitesimal transformations of U arising from transformations $A_t \colon \mathcal{E}^\infty \to \mathcal{E}^\infty$. If $\pi_\infty \circ A_t = \pi_\infty$, then $X \in D^v(\mathcal{E})$ and $V \in \operatorname{im}(\partial^0_{\mathcal{E}})$. It is natural to call such deformations of U *trivial*.

Since, as it was pointed out above, the element U determines the structure of differential equation on the manifold $\mathcal{E}^\infty$ we obtain the following result.

THEOREM 5.4. *The elements of $H^1_{\mathcal{C}}(\mathcal{E})$ are in one-to-one correspondence with the classes of nontrivial infinitesimal vertical deformations of the equation $\mathcal{E}$.*

REMARK 5.2. One can consider deformations of $U_{\mathcal{E}}$ not preserving the verticality condition. Then classes of the corresponding infinitesimal deformations are identified with the elements of the first cohomology module of the complex (5.7) (for $P = \mathcal{E}^\infty$ and $\nabla = \mathcal{C}$). The theory of such deformations is quite interesting but lies beyond the scope of the present book.

REMARK 5.3. Since the operation $[\![\cdot,\cdot]\!]^{\mathrm{fn}}$ defined on $H^1_{\mathcal{C}}(\mathcal{E}^\infty)$ takes its values in $H^2_{\mathcal{C}}(\mathcal{E})$ the elements of the module $H^2_{\mathcal{C}}(\mathcal{E})$ (or a part of them at least) can be interpreted as the obstructions for the deformations of $\mathcal{E}$ (cf. [**78**]).

Local coordinate expressions for the element $U_{\mathcal{E}}$ and for the differentials $\partial_{\mathcal{C}} = \partial^i_{\mathcal{E}}$ in the case $\mathcal{E}^\infty = J^\infty(\pi)$ look as follows.

Let $(x_1, \dots, x_n, u^1, \dots, u^m)$ be local coordinates in $J^0(\pi)$ and p^j_σ, $j = 1, \dots, m$, $|\sigma| \geq 0$, be the corresponding canonical coordinates in $J^\infty(\pi)$. Then from equality (1.35) on p. 26 and (5.3) it follows that

$$U = \sum_{j,\sigma} \omega^j_\sigma \otimes \frac{\partial}{\partial p^j_\sigma}, \tag{5.14}$$

where ω^j_σ are the Cartan forms on $J^\infty(\pi)$ given by (1.27) (see p. 18).

Consider an element $\Theta = \sum_{j,\sigma} \theta^j_\sigma \otimes \partial/\partial p^j_\sigma \in \Lambda^*(\pi) \otimes D^v(\pi)$. Then, due to (5.14) and (4.40), p. 175, we have

$$\partial_\pi(\Theta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{|\sigma|\geq 0} dx_i \wedge \left(\theta^j_{\sigma+1_i} - D_i(\theta^j_\sigma)\right) \otimes \frac{\partial}{\partial p^j_\sigma}, \tag{5.15}$$

where $D_i(\theta)$ is the Lie derivative of the form $\theta \in \Lambda^*(\mathcal{E})$ along the vector field $D_i \in D(\mathcal{E})$.

As it follows from the above said, the cohomology module $H^*_{\mathcal{C}}(\mathcal{E})$ inherits from $\Lambda^* \otimes D_1$ the structure of the graded Lie algebra with respect to the Frölicher–Nijenhuis bracket. In the case when $U = U_\nabla$ is the connection form of a connection $\nabla\colon D(M) \to D(P)$, additional algebraic structures arise in the cohomology modules $H^*_\nabla(P) = \sum_i H^i_\nabla(P)$ of the corresponding vertical complex.

First of all note that for any element $\Omega \in \Lambda^*(P) \otimes D^v(P)$ the identity

$$\Omega \lrcorner U_\nabla = \Omega \tag{5.16}$$

holds. Hence, if $\Theta \in \Lambda^*(P) \otimes D^v(P)$ is a vertical element too, then equality (4.31) on p. 173 acquires the form

$$\Omega \lrcorner \partial_\nabla \Theta + (-1)^\Omega \partial_\nabla(\Omega \lrcorner \Theta) = \partial_\nabla(\Omega) \lrcorner \Theta. \tag{5.17}$$

From (5.17) it follows that

$$\ker(\partial_\nabla) \lrcorner \ker(\partial_\nabla) \subset \ker(\partial_\nabla),$$

$$\ker(\partial_\nabla) \lrcorner \operatorname{im}(\partial_\nabla) \subset \operatorname{im}(\partial_\nabla),$$
$$\operatorname{im}(\partial_\nabla) \lrcorner \ker(\partial_\nabla) \subset \operatorname{im}(\partial_\nabla).$$

Therefore, the contraction operation

$$\lrcorner : \Lambda^i(P) \otimes D^v(P) \times \Lambda^j(P) \otimes D^v(P) \to \Lambda^{i+j-1}(P) \otimes D^v(P)$$

induces an operation

$$\lrcorner : H^i_\nabla(P) \otimes_{\mathbb{R}} H^j_\nabla(P) \to H^{i+j-1}_\nabla(P),$$

which is defined by posing

$$[\Omega] \lrcorner [\Theta] = [\Omega \lrcorner \Theta],$$

where $[\cdot]$ denotes the cohomological class of the corresponding element.

In particular, $H^1_\nabla(P)$ is closed with respect to the contraction operation, and due to (4.31) this operation determines in $H^1_\nabla(P)$ an associative algebra structure. Consider elements $\phi \in H^0_\nabla(P)$ and $\Theta \in H^1_\nabla(P)$. Then one can define an action of Θ on ϕ by posing

$$\mathcal{R}_\Theta(\phi) = \phi \lrcorner \Theta \in H^0_\nabla(P). \tag{5.18}$$

Thus we have a mapping

$$\mathcal{R} \colon H^1_\nabla(P) \to \operatorname{End}_{\mathbb{R}}(H^0_\nabla(P))$$

which is a homomorphism of associative algebras due to (4.31) on p. 173. In particular, taking $P = \mathcal{E}^\infty$ and $\xi = \pi_\infty$, we obtain the following

PROPOSITION 5.5. *For any formally integrable equation $\mathcal{E} \subset J^k(\pi)$ the module $H^1_\mathcal{C}(\mathcal{E})$ is an associative algebra with respect to the contraction operation $\lrcorner$. This algebra acts on $H^0_\mathcal{C}(\mathcal{E}) = \operatorname{sym}(\mathcal{E})$ by means of the representation $\mathcal{R}$ defined by* (5.18).

When (5.16) takes place, equality (4.55), see p. 179, acquires the form

$$\partial_\nabla(\rho \wedge \Omega) = (\mathrm{L}_{U_\nabla} - d\rho) \wedge \Omega + (-1)^\rho \rho \wedge \partial_\nabla(\Omega). \tag{5.19}$$

Let us set

$$d^h_\nabla = d - \mathrm{L}_{U_\nabla} \tag{5.20}$$

and note that

$$(d^h_\nabla)^2 = (\mathrm{L}_{U_\nabla})^2 - \mathrm{L}_{U_\nabla} \circ d - d \circ \mathrm{L}_{U_\nabla} + d^2 = -\mathrm{L}_{U_\nabla} \circ d - d \circ \mathrm{L}_{U_\nabla}.$$

But

$$\mathrm{L}_\Omega \circ d = (-1)^\Omega d \circ \mathrm{L}_\Omega \tag{5.21}$$

and, therefore, $(d^h_\nabla)^2 = 0$. Thus we have the differential

$$d^h_\nabla \colon \Lambda^i(P) \to \Lambda^{i+1}(P), \quad i = 0, 1, \dots,$$

and the corresponding cohomologies

$$H^{h,i}_\nabla(P) = \ker(d^{h,i+1}_\nabla)/\operatorname{im}(d^{h,i}_\nabla).$$

From (5.19) it follows that

$$\begin{aligned}\ker(d_\nabla^h)\wedge\ker(\partial_\nabla)&\subset\ker(\partial_\nabla),\\ \operatorname{im}(d_\nabla^h)\wedge\ker(\partial_\nabla)&\subset\operatorname{im}(\partial_\nabla),\\ \ker(d_\nabla^h)\wedge\operatorname{im}(\partial_\nabla)&\subset\operatorname{im}(\partial_\nabla),\end{aligned}$$

and hence a well-defined wedge product

$$\wedge\colon H_\nabla^{h,i}(P)\otimes_{\mathbb{R}} H_\nabla^j(P)\to H_\nabla^{i+j}(P).$$

Moreover, from (5.20) and (5.21) it follows that

$$\mathrm{L}_\Omega\circ d_\nabla^h=\mathrm{L}_{[\![\Omega,U_\nabla]\!]^{\mathrm{fn}}}+(-1)^\Omega d_\nabla^h\circ\mathrm{L}_\Omega$$

for any $\Omega\in\Lambda^*(P)\otimes D^v(P)$. It means that by posing

$$\mathrm{L}_{[\Omega]}[\omega]=[\mathrm{L}_\Omega\omega],\quad \omega\in\Lambda^*(P)$$

we get a well-defined homomorphism of graded Lie algebras

$$\mathrm{L}\colon H_\nabla^*(P)\to D^{\mathrm{gr}}(H_\nabla^{h,*}(P)),$$

where $H_\nabla^{h,*}(P)=\sum_i H_\nabla^{h,i}(P)$.

If $(x_1,\dots,x_n,y^1,\dots,y^s)$ are local coordinates in P, then an easy computation shows that

$$\begin{aligned}d_\nabla^h(f)&=\sum_i\nabla_i(f)dx_i,\\ d_\nabla^h(dx_i)&=0,\\ d_\nabla^h(dy^j)&=\sum_i d\nabla_i^j\wedge dx_i,\end{aligned}\tag{5.22}$$

where $f\in C^\infty(P)$, $i=1,\dots,n$, $j=1,\dots,s$, while the coefficients ∇_i^j and vector fields ∇_i are given by (5.2). Obviously, the differential d_∇^H is completely defined by (5.22).

2. Spectral sequences and graded evolutionary derivations

In this section, we construct three spectral sequences associated with $\mathcal{C}$-cohomologies of infinitely prolonged equations. One of them is used to compute the algebra $H_\mathcal{C}^*(\pi)=H_\mathcal{C}^*(J^\infty(\pi))$ of the "empty" equation. The result obtained leads naturally to the notion of graded evolutionary derivations which seem to play an important role in the geometry of differential equations.

The first of spectral sequences to be defined originates from a filtration in $\Lambda^*(\mathcal{E})\otimes D^v(\mathcal{E})$ associated with the notion of the degree of horizontality. Namely, an element $\Theta\in\Lambda^p(\mathcal{E})\otimes D^v(\mathcal{E})$ is said to be *i-horizontal* if

$$X_1\lrcorner(X_2\lrcorner\dots(X_{p-i+1}\lrcorner\Theta)\dots)=0$$

for any $X_1,\dots,X_{p-i+1}\in D^v(\mathcal{E})$. Denote by $\mathcal{H}_i^p(\mathcal{E})$ the set of all such elements. Obviously, $\mathcal{H}_i^p(\mathcal{E})\supset\mathcal{H}_{i+1}^p(\mathcal{E})$.

PROPOSITION 5.6. *For any equation $\mathcal{E}$, the embedding*

$$\partial_{\mathcal{C}}(\mathcal{H}_i^p(\mathcal{E})) \subset \mathcal{H}_{i+1}^{p+1}(\mathcal{E})$$

takes place.

To prove this we need some auxiliary facts.

LEMMA 5.7. *For any vector fields $X_1, \dots, X_p \in D^v(\mathcal{E})$ and an element $\Theta \in \Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})$ the equality*

$$\begin{aligned} X_1 \lrcorner \dots \lrcorner X_p \lrcorner \partial_{\mathcal{C}}(\Theta) &= (-1)^p \partial_{\mathcal{C}}(X_1 \lrcorner \dots \lrcorner X_p \lrcorner \Theta) \\ &+ \sum_{i=1}^{p} (-1)^{p+i} X_1 \lrcorner \dots \lrcorner X_{i-1} \lrcorner \partial_{\mathcal{C}}(X_i) \lrcorner X_{i+1} \lrcorner \dots \lrcorner X_p \lrcorner \Theta \end{aligned} \tag{5.23}$$

holds.

PROOF. Recall that for any $\Omega \in \Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})$ one has

$$\Omega \lrcorner U_{\mathcal{E}} = \Omega \tag{5.24}$$

and, by (5.17)

$$\Omega \lrcorner \partial_{\mathcal{C}}(\Theta) = \partial_{\mathcal{C}}(\Omega) \lrcorner \Theta - (-1)^{\Omega} \partial_{\mathcal{C}}(\Omega \lrcorner \Theta). \tag{5.25}$$

In particular, taking $\Omega = X \in D^v(\mathcal{E})$, we get

$$X \lrcorner \partial_{\mathcal{C}}\Theta = \partial_{\mathcal{C}}(X) \lrcorner \Theta - \partial_{\mathcal{C}}(X \lrcorner \Theta). \tag{5.26}$$

This proves (5.23) for $p = 1$. The proof is finished by induction on p starting with (5.26). $\square$

LEMMA 5.8. *Consider vertical vector fields $X_1, \dots, X_{p+1} \in D^v(\mathcal{E})$ and an element $\Theta \in \mathcal{H}_0^p(\mathcal{E}) = \Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})$. Then*

$$X_1 \lrcorner \dots \lrcorner X_{p+1} \lrcorner (\partial_{\mathcal{C}}\Theta) = 0,$$

i.e., $\partial_{\mathcal{C}}(\mathcal{H}_0^p(\mathcal{E})) \subset \mathcal{H}_1^{p+1}(\mathcal{E})$.

This result is a direct consequence of (5.23).

Recall that a form $\theta \in \Lambda^p(\mathcal{E})$ is said to be *horizontal* if the identity $X \lrcorner \theta = 0$ holds for any $X \in D^v(\mathcal{E})$; the set of such forms is denoted by $\Lambda_h^p(\mathcal{E})$. It is easy to see that $\mathcal{H}_i^p(\mathcal{E}) = \Lambda_h^i(\mathcal{E}) \wedge \mathcal{H}_0^{p-i}(\mathcal{E})$, i.e., any element $\Theta \in \mathcal{H}_i^p(\mathcal{E})$ can be represented as

$$\Theta = \sum_s \rho_s \wedge \Theta_s, \tag{5.27}$$

where $\rho_s \in \Lambda_h^p(\mathcal{E})$, $\Theta_s \in \Lambda^{p-i}(\mathcal{E}) \otimes D^v(\mathcal{E})$. Applying (5.19) and (5.20) to (5.27) in the case when ∇ is the Cartan connection $\mathcal{C}$, we get

$$\partial_{\mathcal{C}}(\Theta) = \sum_s \Big(-d_{\mathcal{C}}^h(\rho_s) \wedge \Theta_s + (-1)^i \rho_s \wedge \partial_{\mathcal{C}}(\Theta_s) \Big). \tag{5.28}$$

LEMMA 5.9. *1 Let $\xi\colon P \to M$ be a fiber bundle with a flat connection $\nabla\colon D(M) \to D(P)$ and*

$$\Lambda_h^*(P) = \{\rho \in \Lambda^*(P) \mid Y \lrcorner \rho = 0,\ Y \in D^v(P)\}$$

be the module of horizontal forms on P. Then for any form $\rho \in \Lambda_0^i(P)$ one has

$$d_\nabla^h(\rho) \in \Lambda_h^{i+1}(P).$$

PROOF. Let $\Omega \in \Lambda^*(P) \otimes D(P)$, $\rho \in \Lambda^*(P)$, and $Y \in D(P)$. Then standard computations show that

$$Y \lrcorner (\mathrm{L}_\Omega \rho) = \mathrm{L}_{(Y \lrcorner \Omega)}\rho + (-1)^\Omega \mathrm{L}_\Omega(Y \lrcorner \rho) - (-1)^\Omega [\![\Omega, Y]\!]^{\mathrm{fn}} \lrcorner \rho. \tag{5.29}$$

In particular, if $\Omega = U_\nabla$ and $Y \in D^v(P)$, using (5.16) one has

$$Y \lrcorner (\mathrm{L}_{U_\nabla}\rho) = Y(\rho) - \mathrm{L}_{U_\nabla}(Y \lrcorner \rho) + \partial_\nabla(Y) \lrcorner \rho,$$

from where it follows that

$$Y \lrcorner d_\nabla^h(\rho) = -d_\nabla^h(Y \lrcorner \rho) - \partial_\nabla(Y) \lrcorner \rho,$$

since, by definition, $d_\nabla^h = d - \mathrm{L}_{U_\nabla}$.

Hence, if $Y \in D^v(P)$ and $\rho \in \Lambda_h^*(P)$, then one has $\partial_\nabla(Y) \in \Lambda^1(P) \otimes D^v(P)$ and $Y \lrcorner d_\nabla^h(\rho) = 0$. $\square$

Proposition 5.6 now follows from Lemmas 5.8, 5.9 and identity (5.28).

REMARK 5.4. From the definition of the differential $d_{\mathcal{C}}^h$ it immediately follows that its restriction on $\Lambda_h^*(\mathcal{E})$, denoted by d_h, coincides with the horizontal de Rham complex of the equation $\mathcal{E}$ (see Chapter 2). As it follows from (5.22), in local coordinates this restriction is completely determined by the equalities

$$d_h(f) = \sum_i D_i(f)\,dx_i, \quad d_h(dx_i) = 0, \tag{5.30}$$

where $i = 1, \dots, n$, $f \in \mathcal{F}(\mathcal{E})$ and $D_1, \dots, D_n$ are total derivatives. One can show that the action L of $H_{\mathcal{C}}^*(\mathcal{E})$ can be restricted onto the module $H_h^*(\mathcal{E})$ of horizontal cohomologies. In fact, if $\rho \in \Lambda_0^*(\mathcal{E})$ and $X, Y \in D^v(\mathcal{E})$, then

$$X \lrcorner Y(\rho) = Y(X \lrcorner \rho) + [X, Y] \lrcorner \rho = 0.$$

On the other hand, if $\Omega \in \Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})$, then from (5.29) it follows that

$$Y \lrcorner (\mathrm{L}_\Omega \rho) = \mathrm{L}_{(Y \lrcorner \Omega)}\rho.$$

Hence, by induction,

$$\mathrm{L}_{\left(\Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})\right)}(\Lambda_h^*(\mathcal{E})) \subset \Lambda_h^*(\mathcal{E}).$$

On the other hand, the operator $\mathrm{L}_{U_\mathcal{E}}$ is exactly the Cartan differential of the equation $\mathcal{E}$ (see also Chapter 2).

Let us now define a filtration in $\Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})$ by setting

$$F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})) = \mathcal{H}^p_{p+l}(\mathcal{E}). \tag{5.31}$$

Obviously,

$$F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})) \supset F^{l+1}(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E}))$$

and Proposition 5.6 is equivalent to the fact that

$$\partial_{\mathcal{C}}\Big(F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E}))\Big) \subset F^l(\Lambda^{p+1}(\mathcal{E}) \otimes D^v(\mathcal{E})).$$

Thus (5.31) defines a spectral sequence for the complex (5.9) which we call $\mathcal{H}$-*spectral*. Its term E_0 is of the form

$$E_0^{p,q} = \mathcal{H}^{p+q}_{2p+q}(\mathcal{E})/\mathcal{H}^{p+q}_{2p+q+1}(\mathcal{E}), \tag{5.32}$$

where $p = 0, -1, \dots,\ q = -2p, \dots, -2p+n$.

To express $E_0^{p,q}$ in more suitable terms, let us recall the splitting

$$\Lambda^1(\mathcal{E}) = \Lambda^1_h(\mathcal{E}) \oplus \mathcal{C}\Lambda^1(\mathcal{E}),$$

where $\mathcal{C}\Lambda^1(\mathcal{E})$ is the set of all 1-forms vanishing on the Cartan distribution on $\mathcal{E}$. Let

$$\mathcal{C}^i\Lambda(\mathcal{E}) = \underbrace{\mathcal{C}\Lambda^1(\mathcal{E}) \wedge \dots \wedge \mathcal{C}\Lambda^1(\mathcal{E})}_{i \text{ times}}.$$

Then for any p the module $\Lambda^p(\mathcal{E})$ can be represented as

$$\Lambda^p(\mathcal{E}) = \sum_{i=0}^{p} \mathcal{C}^{p-i}\Lambda(\mathcal{E}) \wedge \Lambda^i_h(\mathcal{E}).$$

Thus

$$\mathcal{H}^p_i(\mathcal{E}) = \Big(\sum_{i=0}^{p} \mathcal{C}^{p-i}\Lambda(\mathcal{E}) \wedge \Lambda^i_h(\mathcal{E})\Big) \otimes D^v(\mathcal{E})$$

from where it follows that

$$E_0^{p,q} = \mathcal{C}^{-p}\Lambda(\mathcal{E}) \wedge \Lambda^{2p+q}_h(\mathcal{E}) \otimes D^v(\mathcal{E}).$$

The configuration of the term E_0 for the $\mathcal{H}$-spectral sequence is presented on Fig. 5.1, where $D^v = D^v(\mathcal{E})$, $\Lambda^i_h = \Lambda^i_h(\mathcal{E})$, etc.

The second spectral sequence to be defined is in a sense complementary to the first one. Namely, we say that an element $\Theta \in \Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})$ is $(p-i+1)$-*Cartan*, if $X_1 \lrcorner \dots \lrcorner X_i \lrcorner \Theta = 0$ for any $X_1, \dots, X_i \in \mathcal{C}D(\mathcal{E})$, and denote the set of all such elements by $\mathcal{C}^p_i(\mathcal{E}) \subset \Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})$. Obviously, $\mathcal{C}^p_i(\mathcal{E}) \subset \mathcal{C}^p_{i+1}(\mathcal{E})$.

PROPOSITION 5.10. *For any equation $\mathcal{E} \subset J^k(\pi)$ one has*

$$\partial_{\mathcal{C}}(\mathcal{C}^p_i(\mathcal{E})) \subset \mathcal{C}^{p+1}_{i+1}(\mathcal{E}).$$

To prove this proposition, we need some preliminary facts.

$\mathcal{C}^2\Lambda \wedge \Lambda_h^n \otimes D^v$			
$\mathcal{C}^2\Lambda \wedge \Lambda_h^{n-1} \otimes D^v$			
$\mathcal{C}^2\Lambda \wedge \Lambda_h^{n-2} \otimes D^v$	$\mathcal{C}^1\Lambda \wedge \Lambda_h^n \otimes D^v$		
$\dots$	$\mathcal{C}^1\Lambda \wedge \Lambda_h^{n-1} \otimes D^v$		
$\dots$	$\mathcal{C}^1\Lambda \wedge \Lambda_h^{n-2} \otimes D^v$	$\Lambda_h^n \otimes D^v$	$q = n$
$\dots$	$\dots$	$\Lambda_h^{n-1} \otimes D^v$	$q = n-1$
$\dots$	$\dots$	$\dots$	$\dots$
$\mathcal{C}^2\Lambda \wedge \Lambda_h^1 \otimes D^v$	$\dots$	$\dots$	$\dots$
$\mathcal{C}^2\Lambda \otimes D^v$	$\mathcal{C}^1\Lambda \wedge \Lambda_h^2 \otimes D^v$	$\dots$	$\dots$
	$\mathcal{C}^1\Lambda \wedge \Lambda_h^1 \otimes D^v$	$\dots$	$\dots$
	$\mathcal{C}^1\Lambda \otimes D^v$	$\Lambda_h^2 \otimes D^v$	$q = 2$
		$\Lambda_h^1 \otimes D^v$	$q = 1$
		D^v	$q = 0$
$p = -2$	$p = -1$	$p = 0$	

FIGURE 5.1. The $\mathcal{H}$-spectral sequence configuration (term E_0).

LEMMA 5.11. *For any vector fields $X_1, \dots, X_p \in \mathcal{C}D(\mathcal{E})$ and an element $\Theta \in \Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})$ the equality*

$$X_1 \rfloor \dots \rfloor X_p \rfloor \partial_{\mathcal{C}}(\Theta) = (-1)^p \partial_{\mathcal{C}}(X_1 \rfloor \dots \rfloor X_p \rfloor \Theta)$$
$$+ \sum_{i=1}^{p} (-1)^{p+i+1} X_1 \rfloor \dots \rfloor X_{i-1} \rfloor [\![X_i, X_{i+1} \rfloor \dots \rfloor X_p \rfloor \Theta]\!]^{\mathrm{fn}} \rfloor U_{\mathcal{E}}. \quad (5.33)$$

holds.

PROOF. We proceed by induction on p. Let $X \in \mathcal{C}D(\mathcal{E})$. Then, since $X \rfloor U_{\mathcal{E}} = 0$ and $[\![X, U_{\mathcal{E}}]\!]^{\mathrm{fn}} = 0$, from equality (4.45) on p. 175 it follows that

$$X \rfloor \partial_{\mathcal{C}}(\Theta) = -\partial_{\mathcal{C}}(X \rfloor \Theta) - [\![X, \Theta]\!]^{\mathrm{fn}} \rfloor U_{\mathcal{E}}, \quad (5.34)$$

which gives us the starting point of induction.

Suppose now that (5.33) is proved for all $s \le r$. Then by (5.34) we have

$$X_1 \rfloor X_2 \rfloor \dots \rfloor X_{r+1} \rfloor \partial_{\mathcal{C}}(\Theta) = X_1 \rfloor (X_2 \rfloor \dots \rfloor X_{r+1} \rfloor \partial_{\mathcal{C}}(\Theta))$$
$$= (-1)^r X_1 \rfloor \partial_{\mathcal{C}}(X_2 \rfloor \dots \rfloor X_{r+1} \rfloor \Theta)$$
$$+ X_1 \rfloor \sum_{i=2}^{r+1} (-1)^{r+i} X_2 \rfloor \dots \rfloor X_{i-1} \rfloor [\![X_i, X_{i+1} \rfloor \dots \rfloor X_{r+1} \rfloor \Theta]\!]^{\mathrm{fn}} \rfloor U_{\mathcal{E}}$$

$$= (-1)^r\Big(-\partial_{\mathcal{C}}(X_1 \rfloor \dots \rfloor X_{r+1} \rfloor \Theta) - [\![X_1, X_2 \rfloor \dots \rfloor X_{r+1} \rfloor \Theta]\!]^{\mathrm{fn}} \rfloor U_{\mathcal{E}}\Big)$$
$$+ \sum_{i=2}^{r+1} (-1)^{r+i} X_1 \rfloor X_2 \rfloor \dots \rfloor X_{i-1} \rfloor [\![X_i, X_{i+1} \rfloor \dots \rfloor X_{r+1} \rfloor \Theta]\!]^{\mathrm{fn}} \rfloor U_{\mathcal{E}}$$
$$= (-1)^{r+1}\partial_{\mathcal{C}}(X_1 \rfloor \dots \rfloor X_{r+1} \rfloor \Theta)$$
$$+ \sum_{i=1}^{r+1} (-1)^{r+i+1} X_1 \rfloor \dots \rfloor X_{i-1} \rfloor [\![X_i, X_{i+1} \rfloor \dots \rfloor X_{r+1} \rfloor \Theta]\!]^{\mathrm{fn}} \rfloor U_{\mathcal{E}},$$

which finishes the proof of lemma. □

LEMMA 5.12. *For any $X \in \mathcal{C}D(\mathcal{E})$ and $\Theta \in \mathcal{C}_i^p(\mathcal{E})$ we have*

(i)
$$X \rfloor \Theta \in \mathcal{C}_{i-1}^{p-1}(\mathcal{E})$$

and

(ii)
$$[\![X, \Theta]\!]^{\mathrm{fn}} \in \mathcal{C}_i^p(\mathcal{E}).$$

PROOF. The first statement is obvious. To prove the second one, note that from equality (4.45) on p. 175 it follows that for any $X, X_1 \in D(\mathcal{E})$ and $\Theta \in \Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})$ one has

$$X_1 \rfloor [\![X, \Theta]\!]^{\mathrm{fn}} = [\![X, X_1 \rfloor \Theta]\!]^{\mathrm{fn}} + [\![X_1, X]\!]^{\mathrm{fn}} \rfloor \Theta.$$

Now, by an elementary induction one can conclude that

$$X_1 \rfloor \dots \rfloor X_i \rfloor [\![X, \Theta]\!]^{\mathrm{fn}} = [\![X, X_1 \rfloor \dots \rfloor X_i \rfloor \Theta]\!]^{\mathrm{fn}}$$
$$+ \sum_{s=1}^{i} X_1 \rfloor \dots \rfloor X_{s-1} \rfloor [\![X_s, X]\!]^{\mathrm{fn}} \rfloor X_s \rfloor \dots \rfloor X_i \rfloor \Theta \quad (5.35)$$

for any $X_1, \dots, X_i \in D(\mathcal{E})$.

Consider vector fields $X, X_1, \dots, X_i \in \mathcal{C}D(\mathcal{E})$ and an element $\Theta \in \mathcal{C}_i^p(\mathcal{E})$. Then, since $[\![X_s, X]\!]^{\mathrm{fn}} = [X_s, X] \in \mathcal{C}D(\mathcal{E})$, all the summands on the right-hand side of (5.35) vanish. □

PROOF OF PROPOSITION 5.10. Consider an element $\Theta \in \mathcal{C}_i^p(\mathcal{E})$ and fields $X_1, \dots, X_{i+1} \in \mathcal{C}D(\mathcal{E})$. Then, by (5.33), one has

$$X_1 \rfloor \dots \rfloor X_{i+1} \rfloor \partial_{\mathcal{C}}(\Theta) = (-1)^{i+1}\partial_{\mathcal{C}}(X_1 \rfloor \dots \rfloor X_{i+1} \rfloor \Theta)$$
$$+ \sum_{s=1}^{i+1} (-1)^{i+s} X_1 \rfloor \dots \rfloor X_{s-1} \rfloor [\![X_s, X_{s+1} \rfloor \dots \rfloor X_{i+1} \rfloor \Theta]\!]^{\mathrm{fn}} \rfloor U_{\mathcal{E}}. \quad (5.36)$$

The first summand on the right-hand side vanishes by definition while the rest of them, due to equality (4.31) on p. 173 and since $U_{\mathcal{E}} \in \Lambda^1(\mathcal{E}) \otimes D^v(\mathcal{E})$, can be represented in the form

$$(-1)^{i+s}\Big(X_1 \rfloor \dots \rfloor X_{s-1} \rfloor [\![X_s, X_{s+1} \rfloor \dots \rfloor X_{i+1} \rfloor \Theta]\!]^{\mathrm{fn}}\Big) \rfloor U_{\mathcal{E}}.$$

Since $\Theta \in \mathcal{C}^p_i(\mathcal{E})$ and $X_1, \dots, X_{i+1} \in \mathcal{C}D(\mathcal{E})$, we have

$$X_{s+1} \lrcorner \dots \lrcorner X_{i+1} \lrcorner \Theta \in \mathcal{C}^{p-i+s-1}_{s-1}(\mathcal{E})$$

and by Lemma 5.12 (ii) the element $[\![X_s, X_{s+1} \lrcorner \dots \lrcorner X_{i+1} \lrcorner \Theta]\!]^{\mathrm{fn}}$ belongs to $\mathcal{C}^{p-i+s-1}_{s-1}(\mathcal{E})$ as well. Hence, all the summands in (5.36) vanish. $\square$

Let us now define a filtration in $\Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})$ by setting

$$F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})) = \mathcal{C}^p_{p-l+1}(\mathcal{E}). \tag{5.37}$$

Obviously,

$$F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})) \supset F^{l+1}(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E}))$$

and, by Proposition 5.10,

$$\partial_{\mathcal{C}}\Big(F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E}))\Big) \subset F^l(\Lambda^{p+1}(\mathcal{E}) \otimes D^v(\mathcal{E})).$$

Thus, filtration (5.37) defines a spectral sequence for the complex (5.9) which we call the *$\mathcal{C}$-spectral sequence for the equation $\mathcal{E}$*.

REMARK 5.5. As it was already mentioned before, $\mathcal{C}$-spectral sequences were introduced by A.M. Vinogradov (see [**102**]). As A.M. Vinogradov noted (a private communication), the $\mathcal{H}$-spectral sequence can also be viewed as a $\mathcal{C}$-spectral sequence constructed with respect to fibers of the bundle $\pi_\infty \colon \mathcal{E}^\infty \to M$. It is similar to the classical Leray–Serre sequence.

The term E_0 of the $\mathcal{C}$-spectral sequence is of the form

$$E_0^{p,q} = \mathcal{C}^{p+q}_{q+1}(\mathcal{E})/\mathcal{C}^{p+q}_q(\mathcal{E}), \quad p = 0, 1, \dots, \ q = 0, 1, \dots, n.$$

To describe these modules explicitly, note that

$$\mathcal{C}^{p+q}_q(\mathcal{E}) = \Big(\sum_{i=p}^{p+q} \mathcal{C}^i\Lambda(\mathcal{E}) \wedge \Lambda_h^{p+q-i}(\mathcal{E})\Big) \otimes D^v(\mathcal{E})$$

while

$$\mathcal{C}^{p+q}_{q+1}(\mathcal{E}) = \Big(\sum_{i=p+1}^{p+q} \mathcal{C}^i\Lambda(\mathcal{E}) \wedge \Lambda_h^{p+q-i}(\mathcal{E})\Big) \otimes D^v(\mathcal{E}).$$

Thus

$$E_0^{p,q} = \mathcal{C}^p\Lambda(\mathcal{E}) \wedge \Lambda^q_h(\mathcal{E}) \otimes D^v(\mathcal{E}).$$

The configuration of the term E_0 for the $\mathcal{C}$-spectral sequence is given on Fig. 5.2.

REMARK 5.6. The 0-th column of the term E_0 coincides with the horizontal de Rham complex for the equation $\mathcal{E}$ with coefficients in the bundle of vertical vector fields. Complexes of such a type were introduced by T. Tsujishita in [**97**].

$q=n$	$\Lambda_h^n \otimes D^v$	$\Lambda_h^n \wedge \mathcal{C}^1\Lambda \otimes D^v$	$\dots$	$\Lambda_h^n \wedge \mathcal{C}^p\Lambda \otimes D^v$	$\dots$
$q=n-1$	$\Lambda_h^{n-1} \otimes D^v$	$\Lambda_h^{n-1} \wedge \mathcal{C}^1\Lambda \otimes D^v$	$\dots$	$\Lambda_h^{n-1} \wedge \mathcal{C}^p\Lambda \otimes D^v$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\Lambda_h^q \otimes D^v$	$\Lambda_h^q \wedge \mathcal{C}^1\Lambda \otimes D^v$	$\dots$	$\Lambda_h^q \wedge \mathcal{C}^p\Lambda \otimes D^v$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$q=1$	$\Lambda_h^1 \otimes D^v$	$\Lambda_h^1 \wedge \mathcal{C}^1\Lambda \otimes D^v$	$\dots$	$\Lambda_h^1 \wedge \mathcal{C}^p\Lambda \otimes D^v$	$\dots$
$q=0$	D^v	$\mathcal{C}^1\Lambda \otimes D^v$	$\dots$	$\mathcal{C}^p\Lambda \otimes D^v$	$\dots$
	$p=0$	$p=1$	$\dots$	$\dots$	$\dots$

FIGURE 5.2. The $\mathcal{C}$-spectral sequence configuration (term E_0).

Consider, as before, a formally integrable equation $\mathcal{E} \subset J^k(\pi)$ and the corresponding algebra $\mathcal{F}(\mathcal{E})$ filtered by its subalgebras $\mathcal{F}_i(\mathcal{E})$.

We say that an element $\Theta \in \Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})$ is *i-vertical* if

$$\mathrm{L}_\Theta|_{\mathcal{F}_{i-k-1}(\mathcal{E})} = 0 \tag{5.38}$$

and denote by $\mathcal{V}_i^p(\mathcal{E})$ the set of all such elements. Obviously, $\mathcal{V}_i^p(\mathcal{E}) \supset \mathcal{V}_{i+1}^p(\mathcal{E})$ and $\mathcal{V}_0^p(\mathcal{E}) = \Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})$.

PROPOSITION 5.13. *For any equation $\mathcal{E}$ the embedding*

$$\partial_{\mathcal{C}}(\mathcal{V}_i^p(\mathcal{E})) \subset \mathcal{V}_{i-1}^{p+1}(\mathcal{E})$$

takes place.

PROOF. Obviously, $\mathrm{L}_{U_\mathcal{E}}(\mathcal{F}_j(\mathcal{E})) \subset \mathcal{F}_{j+1}(\mathcal{E})$ for any $j \geq -k-1$. Consider elements $\Theta \in \mathcal{V}_i^p(\mathcal{E})$ and $\phi \in \mathcal{F}_{i-k-2}(\mathcal{E})$. Then, by definition,

$$\mathrm{L}_{\partial_{\mathcal{C}}(\Theta)}(\phi) = \mathrm{L}_{[\![U_\mathcal{E},\Theta]\!]^{\mathrm{fn}}}(\phi) = \mathrm{L}_{U_\mathcal{E}}(\mathrm{L}_\Theta(\phi)) - (-1)^\Theta \mathrm{L}_\Theta(\mathrm{L}_{U_\mathcal{E}}(\phi)) = 0,$$

which finishes the proof. □

Let us define a filtration in $\Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E})$ by setting

$$F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})) = \mathcal{V}_{l-p}^p(\mathcal{E}). \tag{5.39}$$

Obviously,

$$F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})) \subset F^{l+1}(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E}))$$

and

$$\partial_{\mathcal{C}}\big(F^l(\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E}))\big) \subset F^l(\Lambda^{p+1}(\mathcal{E}) \otimes D^v(\mathcal{E})).$$

Thus, (5.39) defines a spectral sequence for the complex (5.9) which we call *$\mathcal{V}$-spectral*. The term E_0 for this spectral sequence is of the form

$$E_0^{p,q} = \mathcal{V}_{-q}^{p+q}(\mathcal{E})/\mathcal{V}_{1-q}^{p+q}(\mathcal{E}), \quad p = 0, 1, \dots, \ q = 0, -1, \dots, -p.$$

	$p=0$	$p=1$	$\dots$	$\dots$	$\dots$
$q=0$	$\mathcal{F}^V$	$\mathcal{F}^V \otimes \Lambda^1(\pi)$	$\dots$	$\mathcal{F}^V \otimes \Lambda^p(\pi)$	$\dots$
$q=-1$		$\mathcal{F}^V \otimes S^1 D(M)$	$\dots$	$\mathcal{F}^V \otimes \Lambda^{p-1}(\pi) \otimes S^1 D(M)$	$\dots$
$\dots$			$\dots$	$\dots$	$\dots$
$q=-p$				$\mathcal{F}^V \otimes S^p D(M)$	$\dots$
$\dots$					$\dots$

FIGURE 5.3. The $\mathcal{V}$-spectral sequence configuration for $J^\infty(\pi)$ (term E_0).

Now we shall compute the algebra $H^*_{\mathcal{C}}(\mathcal{E}) = H^*_{\mathcal{C}}(\pi)$ for the "empty equation" $J^\infty(\pi)$ using the $\mathcal{V}$-spectral sequence.

First, we shall represent elements of the modules $E_0^{p,q}$ in a more convenient way. Denote $-q$ by r and consider the bundle $\pi_{r,r-1}\colon J^r(\pi) \to J^{r-1}(\pi)$ and the subbundle $\pi_{r,r-1,V}\colon T^v(J^r(\pi)) \to J^r(\pi)$ of the tangent bundle $T(J^r(\pi)) \to J^r(\pi)$ consisting of $\pi_{r,r-1}$-vertical vectors. Then we have the induced bundle:

$$\begin{tikzcd}
\pi^*_{\infty,r}(T^v(J^r(\pi)) \arrow[r] \arrow[d, "\pi^*_{\infty,r}(\pi_{r,r-1,V})"'] & T^v(J^r(\pi)) \arrow[d, "\pi_{r,r-1,V}"] \\
J^\infty(\pi) \arrow[r] & J^r(\pi)
\end{tikzcd}$$

and obviously,

$$E_0^{p,-r} = \Lambda^{p-r}(\pi) \otimes_{\mathcal{F}(\pi)} \Gamma(\pi^*_{\infty,r}(\pi_{r,r-1,V})).$$

On the other hand, the bundle $\pi^*_{\infty,r}(\pi_{r,r-1,V})$ can be described in the following way. Consider the tangent bundle $\tau\colon T(M) \to M$, its rth symmetric power $S^r(\tau)\colon S^r T(M) \to M$ and the bundle

$$\pi_V \otimes \pi^*(S^r(\tau))\colon T^v(J^0(\pi)) \otimes \pi^*(S^r T(M)) \to J^0(\pi),$$

where $\pi_V\colon T^v(J^0(\pi)) \to J^0(\pi)$ is the bundle of π-vertical vectors. Then, at least locally,

$$\pi_{r,r-1,V} \approx \pi^*_{p,0}(\pi_V \otimes \pi^*(S^r(\tau))).$$

It means that locally we have an isomorphism

$$\mu\colon E_0^{p,-r} \approx \mathcal{F}(\pi, \pi_V) \otimes_{\mathcal{F}(\pi)} \Lambda^{p-r}(\pi) \otimes_{C^\infty(M)} S^r(D(M)).$$

Thus the term E_0 of the $\mathcal{V}$-spectral sequence is of the form which is presented on Fig. 5.3, where $\mathcal{F}^V \stackrel{\text{def}}{=} \mathcal{F}(\pi, \pi_V)$.

Let $(x_1, \dots, x_n)$ be local coordinates in M, p^j_σ be the coordinates arising naturally in $J^\infty(\pi)$, and $\xi_1 = \partial/\partial x_1, \dots, \xi_n = \partial/\partial x_n$ be the local basis in $T(M)$ corresponding to $(x_1, \dots, x_n)$. Denote also by v^j, $j = 1, \dots, m$, local vector fields $\partial/\partial u^j$, where $u^j = p^j_{(0,\dots,0)}$ are coordinates along the fiber of the bundle π. Then any element $\Theta \in E_0^{p,-r}$ is of the form

$$\Theta = \sum_{j=1}^m \sum_{|\sigma|=r} \theta^j_\sigma \otimes \frac{\partial}{\partial p^j_\sigma}, \quad \theta^j_\sigma \in \Lambda^{p-r}(\pi),$$

while the identification μ can be represented as

$$\Omega = \mu(\Theta) = \sum_{j=1}^m \sum_{|\sigma|=r} v^j \otimes \theta^j_\sigma \otimes \Big(\frac{1}{\sigma!}\xi^\sigma\Big),$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$, $\sigma! = \sigma_1! \cdot \dots \cdot \sigma_n!$, $\xi^\sigma = \xi_1^{\sigma_1} \cdot \dots \cdot \xi_n^{\sigma_n}$.

Let us now represent Ω in the form

$$\Omega = \sum_{i=0}^{p-r} \rho_i \wedge \omega_i \otimes Q_i,$$

where $\rho_i \in \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^{p-r-i}\Lambda(\pi)$, $\omega_i \in \Lambda^i_h(\pi)$, and $Q_i = Q_i(\xi)$ are homogeneous polynomials in $\xi_1, \dots, \xi_n$ of the power q.

From equality (5.15) it follows that in this representation the differential $\partial_0\colon E_0^{p,-r} \to E_0^{p,-r+1}$ in the following way

$$\partial_0(\Omega) = \sum_{i=0}^{p-r} (-1)^{p-r-i} \rho_i \wedge \sum_{s=1}^n dx_s \wedge \omega_i \otimes \frac{\partial Q}{\partial \xi_s}. \tag{5.40}$$

Thus, the differential ∂_0 reduces to δ-Spencer operators (see [**93**]) from which it follows that all its cohomologies are trivial except for the terms $E_0^{p,0}$. But as it is easily seen from (5.40) and from the previous constructions,

$$E_1^{p,0} = E_0^{p,0}/\partial_0(E_0^{p,1}) = \mathcal{F}(\pi, \pi_V) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^p\Lambda(\pi).$$

Hence, only the 0-th row survives in the term E_1 and it is of the form

$$0 \to \mathcal{F}(\pi, \pi_V) \xrightarrow{\partial_1^{0,0}} \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^1\Lambda(\pi) \to \cdots$$
$$\to \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^p\Lambda(\pi) \xrightarrow{\partial_1^{p,0}} \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^{p+1}\Lambda(\pi) \to \cdots$$

Recall now that ∂_1 is induced by the differential ∂_π and that the latter increases the degree of horizontality for the elements from $\Lambda^*(\pi) \otimes D^v(\pi)$ (Proposition 5.6). Again, we see that ∂_1 is trivial. Thus, we have proved the following

THEOREM 5.14. *The $\mathcal{V}$-spectral sequence for the "empty" equation $\mathcal{E}^\infty = J^\infty(\pi)$ stabilizes at the term E_1, i.e., $E_1 = E_2 = \dots = E_\infty$, and $\mathcal{C}$-cohomologies for this equation are of the form*

$$H^p_{\mathcal{C}}(\pi) \approx \mathcal{F}(\pi, \pi_V) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^p\Lambda(\pi).$$

REMARK 5.7. When π is a vector bundle, then $\mathcal{F}(\pi, \pi_V) \approx \mathcal{F}(\pi, \pi)$ and we have the isomorphism

$$H^p_{\mathcal{C}}(\pi) \approx \mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^p\Lambda(\pi).$$

This result allows to generalize the notion of evolutionary derivations and to introduce *graded* (or *super*) *evolutionary derivations.* Namely, we choose a canonical coordinate system (x, p^j_σ) in $J^\infty(\pi)$ and for any element $\omega = (\omega^1, \dots, \omega^m) \in \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^p\Lambda(\pi)$, $\omega^j \in \mathcal{C}^p\Lambda(\pi)$, set

$$\mathcal{B}_\omega = \sum_{j,\sigma} D_\sigma(\omega^j) \otimes \frac{\partial}{\partial p^j_\sigma} \in \Lambda^p(\pi) \otimes D^v(\pi). \tag{5.41}$$

We call $\mathcal{B}_\omega$ a *graded evolutionary derivation with the generating form* $\omega \in \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^p\Lambda(\pi)$. Denote the set of such derivations by $\kappa^p(\pi)$.

The following local facts are obvious:

(i)
$$\mathrm{L}_{\mathcal{B}_\omega}(\mathcal{F}(\pi)) \subset \mathcal{C}^p\Lambda(\pi),$$

(ii)
$$\mathcal{B}_\omega \in \ker(\partial^p_\pi),$$

(iii) the correspondence $\omega \mapsto \mathcal{B}_\omega$ splits the natural projection
$$\ker(\partial^p_\pi) \to H^p_{\mathcal{C}}(\pi)$$
and thus
$$\ker(\partial^p_\pi) = \operatorname{im}(\partial^{p-1}_\pi) \oplus \kappa^p(\pi).$$

We shall show now that Definition (5.41) is independent of local coordinates. The proposition below, as well as its proof, is quite similar to that one which has been proved in [**60**] for "ordinary" evolutionary derivations (see also Chapter 2).

PROPOSITION 5.15. *Any element* $\Omega \in \Lambda^*(\pi) \otimes D^v(\pi)$ *which satisfies the conditions* (i) *and* (ii) *above, i.e., for which* $\mathrm{L}_\Omega(\mathcal{F}(\pi)) \subset \mathcal{C}^p\Lambda(\pi)$ *and* $\partial_\pi(\Omega) = 0$, *is uniquely determined by the restriction of* L_Ω *onto* $\mathcal{F}_0(\pi) = C^\infty(J^0(\pi))$.

PROOF. First recall that Ω is uniquely determined by the derivation $\mathrm{L}_\Omega \in D^{\mathrm{gr}}(\Lambda^*)$ (see Proposition 4.20). Further, since L_Ω is a graded derivation and due to the fact that

$$\mathrm{L}_\Omega(d\theta) = (-1)^\Omega d(\mathrm{L}_\Omega(\theta)) \tag{5.42}$$

for any $\theta \in \Lambda^*(\pi)$ (Proposition 4.20), L_Ω is uniquely determined by its restriction onto $\mathcal{F}(\pi) = \Lambda^0(\pi)$.

Now, from the equality $\partial_\pi(\Omega) = 0$ it follows that

$$0 = [\![U_\pi, \Omega]\!]^{\mathrm{fn}}(\phi) = \mathrm{L}_{U_\pi}(\mathrm{L}_\Omega(\phi)) - (-1)^\Omega \mathrm{L}_\Omega(\mathrm{L}_{U_\pi}(\phi)). \tag{5.43}$$

Let Ω be such that $\mathrm{L}_\Omega|_{\mathcal{F}_0(\pi)} = 0$ and suppose that we have proved that $\mathrm{L}_\Omega|_{\mathcal{F}_r(\pi)} = 0$. Then taking $\phi = p^j_\sigma$, $|\sigma| = r$, and using equality (5.43), we obtain

$$(-1)^\Omega \mathrm{L}_\Omega\Big(dp^j_\sigma - \sum_{i=1}^n p^j_{\sigma+1_i}\,dx_i\Big) = \mathrm{L}_{U_\pi}(\mathrm{L}_\Omega(p^j_\sigma)) = 0.$$

In other words,

$$\begin{aligned}\mathrm{L}_\Omega\Big(\sum_{i=1}^n p^j_{\sigma+1_i}\,dx_i\Big) &= \sum_{i=1}^n \mathrm{L}_\Omega(p^j_{\sigma+1_i})\,dx_i\\ &= \mathrm{L}_\Omega(dp^j_\sigma) = (-1)^\Omega d(\mathrm{L}_\Omega(p^j_\sigma) = 0.\end{aligned}$$

Since $\mathrm{L}_\Omega(p^j_{\sigma+1_i}) \in \mathcal{C}^*\Lambda(\pi)$, we conclude that $\mathrm{L}_\Omega(p^j_{\sigma+1_i}) = 0$, i.e., we have $\mathrm{L}_\Omega|_{\mathcal{F}_{r+1}(\pi)} = 0$. □

REMARK 5.8. The element $U_\pi = \sum_{j,\sigma}\big(dp^j_\sigma - \sum_i p^j_{\sigma+1_i}\,dx_i\big)\otimes\partial/\partial p^j_\sigma$ itself is an example of an evolutionary derivation: $U_\pi = \mathcal{Э}_\omega$, $\omega = (\omega^1_\varnothing, \dots, \omega^m_\varnothing)$, where $\omega^j_\varnothing = du^j - \sum_i p^j_{1_i}\,dx_i$.

Since

$$\mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^*\Lambda(\pi) = \mathcal{F}(\pi, \pi_V) \otimes \sum_{i\ge 0} \mathcal{C}^i\Lambda(\pi)$$

is identified with the module $H^*_{\mathcal{C}}(\pi)$, it carries the structure of a graded Lie algebra. The corresponding operation in $\mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^*\Lambda(\pi)$ is denoted by $\{\cdot,\cdot\}$ and is called the *graded Jacobi bracket*. Thus, for any elements $\omega \in \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^p\Lambda(\pi)$ and $\theta \in \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^q\Lambda(\pi)$ we have $\{\omega, \theta\} \in \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^{p+q}\Lambda(\pi)$ and

$$\begin{gathered}\{\omega, \theta\} + (-1)^{pq}\{\theta, \omega\} = 0,\\ \oint (-1)^{(p+r)q}\{\omega, \{\theta, \rho\}\} = 0,\end{gathered}$$

where $\rho \in \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^r\Lambda(\pi)$ and $\oint$, as before, denotes the sum of cyclic permutations.

To express the graded Jacobi bracket in more efficient terms we prove the following

PROPOSITION 5.16. *The space $\kappa^*(\pi) = \sum_{i\ge 0}\kappa^i(\pi)$ of super evolutionary derivations is a graded Lie subalgebra in $\Lambda^*(\pi) \otimes D^v(\pi)$, i.e., for any two generating forms $\omega, \theta \in \mathcal{F}(\pi, \pi_V) \otimes \mathcal{C}^*\Lambda(\pi)$ the bracket $[\![\mathcal{Э}_\omega, \mathcal{Э}_\theta]\!]^{\mathrm{fn}}$ is again an evolutionary derivation and*

$$[\![\mathcal{Э}_\omega, \mathcal{Э}_\theta]\!]^{\mathrm{fn}} = \mathcal{Э}_{\{\omega,\theta\}}. \tag{5.44}$$

PROOF. First note that it is obvious that $[\![\mathcal{Э}_\omega, \mathcal{Э}_\theta]\!]^{\mathrm{fn}}$ lies in $\ker(\partial_\pi)$.

Consider a vector field $X \in \mathcal{C}D(\pi)$. Then, since $X \lrcorner \,\partial_\omega = X \lrcorner \,\partial_\theta = 0$, from equality (4.45) on p. 175 it follows that

$$X \lrcorner [\![\partial_\omega, \partial_\theta]\!]^{\mathrm{fn}} = (-1)^{\omega}[\![X, \partial_\omega]\!]^{\mathrm{fn}} \lrcorner \,\partial_\theta - (-1)^{(\omega+1)\theta}[\![X, \partial_\theta]\!]^{\mathrm{fn}} \lrcorner \,\partial_\omega.$$

Let $X = D_i$, where D_i is the total derivative along x_i in the chosen coordinate system. Then we have

$$[\![D_i, \partial_\omega]\!]^{\mathrm{fn}} = \sum_{j,\sigma}\Big(D_\sigma(\omega^j) \otimes \Big[D_i, \frac{\partial}{\partial p_\sigma^j}\Big] + D_{\sigma+1_i}(\omega^j) \otimes \frac{\partial}{\partial p_\sigma^j}\Big) = 0.$$

Since any $X \in \mathcal{C}D(\pi)$ is a linear combination of the fields D_i, one has

$$\mathcal{C}D(\pi) \lrcorner [\![\partial_\omega, \partial_\theta]\!]^{\mathrm{fn}} = 0,$$

i.e., $[\![\partial_\omega, \partial_\theta]\!]^{\mathrm{fn}} \in \mathcal{C}^*\Lambda(\pi) \otimes D^v(\pi)$. Hence, Proposition 5.15 implies that the bracket $[\![\partial_\omega, \partial_\theta]\!]^{\mathrm{fn}}$ is an evolutionary derivation. $\square$

From (5.44) and from Proposition 5.16 it follows that if $(\omega^1, \dots, \omega^m)$ and $(\theta^1, \dots, \theta^m)$ are local representations of ω and θ respectively then

$$\{\omega, \theta\}^i = \sum_{j=1}^{m}\Big(\partial_{\omega^j}(\theta^i) - (-1)^{\omega^j\theta^i}\,\partial_{\theta^i}(\omega^j)\Big), \tag{5.45}$$

where $i = 1, \dots, m$.

For example, if $\omega = \mathrm{L}_{U_\pi}(f) = df - \sum_i D_i(f)\,dx_i$ and $\theta = \mathrm{L}_{U_\pi}(g)$, where $f, g \in \Gamma(\pi)$, then

$$\{\omega, \theta\}^i = \sum_{j,\sigma}\Bigg(\mathrm{L}_{U_\pi}(D_\sigma(f^j)) \wedge \mathrm{L}_{U_\pi}\Big(\frac{\partial g^i}{\partial p_\sigma^j}\Big) + \mathrm{L}_{U_\pi}(D_\sigma(g^j)) \wedge \mathrm{L}_{U_\pi}\Big(\frac{\partial f^i}{\partial p_\sigma^j}\Big)\Bigg),$$

where $i = 1, \dots, m$. In particular,

$$\{\omega_\sigma^i, \omega_\tau^j\} = 0, \tag{5.46}$$

where $\omega_\sigma^i, \omega_\tau^j$ are the Cartan forms (see (1.27) on p. 18).

3. $\mathcal{C}$-cohomologies of evolution equations

Here we give a complete description for $\mathcal{C}$-cohomologies of systems of evolution equations and consider some examples.

Let $\mathcal{E}$ be a system of evolution equations of the form

$$\frac{\partial u^j}{\partial t} = f^j\Big(x, t, u, \dots, \frac{\partial^{|\sigma|}u}{\partial x_\sigma}, \dots\Big), \quad j = 1, \dots, m, \ |\sigma| \le k, \tag{5.47}$$

where $x = (x_1, \dots, x_n)$, $u = (u^1, \dots, u^m)$. Then the functions x, t, p_σ^j, where $j = 1, \dots, m$, $\sigma = (\sigma_1, \dots, \sigma_n)$, can be chosen as internal coordinates on $\mathcal{E}^\infty$. In this coordinate system the element $U_{\mathcal{E}}$ is represented in the form

$$U_{\mathcal{E}} = \sum_{j,\sigma}\Big(dp_\sigma^j - \sum_i p_{\sigma+1_i}^j\,dx_i - D_\sigma(f^j)\,dt\Big) \otimes \frac{\partial}{\partial p_\sigma^j}, \tag{5.48}$$

where $D_\sigma = D_1^{\sigma_1} \circ \dots \circ D_n^{\sigma_n}$, for $\sigma = (\sigma_1, \dots, \sigma_n)$. If

$$\Theta = \sum_{j,\tau} \theta_\tau^j \otimes \frac{\partial}{\partial p_\tau^j} \in \Lambda^*(\mathcal{E}) \otimes D^v(\mathcal{E}), \quad \theta_\tau^j \in \Lambda^*(\mathcal{E}),$$

then, as it follows from (4.40) on p. 175, the differential $\partial_{\mathcal{C}}$ acts in the following way

$$\partial_{\mathcal{C}}(\Theta) = \sum_{j,\tau} \left(\sum_i dx_i \wedge (\theta_{\tau+1_i}^j - D_i(\theta_\tau^j)) \right. \\ \left. + dt \wedge \left(\sum_{s,\sigma} \frac{\partial}{\partial p_\sigma^s} \big(D_\tau(f^j)\big)\theta_\sigma^s - D_t(\theta_\tau^j) \right) \right) \otimes \partial p_\tau^j, \quad (5.49)$$

where

$$D_t = \frac{\partial}{\partial t} + \sum_{j,\mu} D_\mu(f^j) \frac{\partial}{\partial p_\mu^j}.$$

To proceed with computations consider a direct sum decomposition

$$\Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E}) = \Lambda_t^p(\pi) \otimes D^v(\pi) \oplus dt \wedge \Lambda_t^{p-1}(\pi) \otimes D^v(\pi), \quad (5.50)$$

where $\pi \colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is the natural projection with the coordinates $(u^1, \dots, u^m)$ and $(x_1, \dots, x_n)$ in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, while $\Lambda_t^*(\pi)$ denotes the algebra of exterior forms on $J^\infty(\pi)$ with the variable $t \in \mathbb{R}$ as a parameter in their coefficients. From (5.49) and due to (4.45) on p. 175 it follows that if $\Theta \in \Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})$ and

$$\Theta = \Theta^p + dt \wedge \Theta^{p-1}$$

is the decomposition corresponding to (5.50), then

$$\partial_{\mathcal{C}}(\Theta) = \partial_\pi(\Theta^p) + dt \wedge (L_{\mathcal{E}}(\Theta^p) - \partial_\pi(\Theta^{p-1})), \quad (5.51)$$

where

$$L_{\mathcal{E}}(\Theta) = \sum_{j,\tau} \Big(\sum_{s,\sigma} \frac{\partial}{\partial p_\sigma^s} \big(D_\tau(f^j)\big)\theta_\sigma^s - D_t(\theta_\tau^j) \Big) \otimes \frac{\partial}{\partial p_\tau^j}. \quad (5.52)$$

Consider a diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & \Lambda_t^i \otimes D^v \arrow[r, "\partial_\pi^i"] \arrow[d, "dt \wedge L_{\mathcal{E}}"'] & \Lambda_t^{i+1} \otimes D^v \arrow[r] \arrow[d, "dt \wedge L_{\mathcal{E}}"] & \dots \\
\dots \arrow[r] & dt \wedge \Lambda_t^i \otimes D^v \arrow[r, "-\mathrm{id} \wedge \partial_\pi^i"'] \arrow[d] & dt \wedge \Lambda_t^{i+1} \otimes D^v \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 &
\end{tikzcd}
\tag{5.53}
$$

where $\Lambda_t^i \otimes D^v \overset{\text{def}}{=} \Lambda_t^i(\pi) \otimes D^v(\pi)$. From (5.51) and from the fact that $\partial_\pi \circ \partial_\pi = 0$ it follows that (5.53) is a bicomplex whose total differential is $\partial_{\mathcal{C}}$. Thus, from the general theory of bicomplexes (cf. [**70**]) we see that to calculate $H_{\mathcal{C}}^i(\mathcal{E})$ it is necessary:

(i) To compute cohomologies of the upper and lower lines of (5.53). Denote them by $H_{\mathcal{C}}^i(\pi)$ and $H_L^i(\pi)$ respectively.
(ii) To describe the mappings $L_{\mathcal{E}}^i\colon H_{\mathcal{C}}^i(\pi) \to H_L^i(\pi)$ induced by $dt \wedge L_{\mathcal{E}}$.

Then we have

$$H_{\mathcal{C}}^i(\mathcal{E}) = \ker(L_{\mathcal{E}}^i) \oplus \operatorname{coker}(L_{\mathcal{E}}^{i-1}). \tag{5.54}$$

From Theorem 5.14 it follows that $H_{\mathcal{C}}^i(\pi) = \varkappa_t^i(\pi)$ and $H_L^i(\pi) = dt \wedge \varkappa_t^{i-1}(\pi)$, where $\varkappa_t^p(\pi)$ is the set of all evolutionary derivations with generating forms from $\mathcal{F}(\pi,\pi) \otimes \mathcal{C}^p\Lambda_t(\pi)$ parameterized by t (we write $\mathcal{F}(\pi,\pi)$ instead of $\mathcal{F}(\pi,\pi_V)$ since π is a vector bundle in the case under consideration). Let $\omega = (\omega^1,\dots,\omega^m)$ be such a form. Then, as it is easily seen from (5.52),

$$L_{\mathcal{E}}^p(\ni_\omega) = \ni_{\ell_{\mathcal{E}}^{(p)}(\omega)},$$

where

$$\ell_{\mathcal{E}}^{(p)}(\omega) = \sum_j \Big(\sum_{s,\sigma} \frac{\partial f^j}{\partial p_\sigma^s}(D_\sigma \omega^s) - D_t(\omega^j) \Big) \otimes \frac{\partial}{\partial u^j}. \tag{5.55}$$

Comparing (5.55) with equality (2.23) on p. 71, we see that $\ell_{\mathcal{E}}^{(p)}$ is the extension of the universal linearization operator for the equation (5.47) onto the module $\mathcal{F}(\pi,\pi) \otimes \mathcal{C}^p\Lambda_t(\pi)$.

REMARK 5.9. Note that when the operator Δ is the sum of monomials $X_1 \circ \dots \circ X_r$, the action

$$\Delta(\omega) = \sum X_1(X_2(\dots(X_r(\omega))\dots))$$

is well defined for any form ω such that $X_i \lrcorner \omega = 0$, $i = 1, \dots, r$. It is just the case for formula (5.55), since $X \lrcorner \omega = 0$ for any $X \in \mathcal{C}D(\mathcal{E})$ and $\omega \in \mathcal{C}^p\Lambda(\mathcal{E})$.

Thus we have the following generalization of Theorem 2.15 (see p. 72).

THEOREM 5.17. *Let $\mathcal{E}$ be a system of evolution equations of the form* (5.47), *$\ell_{\mathcal{E}} = \ell_{\mathcal{E}}^{(0)}$ be corresponding universal linearization operator restricted onto $\mathcal{E}^\infty$ and $\ell_{\mathcal{E}}^{(p)}$ be the extension of $\ell_{\mathcal{E}}$ onto $\mathcal{F}(\pi,\pi) \otimes \mathcal{C}^p\Lambda_t(\pi)$. Then*

$$H_{\mathcal{C}}^p(\mathcal{E}) \approx \ker(\ell_{\mathcal{E}}^{(p)}) \oplus dt \wedge \operatorname{coker}(\ell_{\mathcal{E}}^{(p-1)}).$$

REMARK 5.10. The result proved is, in fact, valid for all ℓ-normal equations (see Definition 2.16). The proof can be found in [**98**]. Moreover, let us recall that the module $H_{\mathcal{C}}^*(\mathcal{E})$ splits into the direct sum

$$H_{\mathcal{C}}^*(\mathcal{E}) = \bigoplus_{i \geq 0} \bigoplus_{p+q=i} H_{\mathcal{C}}^{p,q}(\mathcal{E}) = \bigoplus_{q=1}^{n} H_{\mathcal{C}}^{*,q}(\mathcal{E}),$$

where the superscripts p and q correspond to the number of Cartan and horizontal components respectively (see decomposition (4.60) on p. 181). As it can be deduced from Proposition 4.29, the component $H_{\mathcal{C}}^{p,0}(\mathcal{E})$ always coincides with $\ker \ell_{\mathcal{E}}^{(p)}$.

As a first example of application of the above theorem, we shall prove that evolution equations in one space variable are 2-trivial objects in the sense of Section 3 of Chapter 4.

PROPOSITION 5.18. *For any evolution equation $\mathcal{E}$ of the form*

$$\frac{\partial u}{\partial t} = f(x,t,u,\dots,u_k), \quad \frac{\partial f}{\partial u_k} \neq 0, \ k > 0,$$

one has $H_{\mathcal{C}}^{2,0}(\mathcal{E}) = 0$.

PROOF. To prove this fact, we need to solve the equation

$$D_t\omega = \sum_{i=1}^{k} \frac{\partial f}{\partial u_i} D_x^i \omega, \tag{5.56}$$

with $\omega = \sum_{\alpha>\beta} \varphi_{\alpha\beta}\omega_\alpha \wedge \omega_\beta$, where $\varphi_{\alpha\beta} \in \mathcal{F}(\mathcal{E})$ and $\omega_\alpha, \omega_\beta$ are the Cartan forms on $\mathcal{E}^\infty$. Let us represent the form ω as

$$\omega = \varphi_{m,m-1}\omega_m \wedge \omega_{m-1} + \sum_{\alpha<m-1} \varphi_{m,\alpha}\omega_m \wedge \omega_\alpha + \sum_{\beta<m-2} \varphi_{m-1,\beta}\omega_{m-1} \wedge \omega_\beta + o[m-1], \tag{5.57}$$

where the term $o[m-2]$ does not contain Cartan forms of degree higher than $m-2$.

Note now that for any Cartan form ω_i one has

$$D_t\omega_i = D_x^i \sum_{\alpha=0}^{k} \frac{\partial f}{\partial u_\alpha}\omega_\alpha = \frac{\partial f}{\partial u_k}\omega_{i+k} + iD_x\left(\frac{\partial f}{\partial u_k}\right)\omega_{i+k-1}$$

and

$$\sum_{\alpha=1}^{k} \frac{\partial f}{\partial u_i} D_x^\alpha \omega_i = \frac{\partial f}{\partial u_k}\omega_{i+k} + \frac{\partial f}{\partial u_{k-1}}\omega_{i+k-1} + o[i+k-2].$$

Substituting (5.57) into (5.56) and using the above decompositions, one can easily see that the coefficients $\varphi_{m,\alpha}$ vanish, from where, by induction, it follows that $\omega = 0$. $\square$

Now we shall look more closely at the module

$$H^1_{\mathcal{C}}(\mathcal{E}) \approx \ker(\ell^{(1)}_{\mathcal{E}}) \oplus dt \wedge \operatorname{coker}(\ell^{(0)}_{\mathcal{E}})$$

and describe infinitesimal deformations of evolution equations in the form ready for concrete computations. From the decomposition given by the previous theorem we see that there are two types of infinitesimal deformations: those ones which lie in $\ker(\ell^{(1)}_{\mathcal{E}})$ and those which originate from $dt \wedge \operatorname{coker}(\ell^{(0)}_{\mathcal{E}})$. The latter ones are represented by the elements of the form

$$U_1 = \sum_j g^j\, dt \otimes \frac{\partial}{\partial u^j} = dt \otimes \theta, \tag{5.58}$$

where $g^j \in \mathcal{F}(\mathcal{E})$. Deformations corresponding to (5.58) are of the form

$$U(\varepsilon) = U_{\mathcal{E}} + U_1\varepsilon + \dots \tag{5.59}$$

But it is easily seen that the first two summands in (5.59) determine an equation of the form

$$u^j_t = f^j + \varepsilon g^j, \quad j = 1, \dots, m, \tag{5.60}$$

which is infinitesimally equivalent to the initial equation if and only if $\theta \in \operatorname{im}(\ell^{(0)}_{\mathcal{E}})$. The deformations (5.60) preserve the class of evolution equations. The other ones lie in $\ker(\ell^{(1)}_{\mathcal{E}})$ and we shall deduce explicit formulas for their computation. For the sake of simplicity we consider the case $\dim(\pi) = m = 1$, $\dim(M) = n = 2$ (one space variable).

Let $\omega_i = dp_i - p_{i+1}\,dx - D_x^i(f)\,dt$, $i = 0, 1, \dots$, be the basis of Cartan forms on $\mathcal{E}^\infty$, where $f = f^1(x, t, p_0, \dots, p_k)$, $x = x_1$, and p_i corresponds to $\partial^i u/\partial x^i$. Then any form $\omega \in \mathcal{C}^1\Lambda(\mathcal{E})$ can be represented as

$$\omega = \sum_{i=0}^{r} \phi^i\omega_i, \quad \phi^i \in \mathcal{F}(\mathcal{E}). \tag{5.61}$$

Thus we have

$$\ell_{\mathcal{E}}^{(1)}(\omega) = \Big(\sum_{j=0}^{k} f_j D_x^j - D_t\Big)\Big(\sum_i \phi^i \omega_i\Big), \tag{5.62}$$

where f_j denotes $\partial f/\partial p_j$. By definition, we have

$$(f_j D_x^j)(\phi^i\omega_i) = f_j(D_x(\dots(D_x(\phi^i\omega_i))\dots)) = f_j \sum_{s=0}^{j} \binom{j}{s} D_x^{j-s}(\phi^i) D_x^s(\omega_i).$$

But

$$D_x(\omega_i) = \omega_{i+1} \tag{5.63}$$

and therefore,

$$(f_j D_x^j)(\phi^i\omega_i) = f_j \sum_{s=0}^{j} \binom{j}{s} D_x^{j-s}(\phi^i)\omega_{i+s}. \tag{5.64}$$

On the other hand,

$$D_t(\phi^i\omega_i) = D_t(\phi^i)\omega_i + \phi^i D_t(\omega_i). \tag{5.65}$$

Since $\omega_i = D_x^i(\omega_0)$ and $[D_t, D_x] = 0$, one has

$$D_t(\omega_i) = D_t(D_x^i(\omega_0)) = D_x^i(D_t(\omega_0)). \tag{5.66}$$

But $\omega_0 = \mathrm{L}_{U_{\mathcal{E}}}(p_0)$ and $[D_t, \mathrm{L}_{U_{\mathcal{E}}}] = 0$. Hence,

$$D_t(\omega_0) = \mathrm{L}_{U_{\mathcal{E}}}(D_t(p_0)) = \mathrm{L}_{U_{\mathcal{E}}}(f) = \sum_{j=0}^{k} f_j\omega_j. \tag{5.67}$$

Combining now (5.62)–(5.67), we find out that the equation $\ell_{\mathcal{E}}^{(1)}(\omega) = 0$ written in the coordinate form looks as

$$\sum_{i=0}^{r}\sum_{j=0}^{k} f_j \sum_{s=0}^{j} \binom{j}{s} D_x^{j-s}(\phi^i)\omega_{i+s} = \sum_{i=0}^{r}\Big(D_t(\phi^i)\omega_i + \phi^i \sum_{j=0}^{k}\sum_{s=0}^{i}\binom{i}{s} D_x^{i-s}(f_j)\omega_{j+s}\Big). \tag{5.68}$$

Taking into account that $\{\omega_i\}_{i\geq 0}$ is the basis in $\mathcal{C}^1\Lambda(\mathcal{E})$ and equating the coefficients at ω_i, we obtain that (5.68) is equivalent to

$$\ell_{\mathcal{E}}(\phi^s) = \sum_{i=0}^{r} \phi^i D_x^i(f_s) + \sum_{l=1}^{s}\Big(\sum_{i=l}^{r}\binom{i}{l}\phi^i D_x^{i-l}(f_{s-l}) - \sum_{j=l}^{k}\binom{j}{l} f_j D_x^{j-l}(\phi^{s-l})\Big) \tag{5.69}$$

where $s = 0, 1, \dots, k+r-1$, which is the final form of (5.62) for the concrete calculations (we set $\phi^i = f_j = 0$ for $i > r$ and $j > k$ in (5.69)).

Consider some examples now.

EXAMPLE 5.1. Let $\mathcal{E}$ be the heat equation

$$u_t = u_{xx}.$$

For this equation (5.69) looks as

$$\begin{aligned}
&D_x^2(\phi^0) = D_t(\phi^0),\\
&D_x^2(\phi^1) + 2D_x(\phi^0) = D_t(\phi^1),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&D_x^2(\phi^r) + 2D_x(\phi^{r-1}) = D_t(\phi^r),\\
&D_x(\phi^r) = 0.
\end{aligned} \tag{5.70}$$

Simple but rather cumbersome computations show that the basis of solutions for (5.70) consists of the functions

$$\begin{aligned}
\phi^0 &= \sum_{j=0}^{s} A^{(j+s)} \frac{x^{2j}}{(2j)!},\\
&\dots\dots\dots\dots\dots\dots\\
\phi^{2i} &= 2^{2i} \sum_{j=0}^{s-i} \binom{i+s-j}{2i} A^{(j+s-i)} \frac{x^{2j}}{(2j)!},\\
\phi^{2i+1} &= 2^{2i+1} \sum_{j=0}^{s-i} \binom{i+s-j}{2i+1} A^{(j+s-i)} \frac{x^{2j+1}}{(2j+1)!},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\phi^{2s} &= 2^{2s} A
\end{aligned}$$

for $r = 2s$ and

$$\begin{aligned}
\phi^0 &= \sum_{j=0}^{s} A^{(j+s+1)} \frac{x^{2j+1}}{(2j+1)!},\\
&\dots\dots\dots\dots\dots\dots\dots\\
\phi^{2i} &= 2^{2i} \sum_{j=0}^{s-i} \binom{i+s-j}{2i} A^{(j+s-i+1)} \frac{x^{2j+1}}{(2j+1)!},\\
\phi^{2i+1} &= 2^{2i+1} \sum_{j=0}^{s-i} \binom{i+s-j+1}{2i+1} A^{(j+s-i)} \frac{x^{2j}}{(2j)!},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\phi^{2s+1} &= 2^{2s+1} A
\end{aligned}$$

for $r = 2s + 1$.

In both cases $A = 1, t, \dots, t^r$ and $A^{(l)}$ denotes $d^l A/dt^l$.

REMARK 5.11. Let $\phi = \sum_j \phi^j \omega_j$ be an element of $H^1_{\mathcal{C}}(\mathcal{E})$ and $\psi \in \mathcal{F}(\pi, \pi)$ be a symmetry of the equation $\mathcal{E}$. Then, as it follows from (5.18), the element $\mathcal{R}_\phi(\psi)$ is a symmetry of $\mathcal{E}$ again. In particular, since the equation under consideration is linear, it possesses the symmetry $\psi = u$. Hence, its symmetries include those of the form

$$\mathcal{R}_\phi(u) = \sum_j \phi^j p_j,$$

where ϕ^j are given by the formulae above.

EXAMPLE 5.2. The second example we consider is the Burgers equation

$$u_t = uu_x + u_{xx}. \tag{5.71}$$

THEOREM 5.19. *The only solution of the equation* $\ell^{(1)}_{\mathcal{E}}(\omega) = 0$ *for the Burgers equation* (5.71) *is* $\omega = \alpha\omega_0$, $\alpha = \text{const}$.

PROOF. Let $\omega = \phi^0\omega_0 + \dots + \phi^r\omega_r$. Then equations (5.69) transform into

$$\begin{gathered}
p_0 D_x(\phi^0) + D_x^2(\phi^0) = D_t(\phi^0) + \sum_{j=1}^{r} p_{j+1}\phi^j, \\
p_0 D_x(\phi^1) + D_x^2(\phi^1) + 2D_x(\phi^0) = D_t(\phi^1) + \sum_{j=2}^{r}(j+1)p_j\phi^j, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
p_0 D_x(\phi^i) + D_x^2(\phi^i) + 2D_x(\phi^{i-1}) = D_t(\phi^i) + \sum_{j=i+1}^{r}\binom{j+1}{i}p_{j-i+1}\phi^j, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
p_0 D_x(\phi^r) + D_x^2(\phi^r) + 2D_x(\phi^{r-1}) = D_t(\phi^r) + rp_1\phi^r, \\
D_x(\phi^r) = 0.
\end{gathered} \tag{5.72}$$

To prove the theorem we apply the same scheme which was used to describe the symmetry algebra of the Burgers equation in Chapter 2.

Denote by $\mathcal{K}_r$ the set of solutions of (5.72). A direct computation shows that

$$\mathcal{K}_1 = \{\alpha\omega_0 \mid \alpha \in \mathbb{R}\} \tag{5.73}$$

and that any element $\omega \in \mathcal{K}_r$, $r > 1$, is of the form

$$\omega = \alpha_r\omega_r + \Big(\frac{r}{2}p_0\alpha_r + \frac{1}{2}x\alpha_r^{(1)} + \alpha_{r-1}\Big)\omega_{r-1} + \Omega(r-2), \tag{5.74}$$

where $\alpha_r = \alpha_r(t)$, $a_{r-1} = a_{r-1}(t)$, $a^{(i)}$ denotes $d^i\alpha/dt^i$ and $\Omega(s)$ is an arbitrary linear combination of $\omega_0, \dots, \omega_s$ with the coefficients in $\mathcal{F}(\mathcal{E})$.

LEMMA 5.20. *For any evolution equation $\mathcal{E}$ one has*

$$[\![\mathrm{sym}(\mathcal{E}), \ker(\ell_{\mathcal{E}}^{(1)})]\!]^{\mathrm{fn}} \subset \ker(\ell_{\mathcal{E}}^{(1)}).$$

PROOF OF LEMMA 5.20. In fact, we know that there exists the natural action of $\mathrm{sym}(\mathcal{E}) = H_{\mathcal{C}}^{0}(\mathcal{E})$ on $H_{\mathcal{C}}^{i}(\mathcal{E})$. On the other hand, if $X = \vartheta_\phi \in \mathrm{sym}(\mathcal{E})$ and $\Theta = \vartheta_\theta \in \ker(\ell_{\mathcal{E}}^{(1)})$ where $\phi \in \mathcal{F}(\mathcal{E})$ and $\theta \in \mathcal{C}^1\Lambda_t(\pi)$, then $[\![X, \Theta]\!]^{\mathrm{fn}} = \vartheta_{\{\phi,\theta\}}$. But the element

$$\{\phi, \theta\} = \vartheta_\phi(\theta) - \vartheta_\theta(\phi) = \Big(\sum_i D_x^i(\phi)\frac{\partial}{\partial p_i}\Big)(\theta) - \Big(\sum_i D_x^i(\theta)\frac{\partial}{\partial p_i}\Big)(\phi)$$

obviously lies in $\mathcal{C}^1\Lambda_t(\pi)$. □

Thus, if $\vartheta_\phi \in \mathrm{sym}(\mathcal{E})$ and $\omega \in \ker(\ell_{\mathcal{E}}^{(1)})$ then $\{\phi, \omega\}$ lies in $\ker(\ell_{\mathcal{E}}^{(1)})$ as well.

Let $\phi = p_1$. Then we have

$$\{p_1, \omega\} = \Big(\sum_i p_{i+1}\frac{\partial}{\partial p_i}\Big)(\omega) - D_x(\omega) = -\frac{\partial}{\partial x}(\omega).$$

If $\omega \in \mathcal{K}_r$ then, since p_1 is a symmetry of $\mathcal{E}$, from (5.74) and from Lemma 5.20 we obtain that

$$\mathrm{ad}_{p_1}^{(r-1)}(\omega) = \alpha_r^{(r-1)}\omega_1 + \Omega(0) \in \mathcal{K}_1,$$

where $\mathrm{ad}_\phi = \{\phi, \cdot\}$. Taking into account (5.73) we get that $\alpha_r^{(r-1)} = 0$, or

$$\alpha_r = a_0 + a_1 t + \dots + a_{r-2}t^{r-2}, \quad a_i \in \mathbb{R}. \tag{5.75}$$

Recall now (see Chapter 2) that

$$\Phi = t^2 p_2 + (t^2 p_0 + tx)p_1 + tp_0 + x$$

is a symmetry of (5.71) and compute $\{\Phi, \omega\}$ for ω of the form (5.74). To do this, we shall need another lemma.

LEMMA 5.21. *For any $\phi \in \mathcal{F}(\mathcal{E})$ the identity*

$$\vartheta_\phi \circ \mathrm{L}_U = \mathrm{L}_U \circ \vartheta_\phi$$

holds, where $U = U_{\mathcal{E}}$.

PROOF OF LEMMA 5.21. In fact,

$$0 = \partial_{\mathcal{C}}(\vartheta_\phi) = [\mathrm{L}_U, \vartheta_\phi] = \mathrm{L}_U \circ \vartheta_\phi - \vartheta_\phi \circ \mathrm{L}_U.$$

□

Consider the form $\omega = \phi^s\omega_s$, $\phi^s \in \mathcal{F}(\mathcal{E})$. Then we have

$$\{\Phi, \phi^s\omega_s\} = \vartheta_\Phi(\phi^s\omega_s) - \vartheta_{(\phi^s\omega_s)}(\Phi) = \vartheta_\Phi(\phi^s)\omega_s + \phi^s\vartheta_\Phi(\omega_s) - \vartheta_{(\phi^s\omega_s)}(\Phi).$$

But

$$\begin{aligned}\vartheta_\Phi(\omega_s) &= \vartheta_\Phi \mathrm{L}_U(p_s) = \mathrm{L}_U\vartheta_\Phi(p_s) = \mathrm{L}_U D_x^s(\Phi)\\ &= \mathrm{L}_U\big(t^2p_{s+2} + (t^2p_0 + tx)p_{s+1} + (s+1)(t^2p_1 + t)p_s\big) + \Omega(s-1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{\partial}_{(\phi^s\omega_s)}(\Phi) = t^2\phi^s\omega_{s+2} + \bigl(2t^2D_x(\phi^s) + (t^2p_0 + tx)\phi^s\bigr)\omega_{s+1} \\ + \bigl(t^2D_x^2(\phi^s) + (t^2p_0 + tx)D_x(\phi^s) + (t^2p_1 + t)\phi^s\bigr)\omega_s.\end{aligned}$$

Thus, we finally obtain

$$\begin{aligned}\{\Phi, \phi^s\omega_s\} = \{\Phi, \phi^s\}\omega_s + (s+1)(t^2p_1 + t)\omega_s \\ - 2t^2D_x(\phi^s)\omega_{s+1} + \Omega(s-1).\end{aligned} \tag{5.76}$$

Applying (5.76) to (5.74), we get

$$\mathrm{ad}_\Phi(\omega) = \{\Phi, \omega\} = (rt\alpha_r - t^2\alpha_r^{(1)})\omega_r + \Omega(r-1). \tag{5.77}$$

Let now $\omega \in \mathcal{K}_r$ and suppose that ω has a nontrivial coefficient α_r of the form (5.75) and a_i is the first nontrivial coefficient in α_r. Then, by (5.77),

$$\mathrm{ad}_\Phi^{r-i}(\omega) = \alpha'_r\omega_r + \Omega(r-1) \in \mathcal{K}_r,$$

where α'_r is a polynomial of the degree $r-1$. This contradicts to (5.75) and thus finishes the proof. □

4. From deformations to recursion operators

The last example of the previous section shows that our theory is not complete so far. In fact, it is well known that the Burgers equation possesses a recursion operator. On the other hand, in Chapter 4 we identified the elements of the group $H_{\mathcal{C}}^{1,0}(\mathcal{E})$ with the algebra of recursion operators. Consequently, the result of Theorem 5.19 contradicts to practical knowledge. The reason is that almost all known recursion operators contain "nonlocal terms" like D_x^{-1}. To introduce terms of such a type into our theory, we need to combine it with the theory of coverings (Chapter 3), introducing necessary nonlocal variables

Let us do this. Namely, let $\mathcal{E}$ be an equation and $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering over its infinite prolongation. Then, due to Proposition 3.1 on p. 102, the triad $\bigl(\mathcal{F}(\mathcal{N}), C^\infty(M), (\pi_\infty \circ \varphi)^*\bigr)$ is an algebra with the flat connection $\mathcal{C}^\varphi$. Hence, we can apply the whole machinery of Chapter 4 to this situation. To stress the fact that we are working over the covering φ, we shall add the symbol φ to all notations introduced in this chapter. Denote by $U_{\mathcal{C}}^\varphi$ the connection form of the connection $\mathcal{C}^\varphi$ (the structural element of the covering φ).

In particular, on $\mathcal{N}$ we have the $\mathcal{C}^\varphi$-differential

$$\partial_{\mathcal{C}}^\varphi = [\![U_{\mathcal{E}}^\varphi, \cdot]\!]^{\mathrm{fn}}\colon D^v(\Lambda^i(\mathcal{N})) \to D^v(\Lambda^{i+1}(\mathcal{N})),$$

whose 0-cohomology $H_{\mathcal{C}}^0(\mathcal{E}, \varphi)$ coincides with the Lie algebra $\mathrm{sym}_\varphi \mathcal{E}$ of nonlocal φ-symmetries, while the module $H_{\mathcal{C}}^{1,0}(\mathcal{E}, \varphi)$ identifies with recursion operators acting on these symmetries and is denoted by $\mathcal{R}(\mathcal{E}, \varphi)$. We also have the horizontal and the Cartan differential d_h^φ and $d_{\mathcal{C}}^\varphi$ on $\mathcal{N}$ and the splitting $\Lambda^i(\mathcal{N}) = \bigoplus_{p+q=i} \mathcal{C}^p\Lambda^p(\mathcal{N}) \otimes \Lambda_h^q(\mathcal{N})$.

Choose a trivialization of the bundle $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ and nonlocal coordinates $w^1, w^2, \dots$ in the fiber. Then any derivation $X \in D^v(\Lambda^i(\mathcal{N}))$ splits to the sum $X = X_\mathcal{E} + X^v$, where $X_\mathcal{E}(w^j) = 0$ and X^v is a φ-vertical derivation.

Lemma 5.22. *Let $\varphi\colon \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty$, $N \le \infty$, be a covering. Then $H^{p,0}_\mathcal{C}(\mathcal{E},\varphi) = \ker \partial^\varphi_\mathcal{C}\big|\,\mathcal{C}^p\Lambda(\mathcal{N})$. Thus the module $H^{p,0}_\mathcal{C}(\mathcal{E},\varphi)$ consists of derivations $\Omega\colon \mathcal{F}(\mathcal{N}) \to \mathcal{C}^p\Lambda(\mathcal{N})$ such that*

$$[\![U^\varphi_\mathcal{E}, \Omega]\!]^{\mathrm{fn}}_\mathcal{E} = 0, \quad \big([\![U^\varphi_\mathcal{E}, \Omega]\!]^{\mathrm{fn}}\big)^v = 0. \tag{5.78}$$

Proof. In fact, due to equality (4.55) on p. 179, any element lying in the image of $\partial^\varphi_\mathcal{C}$ contains at least one horizontal component, i.e.,

$$\partial^\varphi_\mathcal{C}\big(\mathrm{D}^v(\mathcal{C}^p\Lambda(\mathcal{N}))\big) \subset D^v(\mathcal{C}^p\Lambda(\mathcal{N}) \otimes \Lambda^1_h(\mathcal{N})).$$

Thus, equations (5.78) should hold. $\square$

We call the first equation in (5.78) the *shadow equation* while the second one is called the *relation equation*. This is explained by the following result (cf. Theorem 3.7).

Proposition 5.23. *Let $\mathcal{E}$ be an evolution equation of the form*

$$u_t = f(x, t, u, \dots, \frac{\partial^k u}{\partial u^k})$$

and $\varphi\colon \mathcal{N} = \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty$ be a covering given by the vector fields[2]

$$\tilde{D}_x = D_x + X, \quad \tilde{D}_t = D_t + T,$$

where $[\tilde{D}_x, \tilde{D}_t] = 0$ and

$$X = \sum_s X^s \frac{\partial}{\partial w^s}, \quad T = \sum_s T^s \frac{\partial}{\partial w^s},$$

$w^1, \dots, w^s, \dots$ being nonlocal variables in φ. Then the group $H^{p,0}_\mathcal{C}(\mathcal{E},\varphi)$ consists of elements

$$\Psi = \sum_i \Psi_i \otimes \frac{\partial}{\partial u_i} + \sum_s \psi^s \frac{\partial}{\partial w^s} \in D^v(\mathcal{C}^p\Lambda(\mathcal{N}))$$

such that $\Psi_i = \tilde{D}^i_x \Psi_0$ and

$$\tilde{\ell}^{(p)}_\mathcal{E}(\Psi_0) = 0, \tag{5.79}$$

$$\sum_\alpha \frac{\partial X^s}{\partial u_\alpha} \tilde{D}^\alpha_x(\Psi_0) + \sum_\beta \frac{\partial X^s}{\partial w^\beta} \psi^\beta = \tilde{D}_x(\psi^s), \tag{5.80}$$

$$\sum_\alpha \frac{\partial T^s}{\partial u_\alpha} \tilde{D}^\alpha_x(\Psi_0) + \sum_\beta \frac{\partial T^s}{\partial w^\beta} \psi^\beta = \tilde{D}_t(\psi^s), \tag{5.81}$$

$s = 1, 2, \dots$, where $\tilde{\ell}^{(p)}_\mathcal{E}$ is the natural extension of the operator $\ell^{(p)}_\mathcal{E}$ to $\mathcal{N}$.

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[2]To simplify the notations of Chapter 4, we denote the lifting of a $\mathcal{C}$-differential operator Δ to $\mathcal{N}$ by $\tilde{\Delta}$.

PROOF. Consider the Cartan forms

$$\omega_i = du_i - u_{i+1}\,dx - D_x^i(f)\,dt, \quad \theta^s = dw^s - X^s\,dx - T^s\,dt$$

on $\mathcal{N}$. Then the derivation

$$U_{\mathcal{E}}^{\varphi} = \sum_i \omega_i \otimes \frac{\partial}{\partial u_i} + \sum_s \theta^s \otimes \frac{\partial}{\partial w^s}$$

is the structural element of the covering φ. Then, using representation (4.40) on p. 175, we obtain

$$\begin{aligned}
\partial_{\mathcal{C}}^{\varphi}\Psi = dx \wedge \sum_i \bigl(\Psi_{i+1} - \tilde{D}_x(\Psi_i)\bigr) \otimes \frac{\partial}{\partial u_i} \\
+ dt \wedge \sum_i \Bigl(\sum_\alpha \frac{\partial(D_x^i f)}{\partial u_\alpha}\Psi_\alpha - \tilde{D}_t\Psi_i\Bigr) \otimes \frac{\partial}{\partial u_i} \\
+ dx \wedge \sum_s \Bigl(\sum_\alpha \frac{\partial X^s}{\partial u_\alpha}\Psi_\alpha + \sum_\beta \frac{\partial X^s}{\partial w^\beta}\psi^\beta - \tilde{D}_x(\psi^s)\Bigr) \otimes \frac{\partial}{\partial w^s} \\
+ dt \wedge \sum_s \Bigl(\sum_\alpha \frac{\partial T^s}{\partial u_\alpha}\Psi_\alpha + \sum_\beta \frac{\partial T^s}{\partial w^\beta}\psi^\beta - \tilde{D}_t(\psi^s)\Bigr) \otimes \frac{\partial}{\partial w^s},
\end{aligned}$$

which gives the needed result. □

Note that relations $\Psi_i = \tilde{D}_x^i(\Psi_0)$ together with equation (5.79) are equivalent to the shadow equations. In the case $p = 1$, we call the solutions of equation (5.79) the *shadows of recursion operators* in the covering φ. Equations (5.80) and (5.81) are exactly the relation equations on the case under consideration.

Thus, any element of the group $H_{\mathcal{C}}^{1,0}(\mathcal{E},\varphi)$ is of the form

$$\Psi = \sum_i \tilde{D}_x^i(\psi) \otimes \frac{\partial}{\partial u_i} + \sum_s \psi^s \otimes \frac{\partial}{\partial w^s}, \tag{5.82}$$

where the forms $\psi = \Psi_0$, $\psi^s \in \mathcal{C}^1\Lambda(\mathcal{N})$ satisfy the system of equations (5.79)–(5.81).

As a direct consequence of the above said, we obtain the following

COROLLARY 5.24. *Let Ψ be a derivation of the form* (5.82) *with $\psi, \psi^s \in \mathcal{C}^p\Lambda(\mathcal{N})$. Then ψ is a solution of equation* (5.79) *in the covering φ if and only if $\partial_{\mathcal{C}}^{\varphi}(\Psi)$ is a φ-vertical derivation.*

We can now formulate the main result of this subsection.

THEOREM 5.25. *Let $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering, $S \in \mathrm{sym}_\varphi \mathcal{E}$ be a φ-symmetry, and $\psi \in \mathcal{C}^1\Lambda(\mathcal{N})$ be a shadow of a recursion operator in the covering φ. Then $\psi' = \mathrm{i}_S\psi$ is a shadow of a symmetry in φ, i.e., $\tilde{\ell}_{\mathcal{E}}(\psi') = 0$.*

PROOF. In fact, let Ψ be a derivation of the form (5.82). Then, due to identity (4.54) on p. 179, one has

$$\partial_{\mathcal{C}}^{\varphi}(\mathrm{i}_S\Psi) = \mathrm{i}_{\partial_{\mathcal{C}}^{\varphi}S} - \mathrm{i}_S(\partial_{\mathcal{C}}^{\varphi}\Psi) = -\mathrm{i}_S(\partial_{\mathcal{C}}^{\varphi}\Psi),$$

since S is a symmetry. But, by Corollary 5.24, $\partial_{\mathcal{C}}^{\varphi}\Psi$ is a φ-vertical derivation and consequently $\partial_{\mathcal{C}}^{\varphi}(\mathrm{i}_S\Psi) = -\mathrm{i}_S(\partial_{\mathcal{C}}^{\varphi}\Psi)$ is φ-vertical as well. Hence, $\mathrm{i}_S\Psi$ is a φ-shadow by the same corollary. $\square$

Using the last result together with Theorem 3.11, we can describe the process of generating a series of symmetries by shadows of recursion operators. Namely, let ψ be a symmetry and $\omega \in \mathcal{C}^1\Lambda(\mathcal{N})$ be a shadow of a recursion operator in a covering $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$. In particular, ψ is a φ-shadow. Then, by Theorem 3.9, there exists a covering $\varphi_\psi\colon \mathcal{N}_\psi \to \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ where ∂_ψ can be lifted to as a φ_ψ-symmetry. Obviously, ω still remains a shadow in this new covering. Therefore, we can act by ω on ψ and obtain a shadow ψ_1 of a new symmetry on $\mathcal{N}_\psi$. By Theorem 3.11, there exists a covering, where both ψ and ψ_1 are realized as nonlocal symmetries. Thus we can continue the procedure applying ω to ψ_1 and eventually arrive to a covering in which the whole series $\{\psi_k\}$ is realized.

Thus, we can state that classical recursion operators are *nonlocal deformations* of the equation structure. Algorithmically, computation of such deformations fits the following scheme:

1. Take an equation $\mathcal{E}$ and solve the linear equation $\ell_{\mathcal{E}}^{(1)}\omega = 0$, where ω is an arbitrary Cartan form.
2. If solutions are trivial, take a covering $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ and try to find shadows of recursion operators. Usually, such a covering is given by conservation laws of the equation $\mathcal{E}$.
3. If necessary, add another nonlocal variable (perhaps, defined by a nonlocal conservation law), etc.
4. If you succeeded to find a nontrivial solution Ω, then the corresponding recursion operator acts by the rule $\mathcal{R}_\Omega\colon \psi \mapsto \partial_\psi \lrcorner\, \Omega$, where ψ is the generating function of a symmetry.

In the examples below, we shall see how this algorithm works.

REMARK 5.12. Let us establish relation between recursion operators introduced in this chapter with their interpretation as Bäcklund transformations given in Section 8 of Chapter 3.

Let Ω be a shadow of a recursion operator in come covering $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$. Then we can consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{N} \arrow[d, "\varphi"'] & V\mathcal{N} \arrow[l, "\tau^V_{\mathcal{N}}"'] \arrow[d, "V_\varphi"'] \arrow[dr, "V_\Omega"] & \\
\mathcal{E}^\infty & V\mathcal{E} \arrow[l, "\tau^V_{\mathcal{E}}"'] & V\mathcal{E}
\end{tikzcd}$$

where $\tau_{\mathcal{E}}^{V}$ and $\tau_{\mathcal{N}}^{V}$ are the Cartan coverings of the equation and its covering respectively, V_φ is naturally constructed by φ, while the mapping V_Ω is defined by $V_\Omega(v) = v \lrcorner \Omega$, $v \in V\mathcal{N}$. The pair (V_φ, V_Ω) is the Bäcklund transformation corresponding to the recursion operator defined by Ω.

This interpretation is another way to understand why shadows of recursion operators take symmetries to shadows of symmetries (see Section 8 of Chapter 3).

5. Deformations of the Burgers equation

Deformations of the Burgers equation

$$u_t = uu_1 + u_2 \tag{5.83}$$

will be discussed from the point of view of the theory of deformations in coverings. We start with the following theorem (see Theorem 5.19 above):

THEOREM 5.26. *The only solution of the deformation equation*

$$\ell_{\mathcal{E}}^{(1)}(\Omega) = 0$$

for the Burgers equation (5.83) *is* $\omega = \alpha\omega_0$ *where* α *is a constant and*

$$\omega_0 = du - u_1\,dx - (uu_1 + u_2)\,dt$$

i.e., Cartan form associated to u. *This leads to the trivial deformation of* $U_{\mathcal{E}}$ *for* (5.83).

In order to find nontrivial deformations for the Burgers equation, we have to discuss them in the nonlocal setting. So in order to arrive at an augmented system, a situation similar to that one for the construction of nonlocal symmetries (see Section 4 of Chapter 3), we first have to construct conservation laws for the Burgers equation and from this we have to introduce nonlocal variables.

The only conservation law for the Burgers equation is given by

$$D_t(u) = D_x\Big(\frac{1}{2}u^2 + u_1\Big), \tag{5.84}$$

which is just the Burgers equation itself.

In (5.84), the total derivative operators D_x and D_t are given in local coordinates on $\mathcal{E}$, x, t, u, u_1, $u_2, \dots$, by

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + \dots, \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{1t}\frac{\partial}{\partial u_1} + \dots \end{aligned} \tag{5.85}$$

The conservation law (5.84) to the introduction of the new nonlocal variable y, which satisfies formally the additional partial differential equations

$$\begin{aligned} y_x &= u, \\ y_t &= \frac{1}{2}u^2 + u_1. \end{aligned} \tag{5.86}$$

We now start from the covering $\mathcal{E}^1 = \mathcal{E}^\infty \times \mathbb{R}$, where the Cartan distribution, or equivalently the total derivative operators D_x, D_t, is given by

$$\widetilde{D}_x = D_x + u\frac{\partial}{\partial y},$$
$$\widetilde{D}_t = D_t + \Big(\frac{1}{2}u^2 + u_1\Big)\frac{\partial}{\partial y}, \tag{5.87}$$

where y is the (formal) nonlocal variable $y = \int^x u\,dx$ with the associated Cartan form ω_{-1} defined by

$$\omega_{-1} = dy - u\,dx - \Big(\frac{1}{2}u^2 + u_1\Big)\,dt. \tag{5.88}$$

Local coordinates in $\mathcal{E}^1$ are given by

$$(x, t, y, u, u_1, \dots).$$

We now demonstrate the calculations involved in the computations of deformations of a partial differential equation or a system of differential equations.

In order to construct deformations of the Burgers equation (5.83)

$$U = \sum \widetilde{D}_x^i(\Omega) \otimes \frac{\partial}{\partial u_i}, \tag{5.89}$$

we start at the generating form

$$\Omega = F^0\omega_0 + F^1\omega_1 + F^2\omega_2 + F^3\omega_3 + F^{-1}\omega_{-1}, \tag{5.90}$$

where F^i, $i = -1, \dots, 3$, are functions dependent on u, u_1, u_2, u_3, u_4, u_5, y.

The Cartan forms $\omega_{-1}, \dots, \omega_3$ are given by

$$\begin{aligned}
\omega_0 &= du - u_1\,dx - (uu_1 + u_2)\,dt,\\
\omega_1 &= du_1 - u_2\,dx - (u_1^2 + uu_2 + u_3)\,dt,\\
\omega_2 &= du_2 - u_3\,dx - (uu_3 + 3u_1u_2 + u_4)\,dt,\\
\omega_3 &= du_3 - u_4\,dx - (uu_4 + 4u_1u_3 + 3u_2^2 + u_5)\,dt,\\
\omega_{-1} &= dy - u\,dx - \Big(\frac{1}{2}u^2 + u_1\Big)\,dt,
\end{aligned} \tag{5.91}$$

and it is a straightforward computation to show that

$$\begin{aligned}
\widetilde{D}_x(\omega_i) &= \omega_{i+1},\\
\widetilde{D}_t(\omega_{-1}) &= u\omega_0 + \omega_1,\\
\widetilde{D}_t(\omega_0) &= u_1\omega_0 + u\omega_1 + \omega_2,\\
\widetilde{D}_t(\omega_1) &= u_2\omega_0 + 2u_1\omega_1 + u\omega_2 + \omega_3,\\
\widetilde{D}_t(\omega_2) &= u_3\omega_0 + 3u_2\omega_1 + 3u_1\omega_2 + u\omega_3 + \omega_4,\\
\widetilde{D}_t(\omega_3) &= u_4\omega_0 + 4u_3\omega_1 + 6u_2\omega_2 + 4u_1\omega_3 + u\omega_4 + \omega_5,
\end{aligned} \tag{5.92}$$

where $i = -1, 0, \dots$

Now the equation for nonlocal deformations is (4.65), see p. 185,

$$\ell^1_{\mathcal{E}^1}(\Omega) = 0.$$

Since this one amounts to

$$\widetilde{D}_t(\Omega) - u_1\Omega - u\widetilde{D}_x(\Omega) - \widetilde{D}_x^2(\Omega) = 0, \tag{5.93}$$

we are led to an overdetermined system of partial differential equations for the functions $F^{-1}, \dots, F^3$, by equating coefficients of $\omega_{-1}, \dots, \omega_4$ to zero, i.e.,

$$\begin{aligned}
\omega_4: \quad & 0 = -2\widetilde{D}_x(F^3),\\
\omega_3: \quad & 0 = -2\widetilde{D}_x(F^2) + (\widetilde{D}_t - u\widetilde{D}_x - u_1 - \widetilde{D}_x^2)(F^3) + 4u_1F^3,\\
\omega_2: \quad & 0 = -2\widetilde{D}_x(F^1) + (\widetilde{D}_t - u\widetilde{D}_x - u_1 - \widetilde{D}_x^2)(F^2) + 6u_2F^3 + 3u_1F^2,\\
\omega_1: \quad & 0 = -2\widetilde{D}_x(F^0) + (\widetilde{D}_t - u\widetilde{D}_x - u_1 - \widetilde{D}_x^2)(F^1) + 4u_3F^3 + 3u_2F^2\\
& \quad + 2u_1F^1,\\
\omega_0: \quad & 0 = -2\widetilde{D}_x(F^{-1}) + (\widetilde{D}_t - u\widetilde{D}_x - u_1 - \widetilde{D}_x^2)(F^0) + u_4F^3 + u_3F^2\\
& \quad + u_2F^1 + u_1F^0,\\
\omega_{-1}: \quad & 0 = (\widetilde{D}_t - u\widetilde{D}_x - u_1 - \widetilde{D}_x^2)(F^{-1}).
\end{aligned} \tag{5.94}$$

Note that in each coefficient related to $\omega_{-1}, \dots, \omega_3$ there is always a number of terms which together are just

$$\big(\widetilde{D}_t - u\widetilde{D}_x - u_1 - \widetilde{D}_x^2\big)(F^i), \qquad i = -1, 0, 1, 2, 3, \tag{5.95}$$

which arise by action of $\ell^1_{\mathcal{E}^1}$ on the coefficient F^i of the term $F^i\omega_i$ in Ω, (5.90). From these equations we obtain the solution by solving the system in the order as given by the equations in (5.94).

This leads to the following solutions

$$\begin{aligned}
F^3 &= c_1,\\
F^2 &= \frac{3}{2}c_1u + c_2,\\
F^1 &= c_1\Big(\frac{3}{4}u^2 + 3u_1\Big) + c_2u + c_3,\\
F^0 &= c_1\Big(\frac{1}{8}u^3 + \frac{9}{4}uu_1 + 2u_2\Big) + c_2\Big(\frac{1}{4}u^2 + \frac{3}{2}u_1\Big) + \frac{1}{2}c_3u + c_4,\\
F^{-1} &= c_1\Big(\frac{3}{8}u^2u_1 + \frac{3}{4}uu_2 + \frac{3}{4}u_1^2 + \frac{1}{2}u_3\Big) + c_2\Big(\frac{1}{2}uu_1 + \frac{1}{2}u_2\Big) + \frac{1}{2}c_3u_1 + c_5.
\end{aligned} \tag{5.96}$$

Combination of (5.90) and (5.96)) leads to the following independent solutions

$$\begin{aligned}
W^1 &= \omega_0,\\
W^2 &= u_1\omega_{-1} + u_0\omega_0 + 2\omega_1,
\end{aligned}$$

$$\begin{aligned}
W^4 &= 2(uu_1 + u_2)\omega_{-1} + (u^2 + 6u_1)\omega_0 + 4u\omega_1 + 4\omega_2,\\
W^7 &= (3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3)\omega_{-1} + (u^3 + 18uu_1 + 16u_2)\omega_0\\
&\quad + 6(u_1^2 + 4u_1)\omega_1 + 12u\omega_2 + 8\omega_3. \qquad (5.97)
\end{aligned}$$

In case we start from functions F^i, $i = -1, \ldots, 2$, in (5.90), dependent on x, t, u, u_1, u_2, u_3, u_4, u_5, y, and taking $F^3 = 0$, and solving the system of equations (5.96) in a straightforward way, we arrive to

$$\begin{aligned}
F^2 &= c_1(t),\\
F^1 &= \frac{1}{2}c_1'(t)x + c_1(t)u + c_2(t),\\
F^0 &= \frac{1}{8}(c_1''(t)x^2 + 2c_1'(t)xu + 2c_1(t)u^2 + 12c_1(t)u_1 + 4c_2'(t)x\\
&\quad + 4c_2(t)u + c_3(t)),\\
F^{-1} &= \frac{1}{48}(c_1'''(t)x^3 - 6c_1''(t)x + 12c_1'(t)u + 12c_1'(t)xu_1\\
&\quad + 24c_1(t)(uu_1 + u_2) + 6c_2''(t)x^2 + 24c_2(t)u_1 + 24c_3'(t)x + 48c_4(t)). \qquad (5.98)
\end{aligned}$$

Finally, from the last equation in (5.94) we arrive at

$$\begin{aligned}
c_1(t) &= \alpha_1 + \alpha_2 t + \alpha_3 t^2,\\
c_2(t) &= \alpha_4 + \alpha_5 t,\\
c_3(t) &= \alpha_6 + \frac{3}{2}t,\\
c_4(t) &= -\frac{1}{2}c_5, \qquad (5.99)
\end{aligned}$$

which leading to the six independent solutions

$$\begin{aligned}
\alpha_6 :&\quad W^1 = \omega_0,\\
\alpha_4 :&\quad W^2 = u_1\omega_{-1} + u_0\omega_0 + 2\omega_1,\\
\alpha_5 :&\quad W^3 = (tu_1 + 1)\omega_{-1} + (tu + x)\omega_0 + 2t\omega_1,\\
\alpha_1 :&\quad W^4 = 2(uu_1 + u_2)\omega_{-1} + (u^2 + 6u_1)\omega_0 + 4u\omega_1 + 4\omega_2,\\
\alpha_2 :&\quad W^5 = (2tuu_1 + 2tu_2 + xu_1 + u)\omega_{-1},\\
&\quad\quad + (tu^2 + 6tu_1 + xu)\omega_0 + (4tu + 2x)\omega_1 + 4t\omega_2,\\
\alpha_3 :&\quad W^6 = (2t^2(uu_1 + u_2) + 2txu_1 + 2tu + 2x)\omega_{-1},\\
&\quad\quad + (t^2(uu^2 + 6u_1) + 2txu + 6t + x^2)\omega_0,\\
&\quad\quad + (4t^2u + 4tx)\omega_1 + 4t^2\omega_2. \qquad (5.100)
\end{aligned}$$

If we choose the term F^3 in (5.90) to be dependent of x, t, u, u_1, u_2, u_3, u_4, u_5, y too, the general solution of the deformation equation (5.93), or equivalently the resulting overdetermined system of partial differential equations (5.94) for the coefficients F^i, $i = -1, \dots, 3$, is a linear combination of the following ten solutions

$$
\begin{aligned}
W^1 &= \omega_0, \\
W^2 &= u_1\omega_{-1} + u_0\omega_0 + 2\omega_1, \\
W^3 &= (tu_1 + 1)\omega_{-1} + (tu + x)\omega_0 + 2t\omega_1, \\
W^4 &= 2(uu_1 + u_2)\omega_{-1} + (u^2 + 6u_1)\omega_0 + 4u\omega_1 + 4\omega_2, \\
W^5 &= (2tuu_1 + 2tu_2 + xu_1 + u)\omega_{-1} \\
&\quad + (tu^2 + 6tu_1 + xu)\omega_0 + (4tu + 2x)\omega_1 + 4t\omega_2, \\
W^6 &= (2t^2(uu_1 + u_2) + 2txu_1 + 2tu + 2x)\omega_{-1} \\
&\quad + (t^2(u^2 + 6u_1) + 2txu + 6t + x^2)\omega_0 \\
&\quad + (4t^2u + 4tx)\omega_1 + 4t^2\omega_2, \\
W^7 &= (3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3)\omega_{-1} + (u^3 + 18uu_1 + 16u_2)\omega_0 \\
&\quad + 6(u^2 + 4u_1)\omega_1 + 12u\omega_2 + 8\omega_3, \\
W^8 &= (t(3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3) + x(2uu_1 + 2u_2) + u^2)\omega_{-1} \\
&\quad + (t(u^3 + 18uu_1 + 16u_2) + x(u^2 + 6u_1) + 2u)\omega_0 \\
&\quad + (t(6u^2 + 24u_1) + x(4u))\omega_1 \\
&\quad + (12tu + 4x)\omega_2 \\
&\quad + 8t\omega_3, \\
W^9 &= (t^2(3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3) + tx(4uu_1 + 4u_2) + x^2(u_1) \\
&\quad + 2tu^2 + 2xu - 6)\omega_{-1} + (t^2(u^3 + 18uu_1 + 16u_2) + tx(2u^2 + 12u_1) \\
&\quad + x^2u + 4tu - 2x)\omega_0 + (t^2(6u^2 + 24u_1) + 8txu + 2x^2)\omega_1 \\
&\quad + (12t^2u + 8tx)\omega_2 + (8t^2)\omega_3, \\
W^{10} &= (t^3(3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3) + t^2x(6uu_1 + 6u_2) + 3tx^2u_1 \\
&\quad + t^2(3u^2 + 12u_1) + 6txu + 3x^2 + 6t)\omega_{-1} + (t^3(u^3 + 18uu_1 + 16u_2) \\
&\quad + t^2x(3u^2 + 18u_1) + 3tx^2u + x^3 + 18t^2u + 18tx)\omega_0 + (t^3(6u^2 + 24u_1) \\
&\quad + 12t^2xu + 6tx^2 + 24t^2)\omega_1 + (12t^3u + 12t^2x)\omega_2 + (8t^3)\omega_3.
\end{aligned}
\tag{5.101}
$$

In order to compute the classical recursion operators for symmetries resulting from the deformations constructed in (5.100) induced by the characteristic functions $W_1, W_2, \dots$, we use Proposition 4.29. Suppose we start at a (nonlocal) symmetry $\mathit{Э}_X$ of the Burgers equation; its presentation is

$$
\mathit{Э}_X = X_{-1}\frac{\partial}{\partial y} + \sum_i D_x^i(X)\frac{\partial}{\partial u_i}. \tag{5.102}
$$

The nonlocal component X_{-1} is obtained from the invariance of the equations, cf. (5.87)

$$\begin{aligned} y_x &= u, \\ y_t &= \frac{1}{2}u^2 + u_1, \end{aligned}$$

i.e.,

$$D_x(X_{-1}) = X, \tag{5.103}$$

from which we have

$$X_{-1} = D_x^{-1}(X). \tag{5.104}$$

Theorem 4.30, stating that $\mathcal{F}_X \lrcorner U_1$ is a symmetry, yields for the component $\partial/\partial u$,

$$\mathcal{F}_X \lrcorner W^2 = u_1 X_{-1} + uX + 2D_x X = \big(u_1 D_x^{-1} + u + 2D_x\big)X \tag{5.105}$$

and similar for W^3

$$\begin{aligned} \mathcal{F}_X \lrcorner W^3 &= (tu_1 + 1)X_{-1} + (tu + x)X + 2tD_x X \\ &= \big((tu_1 + 1)D_x^{-1} + (tu + x) + 2tD_x\big)X. \end{aligned} \tag{5.106}$$

From formulas (5.105) and (5.106) together with similar results with respect to $W_4, \dots, W_7$ we arrive in a straightforward way at the recursion operators

$$\begin{aligned} R_1 &= \mathrm{id}, \\ R_2 &= u_1 D_x^{-1} + u + 2D_x, \\ R_3 &= t(u_1 D_x^{-1} + u + 2D_x) + x + D_x^{-1}, \\ R_4 &= 2(uu_1 + u_2)D_x^{-1} + (u^2 + 6u_1) + 4uD_x + 4D_x^2, \\ R_5 &= t\big((2uu_1 + 2u_2)D_x^{-1} + (u^2 + 6u_1) + 4uD_x + 4D_x^2\big) \\ &\quad + x\big(u_1 D_x^{-1} + u + 2D_x\big) + uD_x^{-1}, \\ R_6 &= t^2\big((2uu_1 + 2u_2)D_x^{-1} + (u^2 + 6u_1) + 4uD_x + 4D_x^2\big) \\ &\quad + 2tx\big(u_1 D_x^{-1} + u + 2D_x\big) + x^2 \\ &\quad + t(2uD_x^{-1} + 6) + 2xD_x^{-1}, \\ R_7 &= (3u^2 u_1 + 6uu_2 + 6u_1^2 + 4u_3)D_x^{-1} + (u^3 + 18uu_1 + 16u_2) \\ &\quad + 6(u^2 + 4u_1)D_x + 12uD_x^2 + 8D_x^3. \end{aligned} \tag{5.107}$$

The operator R_1 is just the identity operator while R_2 is the first classical recursion operator for the Burgers equation.

This application shows that from the deformations of the Burgers equation one arrives in a straightforward way at the recursion operators for symmetries. It will be shown in forthcoming sections that the representation of recursion operators for symmetries in terms of deformations of the differential equation is more favorable, while it is in effect a more condensed presentation of this recursion operator. Moreover the appearance of formal integrals in these operators is clarified by their derivation.

The deformation of an equation is a geometrical object, as is enlightened in Chapter 6: it is a symmetry in a new type of covering.

6. Deformations of the KdV equation

Motivated by the results obtained for the Burgers equation, we search for deformations in coverings of the KdV equation. In order to do this, we first have to construct conservation laws for the KdV equation

$$u_t = uu_1 + u_3, \tag{5.108}$$

i.e., we have to find functions F^x, F^t, depending on x, t, u, $u_1, \dots$ such that on $\mathcal{E}^\infty$ one has

$$D_t(F^x) = D_x(F^t), \tag{5.109}$$

where D_x, D_t are total derivative operators, which in local coordinates x, t, u, u_1, u_2, $u_3, \dots$ on $\mathcal{E}^\infty$ have the following presentation

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + \cdots, \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{t1}\frac{\partial}{\partial u_1} + u_{t2}\frac{\partial}{\partial u_2} + \cdots \end{aligned} \tag{5.110}$$

Since the KdV equation is graded,

$$\begin{aligned} &\deg(x) = -1, && \deg(t) = -3 \\ &\deg(u) = 2, && \deg(u_1) = 3, \dots, \end{aligned} \tag{5.111}$$

F^x, F^t will be graded too being of degree k and $k+2$ respectively.

In order to avoid trivialities in the construction of these conservation laws, we start at a function F^{triv} which is of degree $k-1$ and remove in the expression $F^x - D_x(F^{\text{triv}})$ special terms by choosing coefficients in F^{triv} in an appropriate way, since the pair $(D_x(F^{\text{triv}}), D_t(F^{\text{triv}}))$ leads to a trivial conservation law.

After this, we restrict ourselves to conservation laws of the type $(F^x - D_x(F^{\text{triv}}), F^t - D_t(F^{\text{triv}}))$. Searching for conservation laws satisfying the condition $\deg(F^x) \le 6$, we find the following three conservation laws

$$\begin{aligned} F_1^x &= u, & F_1^t &= \Big(\frac{1}{2}u^2 + u_2\Big), \\ F_2^x &= \frac{1}{2}u^2, & F_2^t &= \Big(\frac{1}{3}u^3 - \frac{1}{2}u_1^2 + uu_2\Big), \\ F_3^x &= u^3 - 3u_1^2, & F_3^t &= \Big(\frac{3}{4}u^4 + 3u^2u_2 - 6uu_1^2 - 6u_1u_3 + 3u_2^2\Big). \end{aligned} \tag{5.112}$$

We now introduce the new nonlocal variables y_1, y_2, y_3 by the following system of partial differential equations

$$(y_1)_x = u,$$

$$
\begin{aligned}
(y_1)_t &= \frac{1}{2}u^2 + u_2,\\
(y_2)_x &= \frac{1}{2}u^2,\\
(y_2)_t &= \frac{1}{3}u^3 - \frac{1}{2}u_1^2 + uu_2\\
(y_3)_x &= u^3 - 3u_1^2,\\
(y_3)_t &= \frac{3}{4}u^4 + 3u^2u_2 - 6uu_1^2 - 6u_1u_3 + 3u_2^2.
\end{aligned}
\tag{5.113}
$$

The compatibility conditions for these equations (5.113) are satisfied because of (5.109).

If we now repeat the construction of finding conservation laws on $\mathcal{E}^\infty \times \mathbb{R}^3$, where local variables are given by x, t, u, y_1, y_2, y_3, u_1, $u_2, \dots$ and where the system of partial differential equations is given for u, y_1, y_2, y_3 by (5.113), we find yet another conservation law

$$
\begin{aligned}
F_4^x &= y_1,\\
F_4^t &= u_1 + y_2
\end{aligned}
\tag{5.114}
$$

leading to the nonlocal variable y_4, satisfying the partial differential equations

$$
\begin{aligned}
(y_4)_x &= y_1,\\
(y_4)_t &= u_1 + y_2.
\end{aligned}
\tag{5.115}
$$

The conservation law (F_4^x, F_4^t) is in effect equivalent to the well-known classical (x,t)-dependent conservation law for the KdV equation, i.e.,

$$
\begin{aligned}
\bar{F}_4^x &= xu + \frac{1}{2}tu^2,\\
\bar{F}_4^t &= x\Big(\frac{1}{2}u^2 + u_2\Big) + t\Big(\frac{1}{3}u^3 + uu_2 - \frac{1}{2}u_1^2\Big) - u_1.
\end{aligned}
\tag{5.116}
$$

We now start at the four-dimensional covering $\mathcal{E}^\infty \times \mathbb{R}^4$ of the KdV equation $\mathcal{E}^\infty$

$$
u_t = uu_1 + u_3, \tag{5.117}
$$

where the prolongation of the Cartan distribution to $\mathcal{E}^\infty \times \mathbb{R}^4$ is given by

$$
\begin{aligned}
\widetilde{D}_x &= D_x + u\frac{\partial}{\partial y_1} + \frac{1}{2}u^2\frac{\partial}{\partial y_2} + (u^3 - 3u_1^2)\frac{\partial}{\partial y_3} + y_1\frac{\partial}{\partial y_4},\\
\widetilde{D}_t &= D_t + \Big(\frac{1}{2}u^2 + u_2\Big)\frac{\partial}{\partial y_1} + \Big(\frac{1}{3}u^3 - \frac{1}{2}u_1^2 + uu_2\Big)\frac{\partial}{\partial y_2}\\
&\quad + \Big(\frac{3}{4}u^4 + 3u^2u_2 - 6uu_1^2 - 6u_1u_3 + 3u_2^2\Big)\frac{\partial}{\partial y_3} + (u_1 + y_2)\frac{\partial}{\partial y_4},
\end{aligned}
\tag{5.118}
$$

where D_x, D_t are the total derivative operators on $\mathcal{E}^\infty$, (5.110). In fact y_1, y_2, y_3 are just potentials for the KdV equation, i.e.,

$$\begin{aligned}
y_1 &= \int^x u\,dx,\\
y_2 &= \int^x \frac{1}{2}u^2\,dx,\\
y_3 &= \int^x u^3 - 3u_1^2\,dx,
\end{aligned}\tag{5.119}$$

while y_4 is the nonlocal potential

$$y_4 = \int^x y_1\,dx. \tag{5.120}$$

The Cartan forms associated to $y_1, \dots, y_4$ are denoted by $\omega_{-1}, \dots, \omega_{-4}$, while ω_0,$\omega_1, \dots$ are the Cartan forms associated to u_0, $u_1, \dots$ The generating function for the deformation U_1 is defined by

$$\Omega = \sum_{i=0}^{6} F^i\omega_i + F^{-1}\omega_{-1} + F^{-2}\omega_{-2} + F^{-3}\omega_{-3} + F^{-4}\omega_{-4}, \tag{5.121}$$

where F^i, $i = -4, \dots, 6$, are dependent on the variables

$$x, t, u, \dots, u_7, y_1, \dots, y_4.$$

The overdetermined system of partial differential equations resulting from the deformation equation (4.65) on p. 185

$$\ell^{(1)}_{\mathcal{E}^1}(\Omega) = 0,$$

i.e.,

$$\widetilde{D}_t(\Omega) - u_1\Omega - u\widetilde{D}_x(\Omega) - \widetilde{D}_x^3(\Omega) = 0, \tag{5.122}$$

can be solved in a straightforward way which yields the following characteristic functions

$$\begin{aligned}
W_0 &= \omega_0,\\
W_1 &= \frac{2}{3}u\omega_0 + \omega_2 + \frac{1}{3}u_1\omega_{-1},\\
W_2 &= \Big(\frac{4}{9}u^2 + \frac{4}{3}u_2\Big)\omega_0 + 2u_1\omega_1 + \frac{4}{3}u\omega_2 + \omega_4\\
&\quad + \frac{1}{3}(uu_1 + u_3)\omega_{-1} + \frac{1}{9}u_1\omega_{-2},\\
W_3 &= \Big(\frac{8}{27}u^3 + \frac{8}{3}uu_2 + 2u_1^2 + 2u_4\Big)\omega_0 + (4uu_1 + 5u_3)\omega_1\\
&\quad + \Big(\frac{4}{3}u^2 + \frac{20}{3}u_2\Big)\omega_2 + 5u_1\omega_3 + 2u\omega_4 + \omega_6\\
&\quad + \frac{1}{18}(5u^2u_1 + 10uu_3 + 20u_1u_2 + 6u_5)\omega_{-1} + \frac{1}{9}(uu_1 + u_3)\omega_{-2}
\end{aligned}$$

$$+\frac{1}{54}u_1\omega_{-3}. \tag{5.123}$$

Note that the coefficients of ω_{-1}, ω_{-2}, ω_{-3} in (5.123) are just higher symmetries in a agreement with the remark made in the case of the Burgers equation.

From these results it is straightforward to obtain recursion operators for the KdV equation, i.e.,

$$\begin{aligned}
\widetilde{R}_1 &= \frac{2}{3}u + D_x^2 + \frac{1}{3}u_1 D_x^{-1},\\
\widetilde{R}_2 &= \Big(\frac{4}{9}u^2 + \frac{4}{3}u_2\Big) + 2u_1 D_x + \frac{4}{3}uD_x^2 + D_x^4\\
&\quad + \frac{1}{3}(uu_1 + u_3)D_x^{-1} + \frac{1}{9}u_1 D_x^{-1}u,
\end{aligned} \tag{5.124}$$

while

$$\begin{aligned}
\widetilde{R}_3 &= \Big(\frac{8}{27}u^3 + \frac{8}{3}uu_2 + 2u_1^2 + 2u_4\Big) + \Big(4uu_1 + 5u_3)D_x + (\frac{4}{3}u^2 + \frac{20}{3}u_2\Big)D_x^2\\
&\quad + 5u_1D_x^3 + 2uD_x^4 + D_x^6 + \frac{1}{18}(5u^2u_1 + 10uu_3 + 20u_1u_2 + 6u_5)D_x^{-1}\\
&\quad + \frac{1}{9}(uu_1 + u_3)D_x^{-1}u + \frac{1}{54}u_1D_x^{-1}(3u^2 - 6u_1D_x).
\end{aligned} \tag{5.125}$$

The last term in $\widetilde{R}_2$ and the last two terms in $\widetilde{R}_3$ arise due to the invariance of

$$\begin{aligned}
y_2 &= D_x^{-1}\Big(\frac{1}{2}u^2\Big),\\
y_3 &= D_x^{-1}(u^3 - 3u_1^2).
\end{aligned} \tag{5.126}$$

The operators $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{R}_3$ are just classical recursion operators for the KdV equations (5.119). From (5.125) one observes the complexity of the recursion operators in the last two terms of this expression, due to the complexity of the conservation laws. The complexity of these operators increases more if higher nonlocalities are involved.

Remark 5.13 (Linear coverings for the KdV equation). We also considered deformations of the KdV equations in the linear covering and the prolongation coverings, performing computations related to these coverings.

1. *Linear covering* $\mathcal{E}^\infty \times \mathbb{R}^2$. Local coordinates are x, t, u, $u_1, \dots, s_1$, s_2 while the Cartan distribution is given by

$$\begin{aligned}
\widetilde{D}_x &= D_x + \frac{1}{6}s_2\frac{\partial}{\partial s_1} + \Big(-\frac{1}{6}s_2u_1 + \frac{1}{18}s_2u - \frac{1}{9}\lambda s_2\Big)\frac{\partial}{\partial s_2},\\
\widetilde{D}_t &= D_t - (\lambda + u)\frac{\partial}{\partial s_1} + \Big(-s_1u_2 + \frac{1}{6}s_2u_1 - \frac{1}{3}s_1u^2 + \frac{1}{3}\lambda s_1u\\
&\quad + \frac{2}{3}\lambda^2 s_1\Big)\frac{\partial}{\partial s_2}.
\end{aligned} \tag{5.127}$$

The only deformation admitted here is the trivial one. There is however a yet unknown symmetry in this case, i.e.,

$$V = s_1 s_2 \frac{\partial}{\partial u}. \tag{5.128}$$

2. *Prolongation covering* $\mathcal{E}^\infty \times \mathbb{R}^1$. In this case the Cartan distribution is given by

$$\begin{aligned}
\widetilde{D}_x &= D_x + (u + \frac{1}{6}q^2 + \alpha)\frac{\partial}{\partial q},\\
\widetilde{D}_t &= D_t + \Big(u_2 + \frac{1}{3}qu_1 + \frac{1}{3}u^2 + u\big(\frac{1}{8}q^2 - \frac{1}{3}\alpha\big)\\
&\quad - \frac{2}{3}\alpha\big(\frac{1}{6}q^2 + \alpha\big)\Big)\frac{\partial}{\partial q}.
\end{aligned} \tag{5.129}$$

But here no nontrivial results were obtained.

In effect these special coverings did not lead to new interesting deformation structures.

7. Deformations of the nonlinear Schrödinger equation

In this section deformations and recursion operators of the nonlinear Schrödinger (NLS) equation

$$\begin{aligned}
u_t &= -v_2 + kv(u^2 + v^2),\\
v_t &= u_2 - ku(u^2 + v^2)
\end{aligned} \tag{5.130}$$

will be discussed in the nonlocal setting.

In previous sections we explained how to compute conservation laws for partial differential equations and how to construct from them the nonlocal variables, thus "killing" the conservation laws, i.e., in the coverings the conservation laws associated to the nonlocal variables become trivial.

We introduce the nonlocal variables y_1, y_2, y_3 associated to the conservation laws of the NLS equation and given by

$$\begin{aligned}
y_{1x} &= u^2 + v^2,\\
y_{1t} &= 2(-uv_1 + vu_1),\\
y_{2x} &= uv_1,\\
y_{2t} &= -\frac{3}{4}ku^4 - \frac{1}{2}ku^2v^2 + \frac{1}{4}kv^4 + uu_2 - \frac{1}{2}u_1^2 - \frac{1}{2}v_1^2
\end{aligned} \tag{5.131}$$

and

$$\begin{aligned}
y_{3x} &= k(u^2 + v^2)^2 + 2u_1^2 + 2v_1^2,\\
y_{3t} &= 4\big((-kuv_1 + kvu_1)(u^2 + v^2) - u_1v_2 + v_1u_2\big).
\end{aligned} \tag{5.132}$$

In the three-dimensional covering $\mathcal{E}^\infty \times \mathbb{R}^3$ of the NLS equation the Cartan distribution is given by

$$\widetilde{D}_x = D_x + \sum_{i=1}^{3} y_{ix}\frac{\partial}{\partial y_i},$$
$$\widetilde{D}_t = D_t + \sum_{i=1}^{3} y_{it}\frac{\partial}{\partial y_i}, \tag{5.133}$$

while D_x, D_t are total derivative operators on $\mathcal{E}^\infty$, which in internal coordinates x, t, u, v, u_1, $v_1, \dots$ have the representation

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i} + \sum_{i=0}^{\infty} v_{i+1}\frac{\partial}{\partial v_i},$$
$$D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} u_{it}\frac{\partial}{\partial u_i} + \sum_{i=0}^{\infty} v_{it}\frac{\partial}{\partial v_i}. \tag{5.134}$$

Now in order to construct a deformation of the NLS equation, we construct a tuple of characteristic functions

$$W^u = \sum_{i=0}^{3}(f^i\omega_i^u + \overline{f}^i\omega_i^v) + \sum_{i=1}^{3}\tilde{f}^i\omega_{y_i},$$
$$W^v = \sum_{i=0}^{3}(g^i\omega_i^u + \overline{g}^i\omega_i^v) + \sum_{i=1}^{3}\tilde{g}^i\omega_{y_i}, \tag{5.135}$$

where in (5.135) ω_i^u, ω_i^v, ω_{y_i} are the Cartan forms associated to u_i, v_i, y_i respectively; the coefficients f^i, $\overline{f}^i$, $\tilde{f}^i$, g^i, $\overline{g}^i$, $\tilde{g}^i$ are dependent on

$$x, t, u, v, \dots, u_4, v_4, y_1, y_2, y_3.$$

The solution constructed from the deformation equation (4.65) leads to the following nontrivial results.

$$\begin{aligned}
W_1^u &= \frac{1}{k}\omega_1^v - v\omega_{y_1},\\
W_1^v &= -\frac{1}{k}\omega_1^u + u\omega_{y_1},\\
W_2^u &= (u^2+v^2)\omega_0^u + uv\omega_0^v - \frac{1}{2k}\omega_2^u + \frac{1}{2}u_1\omega_{y_1} - v\omega_{y_2},\\
W_2^v &= v^2\omega_0^v - \frac{1}{2k}\omega_2^v + \frac{1}{2}v_1\omega_{y_1} + u\omega_{y_2},\\
W_3^u &= 8uv_1\omega_0^u + 12vv_1\omega_0^v + 4uv\omega_1^u + (4u^2+8v^2)\omega_1^v - \frac{2}{k}\omega_3^v\\
&\quad + 2(-k(u^2+v^2)v + v_2)\omega_{y_1} + 4u_1\omega_{y_2} - v\omega_{y_3},\\
W_3^v &= (-12uu_1 - 4vv_1)\omega_0^u + (-4uv_1 - 8vu_1)\omega_0^v
\end{aligned}$$

$$+ (-8u^2 - 4v^2)\omega_1^u - 4uv\omega_1^v + \frac{2}{k}\omega_3^u$$
$$+ 2(k(u^2 + v^2)u - u_2)\omega_{y_1} + 4v_1\omega_{y_2} + u\omega_{y_3}. \tag{5.136}$$

Suppose we have a shadow of a nonlocal symmetry

$$X = X^u \frac{\partial}{\partial u} + \ldots + X^v \frac{\partial}{\partial v} + \ldots + X_{-1}\frac{\partial}{\partial y_1} + X_{-2}\frac{\partial}{\partial y_2} + X_{-3}\frac{\partial}{\partial y_3}. \tag{5.137}$$

Then the nonlocal component X_{-1} associated to y_1 is obtained from the invariance of the equations

$$\begin{aligned} y_{1x} &= u^2 + v^2, \\ y_{1t} &= 2(-uv_1 + vu_1). \end{aligned} \tag{5.138}$$

So from (5.138) we arrive at the following condition

$$D_x(X_{-1}) = 2uX^u + 2vX^v,$$

or formally

$$X_{-1} = D_x^{-1}(2uX^u + 2vX^v). \tag{5.139}$$

From the invariance of the partial differential equations for y_2, y_3, (5.131), (5.132) we obtain in a similar way

$$\begin{aligned} X_{-2} &= D_x^{-1}\big(uD_x(X^v) + v_1X^u\big), \\ X_{-3} &= D_x^{-1}\big(4k(u^2 + v^2)(uX^u + vX^v) + 4u_1D_x(X^u) + 4v_1D_x(X^v)\big). \end{aligned} \tag{5.140}$$

Using these results, we arrive from W_1^u, W_1^v in a straightforward way at the well-known recursion operator

$$R_1 = \begin{pmatrix} -vD_x^{-1}(2u) & -vD_x^{-1}(2v) + \frac{1}{k}D_x \\ +uD_x^{-1}(2u) - \frac{1}{k}D_x & +uD_x^{-1}(2v) \end{pmatrix} \tag{5.141}$$

Recursion operators resulting from W_i^u, W_i^v, $i = 2, 3, \ldots$, can be obtained similarly, using constructed formulas for X_{-2}, X_{-3}, see (5.140).

8. Deformations of the classical Boussinesq equation

Let us discuss now deformations of Classical Boussinesq equation

$$\begin{aligned} v_t &= u_1 + vv_1, \\ u_t &= u_1v + uv_1 + \sigma v_3. \end{aligned} \tag{5.142}$$

To this end, we start at a four-dimensional covering $\mathcal{E}^\infty \times \mathbb{R}^4$ of the Boussinesq equation, where local coordinates are given by

$$(x, t, v, u, \ldots, y_1, \ldots, y_4)$$

with the Cartan distribution defined by

$$\widetilde{D}_x = D_x + v\frac{\partial}{\partial y_1} + u\frac{\partial}{\partial y_2} + uv\frac{\partial}{\partial y_3} + (u^2 + uv^2 + vv_2\sigma)\frac{\partial}{\partial y_4},$$

$$\begin{aligned}
\widetilde{D}_t = D_t &+ \Big(u + \frac{1}{2}v^2\Big)\frac{\partial}{\partial y_1} + (uv + v_2\sigma)\frac{\partial}{\partial y_2} \\
&+ \Big(\frac{1}{2}u^2 + uv^2 + vv_2\sigma - \frac{1}{2}v_1^2\sigma\Big)\frac{\partial}{\partial y_3} \\
&+ (2u^2v + uv^3 + 2\sigma uv_2 + 2\sigma v^2 v_2 + \sigma v u_2 - \sigma v_1 u_1)\frac{\partial}{\partial y_4}.
\end{aligned} \tag{5.143}$$

The nonlocal variables y_1, y_2, y_3, y_4 satisfy the equations

$$\begin{aligned}
(y_1)_x &= v, \\
(y_1)_t &= u + \frac{1}{2}v^2, \\
(y_2)_x &= u, \\
(y_2)_t &= uv + v_2\sigma, \\
(y_3)_x &= uv, \\
(y_3)_t &= \frac{1}{2}u^2 + uv^2 + vv_2\sigma - \frac{1}{2}v_1^2\sigma, \\
(y_4)_x &= u^2 + uv^2 + vv_2\sigma, \\
(y_4)_t &= 2u^2v + uv^3 + 2\sigma uv_2 + 2\sigma v^2 v_2 + \sigma v u_2 - \sigma v_1 u_1.
\end{aligned} \tag{5.144}$$

We assume the characteristic functions W^v, W^u to be dependent on ω_0^v, $\omega_0^u, \dots, \omega_5^v$, ω_5^u, $\omega_{y_1}, \dots, \omega_{y_4}$, whereas the coefficients are required to be dependent on x, t, v, $u, \dots, v_5$, u_5, $y_1, \dots, y_4$.

Solving the overdetermined system of partial differential equations resulting from the deformation condition (4.65), we arrive at the following nontrivial characteristic functions

$$\begin{aligned}
W_1^v &= v\omega_0^v + 2\omega_0^u + v_1\omega_{y_1}, \\
W_1^u &= 2u\omega_0^v + v\omega_0^u + 2\sigma\omega_2^v + u_1\omega_{y_1}, \\
W_2^v &= (4u + v^2)\omega_0^v + 4v\omega_0^u + 4\sigma\omega_2^v + (2vv_1 + 2u_1)\omega_{y_1} + 2v_1\omega_{y_2}, \\
W_2^u &= (4uv + 6\sigma v_2)\omega_0^v + (4u + v^2)\omega_0^u + 6v_1\sigma\omega_1^v + 4\sigma v\omega_2^v + 4\sigma\omega_2^u \\
&\quad + (2uv_1 + 2vu_1 + 2\sigma v_3)\omega_{y_1} + 2u_1\omega_{y_2}
\end{aligned} \tag{5.145}$$

and two more deformations.

As in the preceding section we use the invariance of the equations

$$\begin{aligned}
(y_1)_x &= v, \\
(y_2)_x &= u
\end{aligned} \tag{5.146}$$

to arrive at the associated recursion operators

$$R_1 = \begin{pmatrix} v + v_1 D_x^{-1} & 2 \\ 2u + 2\sigma D_x^2 + u_1 D_x^{-1} & v \end{pmatrix} \tag{5.147}$$

and

$$R_2 = \left(\begin{array}{l|l} (4u + v^2) + 4\sigma D_x^2 + (2vv_1 + 2u_1)D_x^{-1} & 4v + 2v_1 D_x^{-1} \\ \hline (4uv + 6\sigma_2 v_2) + 6\sigma v_1 D_x + 4\sigma v D_x^2 & (4u + v^2) + 4\sigma D_x^2 \\ +(2uv_1 + 2vu_1 + 2\sigma v_3)D_x^{-1} & +2u_1 D_x^{-1} \end{array} \right) \tag{5.148}$$

Note that R_2 is just equivalent to double action of the operator R_1, i.e.,

$$R_2 = R_1 \circ R_1 = (R_1)^2. \tag{5.149}$$

9. Symmetries and recursion for the Sym equation

The following system of partial differential equations plays an interesting role in some specific areas of geometry [**16**]:

$$\begin{aligned} \frac{\partial u}{\partial x} + (u - v)\frac{\partial w}{\partial x} &= 0, \\ \frac{\partial v}{\partial y} - (u - v)\frac{\partial w}{\partial y} &= 0, \\ uve^{2w} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= 0. \end{aligned} \tag{5.150}$$

The underlying geometry is defined as the manifold of local surfaces which admit nontrivial isometries conserving principal curvatures, the so-called isothermic surfaces.

In this section we shall prove that this system (5.150) admits an infinite hierarchy of commuting symmetries and conservation laws, [**7**]. Results will be computed not for system (5.150), but for a simplified system obtained by the transformation $u \mapsto ue^{-w}$, $v \mapsto ve^{-w}$, i.e.,

$$\begin{aligned} \frac{\partial u}{\partial x} - v\frac{\partial w}{\partial x} &= 0, \\ \frac{\partial v}{\partial y} - u\frac{\partial w}{\partial y} &= 0, \\ uv + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= 0. \end{aligned} \tag{5.151}$$

9.1. Symmetries. In this subsection we discuss higher symmetries for system (5.151):

$$u_x - vw_x = 0, \quad v_y - uw_y = 0, \quad w_{yy} + uv + w_{xx} = 0. \tag{5.152}$$

This system is a graded system of differential equations, i.e.,

$$\begin{aligned} \deg(x) = \deg(y) &= -1, \\ \deg(u) = \deg(v) &= 1, \\ \deg(w) &= 0. \end{aligned} \tag{5.153}$$

All objects of interest for system (5.152), like symmetries and conservation laws, turn out to be homogeneous with respect to this grading, e.g.,

$$\deg(u_{i,j}) = \deg(u) - i\deg(x) - j\deg(y) = 1 + i + j,$$
$$\deg\left(uv\frac{\partial}{\partial w_{xy}}\right) = \deg(u) + \deg(v) - \deg(w_{xy}) = 0, \tag{5.154}$$

whereas in (5.154) $u_{i,j} = u_{\underbrace{x\dots x}_{i\text{ times}}\underbrace{y\dots y}_{j\text{ times}}}$.

For computation of higher symmetries we have to introduce vertical vector fields $\mathfrak{I}_\Phi$ with generating function $\Phi = (\Phi_u, \Phi_v, \Phi_w)$, which has to satisfy the symmetry condition

$$\ell_F(\Phi) = 0, \tag{5.155}$$

where ℓ_F is the universal linearization operator for system (5.152), i.e.,

$$\ell_F = \begin{pmatrix} D_x & -w_x & -vD_x \\ -w_y & D_y & -uD_y \\ v & u & D_x^2 + D_y^2 \end{pmatrix} \tag{5.156}$$

The system $\ell_F(\Phi) = 0$ is homogeneous with respect to the degree, so the symmetry with the generating function $\Phi = (\Phi_u, \Phi_v, \Phi_w)$ is homogeneous with respect to the degree, i.e., $\deg(\Phi_u\partial/\partial u) = \deg(\Phi_v\partial/\partial v) = \deg(\Phi_w\partial/\partial w)$, leading to the required degree of Φ:

$$\deg(\Phi_u) = \deg(\Phi_v) = \deg(\Phi_w) + 1. \tag{5.157}$$

Internal coordinates of $\mathcal{E}^\infty$, where $\mathcal{E}$ is system (5.152), are chosen to be

$$x, y, u, v, w, u_y, v_x, w_x, w_y, u_{yy}, v_{xx}, w_{xx}, w_{xy}, u_{yyy}, v_{xxx}, w_{xxx}, w_{xxy}, \dots. \tag{5.158}$$

Thus $\mathcal{E}^\infty$ is solved for u_x, v_y, w_{yy} and their differential consequences $u_{x\dots x}$, $v_{y\dots y}$, $w_{x\dots xy\dots yy}$. With this choice of internal coordinates, the symmetry equation (5.155) reads

$$\begin{aligned} D_x(\Phi_u) - w_x\Phi_v - vD_x(\Phi_w) &= 0, \\ -w_y\Phi_u + D_y(\Phi_v) - uD_y(\Phi_w) &= 0, \\ v\Phi_u + u\Phi_v + D_x^2(\Phi_w) + D_y^2(\Phi_w) &= 0. \end{aligned} \tag{5.159}$$

The generating function $\Phi = (\Phi_u, \Phi_v, \Phi_w)$ depends on a finite number of internal coordinates, Φ being defined on $\mathcal{E}^\infty$. Dependencies for the generating function are selected with respect to degree, i.e., Φ depends on the internal coordinates of degree n or less. According to (5.157), this means that Φ_w depends on internal coordinates of degree $n-1$ or less.

The results for the generating function Φ depending on the internal coordinates of degree 6 or less are as follows. There are two symmetries of degree 0:

$$\begin{aligned} X^0 &= (0, 0, 1), \\ Y^0 &= (u + xvw_x + yu_y, v + xv_x + yuw_y, xw_x + yw_y). \end{aligned} \tag{5.160}$$

The second symmetry in (5.160) corresponds to the scaling or grading of systems (5.152), (5.153). Other symmetries appear in pairs of degrees 1, 3 and 5. The symmetries of degree 1 are

$$\begin{aligned} X^1 &= (u_y, uw_y, w_y), \\ Y^1 &= (vw_x, v_x, w_x). \end{aligned} \tag{5.161}$$

They are equivalent to the vector fields of $\partial/\partial y$ and $\partial/\partial x$ respectively.

The symmetries of degree 3 are

$$\begin{aligned} X_u^3 &= 6u^2vw_y + 3u^2u_y + 6uw_yw_{xx} + 3u_yw_x^2 - 3u_yw_y^2 + 2u_{yyy}, \\ X_v^3 &= u^3w_y + 3uw_x^2w_y - 3uw_y^3 - 2uw_{xxy} + 2u_yw_{xx} + 2w_yu_{yy}, \\ X_w^3 &= u^2w_y - 2vu_y + 3w_x^2w_y - w_y^3 - 2w_{xxy} \end{aligned} \tag{5.162}$$

and

$$\begin{aligned} Y_u^3 &= 3v^3w_x - 3vw_x^3 + 3vw_xw_y^2 + 2vw_{xxx} + 2w_xv_{xx} - 2w_{xx}v_x, \\ Y_v^3 &= 3v^2v_x - 6vw_xw_{xx} - 3w_x^2v_x + 3w_y^2v_x + 2v_{xxx}, \\ Y_w^3 &= 3v^2w_x - w_x^3 + 3w_xw_y^2 + 2w_{xxx}. \end{aligned} \tag{5.163}$$

Finally the components of the generating functions $\Phi = (\Phi_u, \Phi_v, \Phi_w)$ of the two symmetries of degree 5 are given by

$$\begin{aligned} X_u^5 = &- 60u^4vw_y + 15u^4u_y - 60u^3w_yw_{xx} - 140u^2v^2u_y + 60u^2vw_x^2w_y \\ &- 60u^2vw_y^3 - 80u^2vw_{xxy} + 30u^2u_yw_x^2 + 110u^2u_yw_y^2 - 40u^2w_yv_{xx} \\ &+ 20u^2u_{yyy} - 40uv^2w_yw_{xx} - 200uvu_yw_{xx} - 120uvw_xw_yv_x \\ &- 40uw_xw_y^2w_{xy} - 60uw_y^3w_{xx} + 120uvw_yu_{yy} - 40uu_yw_xv_x \\ &+ 80uu_yu_{yy} + 60uw_x^2w_yw_{xx} - 40uw_yw_{xxxx} - 80uw_{xx}w_{xxy} \\ &- 40v^2u_yw_x^2 + 80vu_y^2w_y + 20u_y^3 + 15u_yw_x^4 - 50u_yw_x^2w_y^2 - 40u_yw_xw_{xxx} \\ &+ 15u_yw_y^4 + 80u_yw_yw_{xxy} - 60u_yw_{xx}^2 + 20u_yw_{xy}^2 + 20w_x^2u_{yyy} \\ &+ 40w_xu_{yy}w_{xy} - 20w_y^2u_{yyy} + 80w_yw_{xx}u_{yy} + 8u_{yyyyy}, \end{aligned}$$

$$\begin{aligned} X_v^5 = &+ 3u^5w_y - 20u^3v^2w_y + 10u^3w_x^2w_y + 10u^3w_y^3 - 4u^3w_{xxy} \\ &- 16u^2vw_yw_{xx} + 12u^2u_yw_{xx} - 8u^2w_xw_yv_x + 20u^2w_yu_{yy} - 16uv^2w_x^2w_y \\ &+ 8uv^2w_{xxy} + 80uvu_yw_y^2 + 24uvv_xw_{xy} + 20uu_y^2w_y + 15uw_x^4w_y \\ &- 50uw_x^2w_y^3 - 20uw_x^2w_{xxy} - 40uw_xw_yw_{xxx} - 40uw_xw_{xx}w_{xy} + 15uw_y^5 \\ &+ 60uw_y^2w_{xxy} - 20uw_yw_{xx}^2 + 20uw_yw_{xy}^2 + 8uw_{xxxxy} - 8v^2u_yw_{xx} \\ &- 24vu_yw_xv_x + 20u_yw_x^2w_{xx} + 20u_yw_y^2w_{xx} - 8u_yw_{xxxx} + 20w_x^2w_yu_{yy} \\ &- 20w_y^3u_{yy} + 8w_yu_{yyyy} + 8w_{xx}u_{yyy} - 8w_{xxy}u_{yy}, \end{aligned}$$

$$\begin{aligned} X_w^5 = &\ 3u^4w_y - 20u^2v^2w_y - 12u^2vu_y + 10u^2w_x^2w_y - 10u^2w_y^3 - 4u^2w_{xxy} \\ &- 16uvw_yw_{xx} - 8uw_xw_yv_x + 8uw_yu_{yy} - 16v^2w_x^2w_y + 8v^2w_{xxy} \end{aligned}$$

$$
\begin{aligned}
&- 20vu_yw_x^2 + 20vu_yw_y^2 - 8vu_{yyy} + 24vv_xw_{xy} - 4u_y^2w_y \\
&+ 8u_yv_{xx} + 15w_x^4w_y - 30w_x^2w_y^3 - 20w_x^2w_{xxy} - 40w_xw_yw_{xxx} \\
&- 40w_xw_{xx}w_{xy} + 3w_y^5 + 20w_y^2w_{xxy} - 20w_yw_{xx}^2 + 20w_yw_{xy}^2 + 8w_{xxxxy}
\end{aligned} \tag{5.164}
$$

and

$$
\begin{aligned}
Y_u^5 = &+ 15v^5w_x - 50v^3w_x^3 + 30v^3w_xw_y^2 + 20v^3w_{xxx} + 60v^2w_xv_{xx} \\
&+ 20v^2w_{xx}v_x + 15vw_x^5 - 50vw_x^3w_y^2 - 60vw_x^2w_{xxx} \\
&+ 15vw_xw_y^4 + 40vw_xw_yw_{xxy} - 20vw_xw_{xx}^2 + 20vw_xv_x^2 + 20vw_xw_{xy}^2 \\
&+ 20vw_y^2w_{xxx} + 40vw_yw_{xx}w_{xy} + 8vw_{xxxxx} - 20w_x^3v_{xx} - 20w_x^2w_{xx}v_x \\
&+ 20w_xw_y^2v_{xx} + 8w_xv_{xxxx} - 20w_y^2w_{xx}v_x - 8w_{xx}v_{xxx} + 8w_{xxx}v_{xx} \\
&- 8v_xw_{xxxx}, \\
Y_v^5 = &+ 15v^4v_x - 60v^3w_xw_{xx} - 90v^2w_x^2v_x + 30v^2w_y^2v_x + 20v^2v_{xxx} \\
&+ 60vw_x^3w_{xx} - 40vw_x^2w_yw_{xy} - 60vw_xw_y^2w_{xx} - 40vw_xw_{xxxx} \\
&- 80vw_{xx}w_{xxx} + 80vv_{xx}v_x + 15w_x^4v_x - 50w_x^2w_y^2v_x - 20w_x^2v_{xxx} \\
&- 80w_xw_{xx}v_{xx} - 80w_xw_{xxx}v_x + 15w_y^4v_x + 20w_y^2v_{xxx} + 40w_yv_{xx}w_{xy} \\
&+ 40w_yv_xw_{xxy} - 60w_{xx}^2v_x + 20v_x^3 + 20v_xw_{xy}^2 + 8v_{xxxxx}, \\
Y_w^5 = &+ 15v^4w_x - 30v^2w_x^3 + 30v^2w_xw_y^2 + 20v^2w_{xxx} + 40vw_xv_{xx} \\
&+ 40vw_{xx}v_x + 3w_x^5 - 30w_x^3w_y^2 - 20w_x^2w_{xxx} + 15w_xw_y^4 + 40w_xw_yw_{xxy} \\
&- 20w_xw_{xx}^2 + 20w_xv_x^2 + 20w_xw_{xy}^2 + 20w_y^2w_{xxx} + 40w_yw_{xx}w_{xy} \\
&+ 8w_{xxxxx}.
\end{aligned} \tag{5.165}
$$

Apart from the second symmetry in (5.160), these symmetries commute, i.e., $[\partial_\Phi, \partial_{\Phi'}] = 0$. The Lie bracket with the second symmetry in (5.160) acts as multiplication by the degree of the symmetry.

Remark 5.14. One should note that for system (5.152) there exists a discrete symmetry

$$
T\colon x \mapsto y,\ y \mapsto x,\ u \mapsto v,\ v \mapsto u,\ w \mapsto w, \tag{5.166}
$$

from which we have

$$
\begin{aligned}
T(X^0) &= X^0, & T(Y^0) &= Y^0, \\
T(X^1) &= Y^1, & T(Y^1) &= X^1, \\
T(X^3) &= Y^3, & T(Y^3) &= X^3, \\
T(X^5) &= Y^5, & T(Y^5) &= X^5.
\end{aligned} \tag{5.167}
$$

9.2. Conservation laws and nonlocal symmetries. As in previous applications, we first construct conservation laws in order to arrive at nonlocal variables and the augmented system of partial differential equations governing them.

To construct conservation laws, we start at functions F^x and F^y, such that

$$D_y(F^x) = D_x(F^y)$$

We construct conservation laws for functions F^x and F^y of degree 0 until 4. For degree 2 we obtained two solutions,

$$\begin{aligned} F^x &= \frac{-v^2 + w_x^2 - w_y^2}{2}, & F^y &= w_x w_y, \\ F^x &= -w_x w_y, & F^y &= \frac{u^2 + w_x^2 - w_y^2}{2}. \end{aligned} \tag{5.168}$$

Degree 4 yields two conservation laws, which are

$$\begin{aligned} F^x =& -(u^2 w_x w_y - u^2 w_{xy} - 2uvw_x w_y + 2uvw_{xy} + w_x^3 w_y - w_x w_y^3 \\ &+ 2w_{xx} w_{xy}), \\ F^y =& (u^4 - 4u^3 v + 4u^2 v^2 + 2u^2 w_x^2 - 6u^2 w_y^2 - 4u^2 w_{xx} + 8uvw_{xx} \\ &+ 8uw_y u_y + w_x^4 - 6w_x^2 w_y^2 + w_y^4 + 4w_{xx}^2 - 4u_y^2 - 4w_{xy}^2)/4, \\ F^x =& -(v^4 - 6v^2 w_x^2 + 2v^2 w_y^2 + 4v^2 w_{xx} + 8vw_x v_x + w_x^4 - 6w_x^2 w_y^2 + w_y^4 \\ &+ 4w_{xx}^2 - 4w_{xy}^2 - 4v_x^2)/4, \\ F^y =& -2uvw_x w_y + v^2 w_x w_y - v^2 w_{xy} - w_x^3 w_y + w_x w_y^3 - 2w_{xx} w_{xy}. \end{aligned} \tag{5.169}$$

Associated to the conservation laws given in (5.168), (5.169), we introduce nonlocal variables.

The conservation laws (5.168) give rise to two nonlocal variables, p and q of degree 1,

$$\begin{aligned} p_x &= \frac{-v^2 + w_x^2 - w_y^2}{2}, & p_y &= w_x w_y, \\ q_x &= -w_x w_y, & q_y &= \frac{u^2 + w_x^2 - w_y^2}{2}. \end{aligned} \tag{5.170}$$

To the conservation laws (5.169) there correspond two nonlocal variables r and s of degree 3:

$$\begin{aligned} r_x =& -u^2 w_x w_y + u^2 w_{xy} + 2uvw_x w_y - 2uvw_{xy} - w_x^3 w_y + w_x w_y^3 - 2w_{xx} w_{xy}, \\ r_y =& (u^4 - 4u^3 v + 4u^2 v^2 + 2u^2 w_x^2 - 6u^2 w_y^2 - 4u^2 w_{xx} + 8uvw_{xx} + 8uw_y u_y \\ &+ w_x^4 - 6w_x^2 w_y^2 + w_y^4 + 4w_{xx}^2 - 4u_y^2 - 4w_{xy}^2)/4, \\ s_x =& (-v^4 + 6v^2 w_x^2 - 2v^2 w_y^2 - 4v^2 w_{xx} - 8vw_x v_x - w_x^4 + 6w_x^2 w_y^2 - w_y^4 \\ &- 4w_{xx}^2 + 4w_{xy}^2 + 4v_x^2)/4, \end{aligned}$$

$$s_y = -2uvw_xw_y + v^2w_xw_y - v^2w_{xy} - w_x^3w_y + w_xw_y^3 - 2w_{xx}w_{xy}. \tag{5.171}$$

We now discuss the existence of symmetries in the covering of (5.152) by nonlocal variables p, q, r, s, i.e., in $\mathcal{E}^\infty \times \mathbb{R}^4$. The system of partial differential equations in this covering is constituted by (5.152), (5.170) and (5.171). Total derivative operators $\widetilde{D}_x$, $\widetilde{D}_y$ are defined on $\mathcal{E}^\infty \times \mathbb{R}^4$, and are given by

$$\begin{aligned}
\widetilde{D}_x &= D_x + \frac{-v^2 + w_x^2 - w_y^2}{2}\frac{\partial}{\partial p} - w_xw_y\frac{\partial}{\partial q} + r_x\frac{\partial}{\partial r} + s_x\frac{\partial}{\partial s},\\
\widetilde{D}_y &= D_y + w_xw_y\frac{\partial}{\partial p} + \frac{u^2 + w_x^2 - w_y^2}{2}\frac{\partial}{\partial q} + r_y\frac{\partial}{\partial r} + s_y\frac{\partial}{\partial s},
\end{aligned} \tag{5.172}$$

where r_x, r_y, s_x, s_y are given by (5.171).

Symmetries ∂_Φ in this nonlocal setting, where the generating function $\Phi = (\Phi_u, \Phi_v, \Phi_w)$ is dependent on the internal coordinates (5.158) as well as on the nonlocal variables p, q, r, s, have to satisfy the symmetry condition

$$\widetilde{\ell_F}(\Phi) = 0, \tag{5.173}$$

where ℓ_F is the universal linearization operator for the augmented system (5.152) together with (5.170), (5.171), i.e.,

$$\widetilde{\ell_F} = \begin{pmatrix} \widetilde{D}_x & -w_x & -v\widetilde{D}_x \\ -w_y & \widetilde{D}_y & -u\widetilde{D}_y \\ v & u & \widetilde{D}_x^2 + \widetilde{D}_y^2 \end{pmatrix} \tag{5.174}$$

This does lead to the following nonlocal symmetry ∂_Z of degree 2, where

$$\begin{aligned}
Z_u = &-2pvw_x - 2qu_y\\
&+ x\left(3v^3w_x - 3vw_x^3 + 3vw_xw_y^2 + 2vw_{xxx} + 2w_xv_{xx} - 2w_{xx}v_x\right)\\
&+ y\left(-6u^2vw_y - 3u^2u_y - 6uw_yw_{xx} - 3u_yw_x^2 + 3u_yw_y^2 - 2u_{yyy}\right)\\
&- 2u^3 - 2uv^2 - 4uw_x^2 + 6uw_y^2 - 2vw_{xx} + 4w_xv_x - 6u_{yy},\\
Z_v = &-2pv_x - 2quw_y\\
&+ x\left(3v^2v_x - 6vw_xw_{xx} - 3w_x^2v_x + 3w_y^2v_x + 2v_{xxx}\right)\\
&+ y\left(-u^3w_y - 3uw_x^2w_y + 3uw_y^3 + 2uw_{xxy} - 2u_yw_{xx} - 2w_yu_{yy}\right)\\
&- 2uw_{xx} + 2v^3 - 6vw_x^2 + 4vw_y^2 - 4u_yw_y + 6v_{xx},\\
Z_w = &-2pw_x - 2qw_y\\
&+ x\left(3v^2w_x - w_x^3 + 3w_xw_y^2 + 2w_{xxx}\right)\\
&+ y\left(-u^2w_y + 2vu_y - 3w_x^2w_y + w_y^3 + 2w_{xxy}\right)\\
&+ 2uv + 4w_{xx}.
\end{aligned} \tag{5.175}$$

One should note that the coefficients at p, q, i.e., $(-2vw_x, -2v_x, -2w_x)$ and $(-2u_y, -2uw_y, -2w_y)$, are just the generating functions of the symmetries

(5.161). This nonlocal symmetry is just the recursion symmetry, acting by the extended Jacobi brackets on generating functions on $\mathcal{E}^\infty \times \mathbb{R}^4$.

There is another symmetry of degree 4, dependent on p, q, r, s. For an explicit formula of this symmetry we refer to [**10**]. Finally we mention that starting from $\mathcal{E}^\infty \times \mathbb{R}^4$, there is an additional *nonlocal* conservation law

$$D_y(p) = D_x(-q).$$

The nonlocal variable associated to this conservation law did not play an essential role in the construction of the nonlocal symmetry (5.175).

9.3. Recursion operator for symmetries. We now arrive at the construction of the classical recursion operator for symmetries of the Sym equation [**7**]

$$\begin{aligned} &\frac{\partial u}{\partial x} - v\frac{\partial w}{\partial x} = 0,\\ &\frac{\partial v}{\partial y} - u\frac{\partial w}{\partial y} = 0,\\ &uv + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0. \end{aligned} \tag{5.176}$$

We could arrive at this recursion operator by the construction of deformations of system (5.176), but we decided not to do so. We shall demonstrate how we can, from the knowledge we have of the nonlocal structure of deformations, arrive at the formal classical recursion operator, which, by means of its presentation as integral differential operator is of a more complex structure. Due to the structure of conservation laws, we can make an ansatz for the recursion operator.

We expect that as in the previous problems, in the deformation structure of our system (5.176) the Cartan forms associated to the nonlocal variables p, q, i.e.,

$$\begin{aligned} \omega_p &= dp - \frac{-v^2 + w_x^2 - w_y^2}{2}\,dx - w_x w_y\,dy,\\ \omega_q &= dq + w_x w_y\,dx - \frac{u^2 + w_x^2 - w_y^2}{2}\,dy \end{aligned} \tag{5.177}$$

play an essential role. According to this, the associated nonlocal components of the symmetries play a significant role too. These components have to be constructed from the invariance of the associated differential equations for p and q. Since the system at hand is not of evolutionary type, we have a choice to compute these components from the invariance of either p_x or p_y and similar for the q_x and q_y.

Due to the discrete symmetry (5.166), we choose the invariance of the following equations

$$\begin{aligned} p_y &= w_x w_y,\\ q_x &= -\,w_x w_y. \end{aligned}$$

From these invariances, we obtain for the generating function of a symmetry $\Phi = (\Phi_u, \Phi_v, \Phi_w)$, terms like

$$
\begin{aligned}
\Phi_p &= D_y^{-1}(w_x D_y(\Phi_w) + w_y D_x(\Phi_w)),\\
\Phi_q &= D_x^{-1}(w_x D_y(\Phi_w) + w_y D_x(\Phi_w)).
\end{aligned}
\tag{5.178}
$$

From the above considerations we expect the recursion operator to contain terms like $D_y^{-1}(w_x D_y(\cdot) + w_y D_x(\cdot)), D_x^{-1}(w_x D_y(\cdot) + w_y D_x(\cdot))$.

Moreover from the expected degree of the operator, which probably will be equal to 2, due to the degrees of the symmetries of the previous subsection, we arrive at the *ansatz* for the recursion operator for symmetries. From this ansatz we arrive at the following expression for $\mathcal{R}$:

$$
\begin{aligned}
\mathcal{R} = &\begin{pmatrix}
D_y^2 + u^2 + w_x^2 - w_y^2 & -w_x D_x + uv + w_{xx} & uw_x D_x - 2uw_y D_y\\
w_y D_y + w_{xx} & -D_x^2 - v^2 + w_x^2 - w_y^2 & 2vw_x D_x - vw_y D_y\\
0 & u & D_y^2
\end{pmatrix}\\
&+ \begin{pmatrix}
0 & 0 & vw_x D_y^{-1}(w_y D_x + w_x D_y) - u_y D_x^{-1}(w_y D_x + w_x D_y)\\
0 & 0 & v_x D_y^{-1}(w_y D_x + w_x D_y) - uw_y D_x^{-1}(w_y D_x + w_x D_y)\\
0 & 0 & w_x D_y^{-1}(w_y D_x + w_x D_y) - w_y D_x^{-1}(w_y D_x + w_x D_y)
\end{pmatrix}
\end{aligned}
\tag{5.179}
$$

It is a straightforward check that the operator $\mathcal{R}$ is a recursion operator for higher symmetries since

$$
\ell_F \circ \mathcal{R} = \mathcal{S} \circ \ell_F,
\tag{5.180}
$$

where the matrix operator $\mathcal{S}$ is given by

$$
\mathcal{S} = \begin{pmatrix}
D_y^2 + u^2 + w_x^2 - w_y^2 & -D_x^2 - v^2 + w_x^2 - w_y^2 & -uD_y - vw_y\\
-w_y D_x - 2w_{xy} & w_x D_y + 2w_{xy} & uw_x\\
\mathcal{S}_{31} & \mathcal{S}_{32} & \mathcal{S}_{33}
\end{pmatrix}
\tag{5.181}
$$

where $\mathcal{S}_{31}$, $\mathcal{S}_{32}$, $\mathcal{S}_{33}$ are given by

$$
\begin{aligned}
\mathcal{S}_{31} &= 2(uv + w_{xx})D_x^{-1}u - w_y D_x^{-1} D_y u,\\
\mathcal{S}_{32} &= 2w_{xx} D_y^{-1} v + w_x D_y^{-1} D_x v,\\
\mathcal{S}_{33} &= 2w_{xx} D_y^{-1} w_y + w_x D_y^{-1} D_x w_y + 2(uv + w_{xx}) D_x^{-1} w_x\\
&\quad - w_y D_x^{-1} D_y w_x + D_y^2 + w_x^2 - w_y^2.
\end{aligned}
\tag{5.182}
$$

It would have been possible not to start from the invariance of p_y, q_x, but from the invariance of for instance p_x, q_x, but in that case we had to incorporate terms like $D_x^{-1}v$, $D_x^{-1}w_x D_x$, $D_x^{-1}w_y D_y$ into the matrix recursion operator $\mathcal{R}$.

CHAPTER 6

Super and graded theories

We shall now generalize the material of the previous chapters to the case of *super* (or *graded*) partial differential equations. We confine ourselves to the case when only dependent variables admit odd gradings and develop a theory closely parallel to that exposed in Chapters 1–5.

We also show here that the cohomological theory of recursion operators may be considered as a particular case of the symmetry theory for graded equations, which, in a sense, explains the main result of Chapter 5, i.e., $H_{\mathcal{C}}^{p,0}(\mathcal{E}) = \ker \ell_{\mathcal{E}}^{(p)}$. It is interesting to note that this reduction is accomplished using an odd analog of the Cartan covering introduced in Example 3.3 of Chapter 3.

Our main computational object is a *graded extension* of a classical partial differential equation. We discuss the principles of constructing nontrivial extensions of such a kind and illustrate them in a series of examples. Other applications are considered in Chapter 7.

1. Graded calculus

Here we redefine the Frölicher–Nijenhuis bracket for the case of n-graded commutative algebras. All definitions below are obvious generalizations of those from 4. Proofs also follow the same lines and are usually omitted.

1.1. Graded polyderivations and forms. Let R be a commutative ring with a unit $1 \in R$ and $\mathcal{A}$ be a commutative n-graded unitary algebra over R, i.e.,

$$\mathcal{A} = \sum_{i \in \mathbb{Z}^n} \mathcal{A}_i, \quad \mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{i+j}$$

and

$$ab = (-1)^{a \cdot b} ba$$

for any homogeneous elements $a, b \in \mathcal{A}$. Here and below the notation $(-1)^{a \cdot b}$ means $(-1)^{i_1 j_1 + \dots i_n j_n}$, where $i = (i_1, \dots, i_n)$, $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$ are the gradings of the elements a and b respectively. We also use the notation $a \cdot b$ for the scalar product of the gradings of elements a and b. In what follows, one can consider $\mathbb{Z}_2^n$-graded objects as well. We consider the category of n-graded (left) $\mathcal{A}$-modules $\mathcal{M}od = \mathcal{M}od(\mathcal{A})$ and introduce the functors

$$D_i \colon \mathcal{M}od(\mathcal{A}) \Rightarrow \mathcal{M}od(\mathcal{A})$$

as follows (cf. [**54, 58**]):

$$D_0(P) = P$$

for any $P \in Ob(\mathcal{M}od)$, $P = \sum_{i \in \mathbb{Z}^n} P_i$, and

$$\begin{aligned} D_{1,j}(P) = \{\Delta \in \hom_R(A, P) \mid \Delta(\mathcal{A}_i) \subset P_{i+j}, \Delta(ab) \\ = \Delta(a)b + (-1)^{\Delta \cdot a} a\Delta(b)\}, \end{aligned}$$

where $j = (j_1, \dots, j_n) = \mathrm{gr}(\Delta) \in \mathbb{Z}^n$ is the grading of Δ; we set

$$D_1(P) = \sum_{i \in \mathbb{Z}^n} D_{1,j}(P).$$

REMARK 6.1. We can also consider objects of $\mathcal{M}od(\mathcal{A})$ as *right* $\mathcal{A}$-modules by setting $pa = (-1)^{a \cdot p} ap$ for any homogeneous $a \in \mathcal{A}$, $p \in P$. In a similar way, for any graded homomorphism $\varphi \in \hom_R(P, Q)$, the *right* action of φ can be introduced by $(p)\varphi = (-1)^{p\varphi}\varphi(p)$.

Further, if $D_0, \dots, D_s$ are defined, we set

$$\begin{aligned} D_{s+1,j}(P) = \{\Delta \in \hom_R(\mathcal{A}, D_s(P)) \mid \Delta(\mathcal{A}_i) \subset D_{s,i+j}(P), \\ \Delta(ab) = \Delta(a)b + (-1)^{\Delta \cdot a} a\Delta(b), \Delta(a, b) + (-1)^{a \cdot b}\Delta(b, a) = 0\} \end{aligned}$$

and

$$D_{s+1}(P) = \sum_{j \in \mathbb{Z}^n} D_{s+1,j}(P).$$

Elements of $D_s(P)$ a called *graded P-valued s-derivations* of $\mathcal{A}$ and elements of $D_*(P) = \sum_{s \geq 0} D_s(P)$ are called graded P-valued polyderivations of $\mathcal{A}$.

PROPOSITION 6.1. *The functors D_s, $s = 0, 1, 2, \dots$, are representable in the category $\mathcal{M}od(\mathcal{A})$, i.e., there exist n-graded modules $\Lambda^0, \Lambda^1, \dots, \Lambda^s, \dots$, such that*

$$D_s(P) = \hom_{\mathcal{A}}(\Lambda^s, P)$$

for all $P \in Ob(\mathcal{M}od)$.

Elements of the module $\Lambda^s = \Lambda^s(\mathcal{A})$ are called *graded differential forms* of degree s.

Our local target is the construction of *graded calculus* in the limits needed for what follows. By *calculus* we mean the set of basic operations related to the functors D_s and to modules Λ^s as well as most important identities connecting these operations. In further applications, we shall need the following particular case:

(i) $\mathcal{A}_{\mathbf{0}} = C^\infty(M)$ for some smooth manifold M, where $\mathbf{0} = (0, \dots, 0)$;
(ii) All homogeneous components P_i of the modules under consideration are projective $\mathcal{A}_{\mathbf{0}}$-modules of finite type.

REMARK 6.2. In fact, the entire scheme of calculus over commutative algebras is carried over to the graded case. For example, to define *graded linear differential operators*, we introduce the action $\delta_a\colon \hom_R(P,Q) \to \hom_R(P,Q)$, $a \in \mathcal{A}$, by setting $\delta_a\varphi = a\varphi - (-1)^{a\varphi}\varphi \cdot a$, $\varphi \in \hom_R(P,Q)$, and say that φ is an operator of order $\leq k$, if

$$(\delta_{a_0} \circ \cdots \circ \delta_{a_k})\varphi = 0$$

for all $a_0, \ldots, a_k \in \mathcal{A}$, etc. A detailed exposition of graded calculus can be found in [**106, 52**].

1.2. Wedge products. Let us now consider some essential algebraic structures in the above introduced objects.

PROPOSITION 6.2. *Let $\mathcal{A}$ be an n-graded commutative algebra. Then*:

(i) *There exists a derivation $d\colon \mathcal{A} \to \Lambda^1$ of grading $\mathbf{0}$ such that for any $\mathcal{A}$-module P and any graded derivation $\Delta\colon \mathcal{A} \to P$ there exists a uniquely defined morphism $f_\Delta\colon \Lambda^1 \to P$ such that $f_\Delta \circ d = \Delta$.*

(ii) *The module Λ^1 is generated over $\mathcal{A}$ by the elements $da = d(a)$, $a \in \mathcal{A}$, with the relations*

$$d(\alpha a + \beta b) = \alpha da + \beta db, \quad d(ab) = (da)b + adb, \qquad a, b \in \mathcal{A}.$$

The j-th homogeneous component of Λ^1 is of the form

$$\Lambda^1_j = \{\textstyle\sum adb \mid a, b \in A, \mathrm{gr}(a) + \mathrm{gr}(b) = j\},$$

(iii) *The modules Λ^s are generated over $\mathcal{A}$ by the elements of the form*

$$\omega_1 \wedge \cdots \wedge \omega_s, \quad \omega_1, \ldots, \omega_s \in \Lambda^1,$$

with the relations

$$\omega \wedge \theta + (-1)^{\omega\theta}\theta \wedge \omega = 0, \quad \omega \wedge a\theta = \omega a \wedge \theta, \qquad \omega, \theta \in \Lambda^1,\ a \in \mathcal{A}.$$

The j-th homogeneous component of Λ^s is of the form

$$\Lambda^s_j = \{\textstyle\sum \omega_1 \wedge \cdots \wedge \omega_s \mid \omega_i \in \Lambda^1, \mathrm{gr}(\omega_1) + \cdots + \mathrm{gr}(\omega_s) = j\}.$$

(iv) *Let $\omega \in \Lambda^s_j$, $j = (j_1, \ldots, j_n)$. Set $\mathrm{gr}^1(\omega) = (j_1, \ldots, j_n, s)$. Then*

$$\Lambda^* = \sum_{s \geq 0} \Lambda^s = \sum_{s \geq 0} \sum_{j \in \mathbb{Z}^n} \Lambda^s_j$$

is an $(n+1)$-graded commutative algebra with respect to the wedge product

$$\omega \wedge \theta = \omega \wedge \cdots \wedge \omega_s \wedge \theta_1 \wedge \cdots \wedge \theta_r, \quad \omega \in \Lambda^s,\ \theta \in \Lambda^r, \quad \omega_\alpha, \theta_\beta \in \Lambda^1,$$

i.e.,

$$\omega \wedge \theta = (-1)^{\omega \cdot \theta + sr}\theta \wedge \omega,$$

where $\omega \cdot \theta$ in the power of (-1) denotes scalar product of gradings inherited by ω and θ from $\mathcal{A}$.

REMARK 6.3. When working with the algebraic definition of differential forms in the graded situation, one encounters the same problems as in a pure commutative setting, i.e., the problem of ghost elements. To kill ghosts, the same procedures as in Chapter 4 (see Remark 4.4) are to be used.

A similar wedge product can be defined in $D_*(\mathcal{A})$. Namely for $a, b \in D_0(\mathcal{A}) = \mathcal{A}$ we set

$$a \wedge b = ab$$

and then by induction define

$$(\Delta \wedge \nabla)(a) \stackrel{\text{def}}{=} \Delta \wedge \nabla(a) + (-1)^{\nabla \cdot a + r} \Delta(a) \wedge \nabla, \tag{6.1}$$

where $a \in \mathcal{A}$, $\Delta \in D_s(\mathcal{A})$, $\nabla \in D_r(\mathcal{A})$ and ∇ in the power of (-1) denotes the grading of ∇ in the sense of the previous subsection.

PROPOSITION 6.3. *For any n-graded commutative algebra $\mathcal{A}$ the following statements are valid*:

(i) *Definition* (6.1) *determines a mapping*

$$\wedge\colon D_s(\mathcal{A}) \otimes_{\mathcal{A}} D_r(\mathcal{A}) \to D_{s+r}(\mathcal{A}),$$

which is in agreement with the graded structure of polyderivations:

$$D_{s,i}(\mathcal{A}) \wedge D_{r,j}(\mathcal{A}) \subset D_{s+r,i+j}(\mathcal{A}).$$

(ii) *The module $D_*(\mathcal{A}) = \sum_{s \geq 0} \sum_{j \in \mathbb{Z}^n} D_{s,j}$ is an $(n+1)$-graded commutative algebra with respect to the wedge product*:

$$\Delta \wedge \nabla = (-1)^{\Delta \cdot \nabla + rs} \nabla \wedge \Delta$$

for any $\Delta \in D_s(\mathcal{A})$, $\nabla \in D_r(\mathcal{A})$.[1]

(iii) *If $\mathcal{A}$ satisfies conditions* (i), (ii) *on page* 244, *then the module $D_*(\mathcal{A})$ is generated by $D_0(\mathcal{A}) = \mathcal{A}$ and $D_1(\mathcal{A})$, i.e., any $\Delta \in D_s(\mathcal{A})$ is a sum of the elements of the form*

$$a\Delta_1 \wedge \dots \wedge \Delta_s, \quad \Delta_i \in D_1(\mathcal{A}),\ a \in \mathcal{A}.$$

REMARK 6.4. One can define a wedge product $\wedge\colon D_i(\mathcal{A}) \otimes_{\mathcal{A}} D_j(P) \to D_{i+j}(P)$ with respect to which $D_*(P)$ acquires the structure of an $(n+1)$-graded $D_*(\mathcal{A})$-module (see [**54**]), but it will not be needed below.

1.3. Contractions and graded Richardson–Nijenhuis bracket. We define a *contraction* of a polyderivation $\Delta \in D_s(\mathcal{A})$ into a form $\omega \in \Lambda^r$ in the following way

$$\begin{aligned}
&\mathrm{i}_\Delta \omega \equiv \Delta \lrcorner\, \omega = 0, \ \text{if } s > r,\\
&\mathrm{i}_\Delta \omega = \Delta(\omega), \ \text{if } s = r, \ \text{due to the definition of } \Lambda^r,\\
&\mathrm{i}_a \omega = a\omega, \ \text{if } a \in \mathcal{A} = D_0(\mathcal{A}),
\end{aligned}$$

[1] This distinction between first n gradings and additional $(n+1)$-st one will be preserved both for graded forms and graded polyderivations throughout the whole chapter.

and for $r > s$ set by induction

$$\mathrm{i}_\Delta(da \wedge \omega) = \mathrm{i}_{\Delta(a)}(\omega) + (-1)^{\Delta\cdot a+s} da \wedge \mathrm{i}_\Delta(\omega). \tag{6.2}$$

PROPOSITION 6.4. *Let $\mathcal{A}$ be an n-graded commutative algebra.*

(i) *For any $\Delta \in D_s(\mathcal{A})$ definition* (6.2) *determines an $(n+1)$-graded differential operator*

$$\mathrm{i}_\Delta \colon \Lambda^* \to \Lambda^*$$

of the order s.

(ii) *In particular, if $\Delta \in D_1(\mathcal{A})$, then i_Δ is a graded derivation of Λ^*:*

$$\mathrm{i}_\Delta(\omega \wedge \theta) = \mathrm{i}_\Delta(\omega) \wedge \theta + (-1)^{\Delta\cdot\omega+r} \omega \wedge \mathrm{i}_\Delta \theta, \quad \omega \in \Lambda^r, \theta \in \Lambda^*.$$

Now we consider tensor products of the form $\Lambda^r \otimes_{\mathcal{A}} D_s(\mathcal{A})$ and generalize contraction and wedge product operations as follows

$$(\omega \otimes \Delta) \wedge (\theta \otimes \Delta) = (-1)^{\Delta\cdot\theta}(\omega \wedge \theta) \otimes (\Delta \wedge \nabla),$$
$$\mathrm{i}_{\omega\otimes\Delta}(\theta \otimes \nabla) = \omega \wedge i_\Delta(\theta) \otimes \nabla,$$

where $\omega, \theta \in \Lambda^*$, $\Delta, \nabla \in D_*(\mathcal{A})$. Let us define the *Richardson–Nijenhuis bracket* in $\Lambda^* \otimes D_s(\mathcal{A})$ by setting

$$[\![\Omega, \Theta]\!]_s^{\mathrm{rn}} = \mathrm{i}_\Omega(\Theta) - (-1)^{(\omega+\Delta)\cdot(\Theta+\nabla)+(q-s)(r-s)} \mathrm{i}_\Theta(\Omega), \tag{6.3}$$

where $\Omega = \omega \otimes \Delta \in \Lambda^r \otimes D_s(\mathcal{A})$, $\Theta = \theta \otimes \nabla \in \Lambda^q \otimes D_s(\mathcal{A})$. In what follows, we confine ourselves with the case $s = 1$ and introduce an $(n+1)$-graded structure into $\Lambda^* \otimes D_1(\mathcal{A})$ by setting

$$\mathrm{gr}(\omega \otimes X) = (\mathrm{gr}(\omega) + \mathrm{gr}(X), r), \tag{6.4}$$

where $\mathrm{gr}(\omega)$ and $\mathrm{gr}(X)$ are initial n-gradings of the elements $\omega \in \Lambda^r$, $X \in D_1(\mathcal{A})$. We also denote by Ω and Ω_1 the first n and $(n+1)$-st gradings of Ω respectively in the powers of (-1).

PROPOSITION 6.5. *Let $\mathcal{A}$ be an n-graded commutative algebra. Then*:

(i) *For any two elements $\Omega, \Theta \in \Lambda^* \otimes D_1(\mathcal{A})$ one has*

$$[\mathrm{i}_\Omega, \mathrm{i}_\Theta] = \mathrm{i}_{[\![\Omega,\Theta]\!]_1^{\mathrm{rn}}}.$$

Hence, the Richardson–Nijenhuis bracket $[\![\cdot,\cdot]\!]^{\mathrm{rn}} = [\![\cdot,\cdot]\!]_1^{\mathrm{rn}}$ determines in $\Lambda^* \otimes D_1(\mathcal{A})$ the structure of $(n+1)$-graded Lie algebra with respect to the grading in which $(n+1)$-st component is shifted by 1 with respect to (6.4), *i.e.,*

(ii) $[\![\Omega, \Theta]\!]^{\mathrm{rn}} + (-1)^{\Omega\cdot\Theta+(\Omega_1+1)(\Theta_1+1)}[\![\Theta, \Omega]\!]^{\mathrm{rn}} = 0,$

(iii) $\oint(-1)^{\Theta\cdot(\Omega+\Xi)+(\Theta_1+1)(\Omega_1+\Xi_1)}[\![[\![\Omega, \Theta]\!]^{\mathrm{rn}}, \Xi]\!]^{\mathrm{rn}} = 0$, *where, as before, $\oint$ denotes the sum of cyclic permutations.*

(iv) *Moreover, if $\rho \in \Lambda^*$, then*

$$[\![\Omega, \rho \wedge \theta]\!]^{\mathrm{rn}} = (\Omega \lrcorner \rho) \wedge \Theta + (-1)^{\Omega\cdot\rho} \wedge [\![\Omega, \Theta]\!]^{\mathrm{rn}}.$$

(v) *In conclusion, the composition of two contractions is expressed by*

$$\mathrm{i}_\Omega \circ \mathrm{i}_\Theta = \mathrm{i}_{\Omega\lrcorner\Theta} + (-1)^{\Omega_1} \mathrm{i}_{\Omega\wedge\Theta}.$$

1.4. De Rham complex and Lie derivatives. The *de Rham differential* $d\colon \Lambda^r \to \Lambda^{r+1}$ is defined as follows. For $r = 0$ it coincides with the derivation $d\colon \mathcal{A} \to \Lambda^1$ introduced in Proposition 6.2. For any $adb \in \Lambda^1$, $a, b \in \mathcal{A}$, we set

$$d(adb) = da \wedge db$$

and for a decomposable form $\omega = \theta \wedge \rho \in \Lambda^r$, $\theta \in \Lambda^{r'}$, $\rho \in \Lambda^{r''}$, $r > 1, r', r'' < r$, set

$$d\omega = d\theta \wedge \rho + (-1)^{\theta_1} \theta \wedge d\rho.$$

By definition, $d\colon \Lambda^* \to \Lambda^*$ is a derivation of grading $(\mathbf{0}, 1)$ and, obviously,

$$d \circ d = 0.$$

Thus, one gets a complex

$$0 \to \mathcal{A} \xrightarrow{d} \Lambda^1 \to \cdots \to \Lambda^r \to d\Lambda^{r+1} \to \cdots,$$

which is called the *de Rham complex* of $\mathcal{A}$.

Let $X \in D_1(A)$ be a derivation. A *Lie derivative* $\mathrm{L}_X\colon \Lambda^* \to \Lambda^*$ is defined as

$$\mathrm{L}_X = [\mathrm{i}_X, d] = \mathrm{i}_X \circ d + d \circ \mathrm{i}_X. \tag{6.5}$$

Thus for any $\omega \in \Lambda^*$ one has

$$\mathrm{L}_X \omega = X \lrcorner\, d\omega + d(X \lrcorner\, \omega).$$

The basic properties of L_X are described by

PROPOSITION 6.6. *For any commutative n-graded algebra $\mathcal{A}$ one has*

(i) *If $\omega, \theta \in \Lambda^*$, then*

$$\mathrm{L}_X(\omega \wedge \theta) = \mathrm{L}_X\omega \wedge \theta + (-1)^{X \cdot \omega} \omega \wedge \mathrm{L}_X \theta,$$

i.e., L_X *really is a derivation of grading* $(\mathrm{gr}(X), 0)$.

(ii) $[\mathrm{L}_X, d] = \mathrm{L}_X \circ d - d \circ \mathrm{L}_X = 0$.

(iii) *For any $a \in \mathcal{A}$ and $\omega \in \Lambda^*$ one has*

$$\mathrm{L}_{aX}(\omega) = a\mathrm{L}_X\omega + da \wedge \mathrm{i}_X(\omega).$$

(iv) $[\mathrm{L}_X, \mathrm{i}_Y] = [\mathrm{i}_X, \mathrm{L}_Y] = \mathrm{i}_{[X,Y]}$.

(v) $[\mathrm{L}_X, \mathrm{L}_Y] = \mathrm{L}_{[X,Y]}$.

Now we extend the classical definition of Lie derivative onto the elements of $\Lambda^* \otimes D_1(\mathcal{A})$ and for any $\Omega \in \Lambda^* \otimes D_1(\mathcal{A})$ define

$$\mathrm{L}_\Omega = [\mathrm{i}_\Omega, d] = \mathrm{i}_\Omega \circ d + (-1)^{\Omega_1} d \circ \mathrm{i}_\Omega.$$

If $\Omega = \omega \otimes X$, then one has

$$\mathrm{L}_{\omega \otimes X} = \omega \wedge \mathrm{L}_X + (-1)^{\omega_1} d\omega \wedge \mathrm{i}_X.$$

PROPOSITION 6.7. *For any n-graded commutative algebra $\mathcal{A}$ the following statements are valid*:

(i) *For any* $\Omega \in \Lambda^* \otimes D_1(\mathcal{A})$ *one has*

$$\mathrm{L}_\Omega(\rho \wedge \theta) = L_\Omega(\rho) \wedge \theta + (-1)^{\Omega\cdot\rho+\Omega_1\cdot\rho_1}\rho \wedge \mathrm{L}_\Omega\theta, \quad \rho, \theta \in \Lambda^*,$$

i.e., L_Ω *is a derivation of* Λ^* *whose grading coincides with that of* Ω.

(ii) $[\mathrm{L}_\Omega, d] = \mathrm{L}_\Omega \circ d - (-1)^{\Omega_1} d \circ \mathrm{L}_\Omega = 0$.

(iii) $\mathrm{L}_{\rho\wedge\Omega} = \rho \wedge \mathrm{L}_\Omega + (-1)^{\rho_1+\Omega_1} d\rho \wedge \mathrm{i}_\Omega$, $\rho \in \Lambda^*$.

To formulate properties of L_Ω similar to (iv) and (v) of Proposition 6.6, one needs a new notion.

1.5. Graded Frölicher–Nijenhuis bracket. We shall now study the commutator of two Lie derivatives.

Proposition 6.8. *Let, as before,* $\mathcal{A}$ *be an* n*-graded commutative algebra.*

(i) *For any two elements* $\Omega, \Theta \in \Lambda^* \otimes D_1(\mathcal{A})$, *the commutator of corresponding Lie derivatives* $[\mathrm{L}_\Omega, \mathrm{L}_\Theta]$ *is of the form* L_Ξ *for some* $\Xi \in \Lambda^* \otimes D_1(\mathcal{A})$.

(ii) *The correspondence* $\mathrm{L}\colon \Lambda^* \otimes D_1(\mathcal{A}) \to D_1(\Lambda^*)$, $\Omega \mapsto \mathrm{L}_\Omega$, *is injective and hence* Ξ *in* (i) *is defined uniquely. It is called the* (graded) Frölicher–Nijenhuis bracket *of the elements* Ω, Θ *and is denoted by* $\Xi = [\![\Omega, \Theta]\!]^{\mathrm{fn}}$. *Thus, by definition, one has*

$$[\mathrm{L}_\Omega, \mathrm{L}_\Theta] = \mathrm{L}_{[\![\Omega,\Theta]\!]^{\mathrm{fn}}}.$$

(iii) *If* Ω *and* Θ *are of the form*

$$\Omega = \omega \otimes X, \ \Theta = \theta \otimes Y, \quad \omega, \theta \in \Lambda^*, \ X, Y \in D_1(\mathcal{A}),$$

then

$$\begin{aligned}
[\![\Omega, \Theta]\!]^{\mathrm{fn}} &= (-1)^{X\cdot\theta}\omega \wedge \theta \otimes [X, Y] + \omega \wedge \mathrm{L}_X\theta \otimes Y \\
&\quad + (-1)^{\Omega_1} d\omega \wedge (X \lrcorner\, \theta) \otimes Y \\
&\quad - (-1)^{\Omega\cdot\Theta+\Omega_1\cdot\Theta_1}\theta \wedge \mathrm{L}_Y\omega \otimes X \\
&\quad - (-1)^{\Omega\cdot\Theta+(\Omega_1+1)\cdot\Theta_1} d\theta \wedge (Y \lrcorner\, \omega) \otimes X \\
&= (-1)^{X\cdot\theta}\omega \wedge \theta \otimes [X, Y] + \mathrm{L}_\Omega(\theta) \otimes Y \\
&\quad - (-1)^{\Omega\cdot\Theta+\Omega_1\cdot\Theta_1}\mathrm{L}_\Theta(\omega) \otimes X.
\end{aligned} \tag{6.6}$$

(iv) *If* $\Omega = X$, $\Theta = Y \in D_1(\mathcal{A}) = \Lambda^0 \otimes D_1(\mathcal{A})$, *then the graded Frölicher–Nijenhuis bracket of* Ω *and* Θ *coincides with the graded commutator of vector fields*:

$$[\![X, Y]\!]^{\mathrm{fn}} = [X, Y].$$

The main properties of the Frölicher–Nijenhuis bracket are described by

Proposition 6.9. *For any* $\Omega, \Theta, \Xi \in \Lambda^* \otimes D_1(\mathcal{A})$ *and* $\rho \in \Lambda^*$ *one has*

(i)

$$[\![\Omega, \Theta]\!]^{\mathrm{fn}} + (-1)^{\Omega\cdot\Theta+\Omega_1\cdot\Theta_1}[\![\Theta, \Omega]\!]^{\mathrm{fn}} = 0. \tag{6.7}$$

(ii)

$$\oint(-1)^{(\Omega+\Xi)\cdot\Theta+(\Omega_1+\Xi_1)\cdot\Theta_1}[\![\Omega,[\![\Theta,\Xi]\!]^{\mathrm{fn}}]\!]^{\mathrm{fn}}=0, \tag{6.8}$$

i.e., $[\![\cdot,\cdot]\!]^{\mathrm{fn}}$ *defines a graded Lie algebra structure in* $\Lambda^*\otimes D_1(\mathcal{A})$.

(iii)

$$[\![\Omega,\rho\wedge\Theta]\!]^{\mathrm{fn}}=\mathrm{L}_\Omega(\rho)\wedge\Theta-(-1)^{\Omega\cdot(\Theta+\rho)+(\Omega_1+1)\cdot(\Theta_1+\rho_1)}d\rho\wedge\mathrm{i}_\Theta\Omega \\ +(-1)^{\Omega\cdot\rho+\Omega_1\cdot\rho_1}\cdot\rho\wedge[\![\Omega,\Theta]\!]^{\mathrm{fn}}. \tag{6.9}$$

(iv)

$$[\mathrm{L}_\Omega,\mathrm{i}_\Theta]+(-1)^{\Omega\cdot\Theta+\Omega_1\cdot(\Theta_1+1)}\mathrm{L}_{\Theta\lrcorner\Omega}=\mathrm{i}_{[\![\Omega,\Theta]\!]^{\mathrm{fn}}}. \tag{6.10}$$

(v)

$$\mathrm{i}_\Xi[\![\Omega,\Theta]\!]^{\mathrm{fn}}=[\![\mathrm{i}_\Xi\Omega,\Theta]\!]^{\mathrm{fn}}+(-1)^{\Omega\cdot\Xi+\Omega_1\cdot(\Xi_1+1)}[\![\Omega,\mathrm{i}_\Xi\Theta]\!]^{\mathrm{fn}} \\ +(-1)^{\Omega_1}\mathrm{i}_{[\![\Xi,\Omega]\!]^{\mathrm{fn}}}\Theta-(-1)^{\Omega\cdot\Theta+(\Omega_1+1)\cdot\Theta_1}\mathrm{i}_{[\![\Xi,\Theta]\!]^{\mathrm{fn}}}\Omega. \tag{6.11}$$

REMARK 6.5. Similar to the commutative case, identity (6.11) can be taken for the inductive definition of the graded Frölicher–Nijenhuis bracket.

Let now U be an element of $\Lambda^1\otimes D_1(\mathcal{A})$ and let us define the operator

$$\partial_U=[\![U,\cdot]\!]^{\mathrm{fn}}\colon\Lambda^r\otimes D_1(\mathcal{A})\to\Lambda^{r+1}\otimes D_1(\mathcal{A}). \tag{6.12}$$

Then from the definitions it follows that

$$\partial_U(U)=[\![U,U]\!]^{\mathrm{fn}}=(1+(-1)^{U\cdot U})\mathrm{L}_U\circ L_U \tag{6.13}$$

and from (6.7) and (6.8) one has

$$(1+(-1)^{U\cdot U})\partial_U(\partial_U\Omega)+(-1)^{U\cdot U}[\![\Omega,[\![U,U]\!]^{\mathrm{fn}}]\!]^{\mathrm{fn}}=0$$

for any $\Omega\in\Lambda^*\otimes D_1(\mathcal{A})$.

We are interested in the case when (6.12) is a complex, i.e., $\partial_U\circ\partial_U=0$, and give the following

DEFINITION 6.1. An element $U\in\Lambda^1\otimes D_1(\mathcal{A})$ is said to be *integrable*, if

(i) $[\![U,U]\!]^{\mathrm{fn}}=0$ and
(ii) $(-1)^{U\cdot U}$ equals 1.

From the above said it follows that for an integrable element U one has $\partial_U\circ\partial_U=0$, and we can introduce the corresponding cohomologies by

$$H_U^r(\mathcal{A})=\frac{\ker(\partial_U\colon\Lambda^r\otimes D_1(\mathcal{A})\to\Lambda^{r+1}\otimes D_1(\mathcal{A}))}{\operatorname{im}(\partial_U\colon\Lambda^{r-1}\otimes D_1(\mathcal{A})\to\Lambda^r\otimes D_1(\mathcal{A}))}.$$

The main properties of ∂_U are described by

PROPOSITION 6.10. *Let* $U\in\Lambda^1\otimes D_1(\mathcal{A})$ *be an integrable element and* $\Omega,\Theta\in\Lambda^*\otimes D_1(\mathcal{A})$, $\rho\in\Lambda^*$. *Then*

(i) $\partial_U(\rho\wedge\Omega)=\mathrm{L}_U(\rho)\wedge\Omega-(-1)^{U\cdot(\Omega+\rho)}d\rho\wedge\mathrm{i}_\Omega U+(-1)^{U\cdot\rho+\rho_1}\rho\wedge\partial_U\Omega$.

(ii) $[\mathrm{L}_U, \mathrm{i}_\Omega] = \mathrm{i}_{\partial_U\Omega} + (-1)^{U\cdot\Omega+\Omega_1}\mathrm{L}_{\Omega \lrcorner U}$.
(iii) $[\mathrm{i}_\Omega, \partial_U]\Theta + (-1)^{U\cdot\Theta}\mathrm{i}_{[\![\Omega,\Theta]\!]^{\mathrm{fn}}}U = [\![\mathrm{i}_\Omega U, \Theta]\!]^{\mathrm{fn}} + (-1)^{U\cdot\Omega+\Omega_1}\mathrm{i}_{\partial_U\Omega}\Theta$.
(iv) $\partial_U[\![\Omega,\Theta]\!]^{\mathrm{fn}} = [\![\partial_U\Omega,\Theta]\!]^{\mathrm{fn}} + (-1)^{U\cdot\Omega+\Omega_1}[\![\Omega,\partial_U\Theta]\!]^{\mathrm{fn}}$.

From the last equality it follows that the Frölicher–Nijenhuis bracket is inherited by the module $H^*_U(\mathcal{A}) = \sum_{r\geq 0} H^r_U(\mathcal{A})$ and thus the latter forms an $(n+1)$-graded Lie algebra with respect to this bracket.

2. Graded extensions

In this section, we adapt the cohomological theory of recursion operators constructed in Chapter 5 (see also [**55, 58**]) to the case of graded (in particular, super) differential equations. Our first step is an appropriate definition of graded equations (cf. [**87**] and the literature cited there). In what follows, we still assume all the modules to be projective and of finite type over the main algebra $\mathcal{A}_0 = C^\infty(M)$ or to be filtered by such modules in a natural way.

2.1. General construction. Let R be a commutative ring with a unit and $A_{-1} \subset A_0$ be two unitary associative commutative $\mathbb{Z}^n$-graded R-algebras. Let $\mathcal{D} = \mathcal{D}_0 \subset D(A_{-1}, A_0)$ be an A_0-submodule in the module

$$\begin{aligned} D(A_{-1}, A_0) = \{\partial \in \hom_R(A_{-1}, A_0) \mid \partial(aa')\\ = \partial a \cdot a' + (-1)^{a\cdot\partial} a \cdot \partial a',\ a, a' \in A_{-1}\}. \end{aligned}$$

Let us define a $\mathbb{Z}^n$-graded A_0-algebra A_1 by the generators

$$[\partial, a], \quad a \in A_0,\ \partial \in D_0,\ \mathrm{gr}[\partial, a] = \mathrm{gr}(\partial) + \mathrm{gr}(a),$$

with the relations

$$\begin{aligned} [\partial, a_0] &= \partial a_0,\\ [\partial, a + a'] &= [\partial, a] + [\partial, a'],\\ [a'\partial' + a''\partial'', a] &= a'[\partial', a] +' [\partial'', a],\\ [\partial, aa'] &= [\partial, a]\cdot a' + (-1)^{\partial\cdot a} a \cdot [\partial, a'], \end{aligned}$$

where $a_0 \in A_{-1}$, $a, a', a'' \in A$, $\partial, \partial', \partial'' \in \mathcal{D}_0$.

For any $\partial \in \mathcal{D}_0$ we can define a derivation $\partial^{(1)} \in D(A_0, A_1)$ by setting

$$\partial^{(1)}(a) = [\partial, a], \quad a \in A_1.$$

Obviously, $\partial^{(1)} a = \partial a$ for $a \in A_0$. Denoting by $\mathcal{D}_1$ the A_1-submodule in $D(A_0, A_1)$ generated by the elements of the form $\partial^{(1)}$, one gets the triple

$$\{A_0, A_1, \mathcal{D}_1\}, \quad A_0 \subset A_1,\ \mathcal{D}_1 \subset D(A_0, A_1),$$

which allows one to construct $\{A_1, A_2, \mathcal{D}_2\}$, etc. and to get two infinite sequences of embeddings

$$A_{-1} \to A_0 \to \cdots \to A_i \to A_{i+1} \to \cdots$$

and

$$\mathcal{D}_0 \to \mathcal{D}_1 \to \cdots \to \mathcal{D}_i \to \mathcal{D}_{i+1} \to \cdots ,$$

where $A_{i+1} = (A_i)_1$, $\mathcal{D}_{i+1} = (\mathcal{D}_i)_1 \subset D(A_{i-1}, A_i)$, and $\mathcal{D}_i \to \mathcal{D}_{i+1}$ is a morphism of A_{i+1}-modules.

Let us set

$$A_\infty = \operatorname{inj}\lim_{i\to\infty} A_i, \quad \mathcal{D}_\infty = \operatorname{inj}\lim_{i\to\infty} \mathcal{D}_i.$$

Then $\mathcal{D}_\infty \subset D(A_\infty)$ and any element $\partial \in \mathcal{D}_0$ determines a derivation $\mathcal{D}(\partial) = \partial^{(\infty)} \in D(A_\infty)$. The correspondence $\mathcal{D}\colon \mathcal{D}_0 \to D(A_\infty)$ possesses the following properties

$$\mathcal{D}(X)(a) = X(a) \text{ for } a \in A_{-1},$$
$$\mathcal{D}(aX) = a\mathcal{D}(X) \text{ for } a \in A_0.$$

Moreover, by definition one has

$$[\mathcal{D}(X), \mathcal{D}(Y)](a) = \mathcal{D}(X)(Y(a)) - (-1)^{X\cdot Y}\mathcal{D}(Y)(X(a)),$$

$a \in A_{-1}$, $X, Y \in \mathcal{D}_0$.

2.2. Connections. Similar to Chapter 5, we introduce the notion of a connection in the graded setting.

Let A and B be two n-graded algebras, $A \subset B$. Consider modules the of derivations $D(A, B)$ and $D(B)$ and a B-linear mapping

$$\nabla\colon D(A, B) \to D(B).$$

The mapping ∇ is called a *connection* for the pair (A, B), or an (A, B)-connection, if

$$\nabla(X)|_A = X.$$

From the definition it follows that ∇ is of degree $\mathbf{0}$ and that for any derivations $X, Y \in D(A, B)$ the element

$$\nabla(X) \circ Y - (-1)^{X\cdot Y}\nabla(Y) \circ X$$

again lies in $D(A, B)$. Thus one can define the element

$$R_\nabla(X, Y) = [\nabla(X), \nabla(Y)] - \nabla(\nabla(X) \circ Y - (-1)^{X\cdot Y}\nabla(Y) \circ X)$$

which is called the *curvature* of the connection ∇ and possesses the following properties

$$R_\nabla(X, Y) + (-1)^{X\cdot Y}R_\nabla(Y, X) = 0, \quad X, Y \in D(A, B),$$
$$R_\nabla(aX, Y) = aR_\nabla(X, Y), \quad a \in B,$$
$$R_\nabla(X, bY) = (-1)^{X\cdot b}bR_\nabla(X, Y), \quad b \in B.$$

A connection ∇ is called *flat*, if $R_\nabla(X, Y) = 0$ for all $X, Y \in D(A, B)$.

Evidently, when the grading is trivial, the above introduced notions coincide with the ones from Chapter 5.

2.3. Graded extensions of differential equations. Let now M be a smooth manifold and $\pi\colon E \to M$ be a smooth locally trivial fibre bundle over M. Let $\mathcal{E} \subset J^k(\pi)$ be a k-th order differential equation represented as a submanifold in the manifold of k-jets for the bundle π. We assume $\mathcal{E}$ to be formally integrable and consider its infinite prolongation $\mathcal{E}^i \subset J^\infty(\pi)$.

Let $\mathcal{F}(\mathcal{E})$ be the algebra of smooth functions on $\mathcal{E}^\infty$ and $\mathcal{C}D(\mathcal{E}) \subset D(\mathcal{E}) = D(\mathcal{F}(\mathcal{E}))$ be the Lie algebra generated by total derivatives $\mathcal{C}X$, $X \in D(M)$, $\mathcal{C}\colon D(M) \to D(\mathcal{E})$ being the Cartan connection on $\mathcal{E}^\infty$ (see Chapter 2).

Let $\mathcal{F}$ be an n-graded commutative algebra such that $\mathcal{F}_{\mathbf{0}} = \mathcal{F}(\mathcal{E})$. Denote by $\mathcal{C}D_0(\mathcal{E})$ the $\mathcal{F}$-submodule in $D(\mathcal{F}(\mathcal{E}), \mathcal{F})$ generated by $\mathcal{C}D(\mathcal{E})$ and consider the triple $(\mathcal{F}(\mathcal{E}), \mathcal{F}, \mathcal{C}D_0(\mathcal{E}))$ as a starting point for the construction from Subsection 2.1. Then we shall get a pair $(\mathcal{F}_\infty, \mathcal{C}D_\infty(\mathcal{E}))$, where $\mathcal{C}D_\infty(\mathcal{E}) \overset{\text{def}}{=} (\mathcal{C}D_0(\mathcal{E}))_\infty$. We call the pair $(\mathcal{F}_\infty, \mathcal{C}D_\infty(\mathcal{E}))$ a *free differential $\mathcal{F}$-extension* of the equation $\mathcal{E}$.

The algebra $\mathcal{F}_\infty$ is filtered by its graded subalgebras $\mathcal{F}_i$, $i = -1, 0, 1, \dots$, and we consider its filtered graded $\mathcal{C}D_\infty(\mathcal{E})$-stable ideal I. Any vector field (derivation) $X \in \mathcal{C}D_\infty(\mathcal{E})$ determines a derivation $X_I \in D(\mathcal{F}_I)$, where $\mathcal{F}_I = \mathcal{F}/I$. Let $\mathcal{C}D_I(\mathcal{E})$ be an $\mathcal{F}_I$-submodule generated by such derivations. Obviously, it is closed with respect to the Lie bracket. We call the pair $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$ a *graded extension* of the equation $\mathcal{E}$, if $I \cap \mathcal{F}(\mathcal{E}) = 0$, where $\mathcal{F}(\mathcal{E})$ is considered as a subalgebra in $\mathcal{F}_\infty$.

Let $\mathcal{F}_{-\infty} = C^\infty(M)$. In an appropriate algebraic setting, the Cartan connection $\mathcal{C}\colon D(\mathcal{F}_{-\infty}) \to D(\mathcal{F}(\mathcal{E}))$ can be uniquely extended up to a connection

$$\mathcal{C}_I\colon D(\mathcal{F}_{-\infty}, \mathcal{F}_I) \to \mathcal{C}D_I(\mathcal{E}) \subset D(\mathcal{F}_I).$$

In what follows we call graded extensions which admit such a connection *$\mathcal{C}$-natural.* From the flatness of the Cartan connection and from the definition of the algebra $\mathcal{C}D_\infty(\mathcal{E})$ (see Subsection 2.1) it follows that $\mathcal{C}_I$ is a flat connection as well, i.e.,

$$R_{\mathcal{C}_I}(X, Y) = 0,$$

where $X, Y \in D(\mathcal{F}_{-\infty}, \mathcal{F}_I)$, for any $\mathcal{C}$-natural graded extension $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$.

2.4. The structural element and $\mathcal{C}$-cohomologies. Let us consider a $\mathcal{C}$-natural graded extension $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$ and define a homomorphism $U_I \in \hom_{\mathcal{F}_I}(D(\mathcal{F}_I), D(\mathcal{F}_I))$ by

$$U_I(X) = X - \mathcal{C}_I(X_{-\infty}), \quad X \in D(\mathcal{F}_I),\ X_{-\infty} = X|_{\mathcal{F}_{-\infty}}. \tag{6.14}$$

The element U_I is called the *structural element* of the graded extension $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$.

Due to the assumptions formulated above, U_I is an element of the module $D_1(\Lambda^*(\mathcal{F}_I))$, where $\mathcal{F}_I$ is finitely smooth (see Chapter 4) graded algebra, and consequently can be treated in the same way as in the nongraded situation.

THEOREM 6.11. *For any $\mathcal{C}$-natural graded extension $(\mathcal{F}_I(\mathcal{E}), \mathcal{C}D_I(\mathcal{E}))$, the equation $\mathcal{E}$ being formally integrable, its structural element is integrable*:

$$[\![U_I, U_I]\!]^{\mathrm{fn}} = 0.$$

PROOF. Let $X, Y \in D(\mathcal{F}_I)$ and consider the bracket $[\![U_I, U_I]\!]^{\mathrm{fn}}$ as an element of the module $\hom_{\mathcal{F}_I}(D_I(\mathcal{E}) \wedge D_I(\mathcal{E}), D_I(\mathcal{E}))$. Then applying (6.11) twice, one can see that

$$[\![U_I, U_I]\!]^{\mathrm{fn}}(X, Y) = \varepsilon\Big((-1)^{U\cdot Y}[U_I(X), U_I(Y)] - (-1)^{U\cdot Y}U_I([U_I(X), Y]) \\ - U_I([X, U_I(Y)]) + U_I^2([X, Y])\Big), \quad (6.15)$$

where $\varepsilon = (-1)^{X\cdot Y}(1+(-1)^{U\cdot U})$. Expression (6.15) can be called the *graded Nijenhuis torsion* (cf. [**49**]).

From (6.14) if follows that the grading of U_I is $\mathbf{0}$, and thus (6.15) transforms to

$$[\![U_I, U_I]\!]^{\mathrm{fn}}(X, Y) = (-1)^{X\cdot Y} \cdot 2\Big([U_I(X), U_I(Y)] - U_I[U_I(X), Y] \\ - U_I[X, U_I(Y)] + U_I^2[X, Y]\Big). \quad (6.16)$$

Now, using definition (6.14) of U_I, one gets from (6.16):

$$[\![U_I, U_I]\!]^{\mathrm{fn}}(X, Y) \\ = (-1)^{X\cdot Y} \cdot 2\Big([\mathcal{C}_I(X_{-\infty}), \mathcal{C}_I(Y_{-\infty})] - \mathcal{C}_I([\mathcal{C}_I(X_{-\infty}), Y]_{-\infty}) \\ - \mathcal{C}_I([X, \mathcal{C}_I(Y_{-\infty}]_{-\infty}) + \mathcal{C}_I((\mathcal{C}_I([X, Y]_{-\infty}))_{-\infty}\Big).$$

But for any vector fields $X, Y \in D(\mathcal{F}_I)$ one has

$$(\mathcal{C}_I(X_{-\infty}))_{-\infty} = X_{-\infty}.$$

and

$$[X, Y]_{-\infty} = X \circ Y_{-\infty} - (-1)^{X\cdot Y} Y \circ X_{\infty}.$$

Hence,

$$[\![U_I, U_I]\!]^{\mathrm{fn}}(X, Y) = (-1)^{X\cdot Y} \cdot 2\Big([\mathcal{C}_I(X_{-\infty}), \mathcal{C}_I(Y_{-\infty})] \\ - \mathcal{C}_I(\mathcal{C}_I(X_{-\infty}) \circ Y_{-\infty} - (-1)^{X\cdot Y}\mathcal{C}_I(Y_{-\infty}) \circ X_{-\infty})\Big) \\ = (-1)^{X\cdot Y} 2R_{\mathcal{C}_I}(X, Y) = 0.$$

□

Hence, with any $\mathcal{C}$-natural graded $\mathcal{E}$-equation, in an appropriate algebraic setting, one can associate a complex

$$0 \to D(\mathcal{F}_I) \to \Lambda^1(\mathcal{F}_I) \otimes D(\mathcal{F}_I) \to \cdots \\ \cdots \to \Lambda^r(\mathcal{F}_I) \otimes D(\mathcal{F}_I) \xrightarrow{\partial_I} \Lambda^{r+1}(\mathcal{F}_I) \otimes D(\mathcal{F}_I) \to \cdots, \quad (6.17)$$

where $\partial_I(\Omega) = [\![U_I, \Omega]\!]^{\mathrm{fn}}$, $\Omega \in \Lambda^r(\mathcal{F}_I) \otimes D(\mathcal{F}_I)$, with corresponding cohomology modules.

Like in Chapters 4 and 5, we confine ourselves with a subtheory of this cohomological theory.

2.5. Vertical subtheory.

DEFINITION 6.2. An element $\Omega \in \Lambda^*(\mathcal{F}_I) \otimes D(\mathcal{F}_I)$ is called *vertical*, if $\mathrm{L}_\Omega(\varphi) = 0$ for any $\varphi \in \mathcal{F}_{-\infty} \subset \mathcal{F}_I = \Lambda^0(\mathcal{F}_I)$.

Denote by $D^v(\mathcal{F}_I)$ the set of all vertical vector fields from $D(\mathcal{F}_I) = \Lambda^0(\mathcal{F}_I) \otimes D(\mathcal{F}_I)$.

PROPOSITION 6.12. *Let $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$ be a $\mathcal{C}$-natural graded extension of an equation $\mathcal{E}$. Then*

(i) *The set of vertical elements in $\Lambda^r(\mathcal{F}_I) \otimes D(\mathcal{F}_I)$ coincides with the module $\Lambda^r(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$.*
(ii) *The module $\Lambda^*(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$ is closed with respect to the Frölicher–Nijenhuis bracket as well as with respect to the contraction operation:*

$$[\![\Lambda^r(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I), \Lambda^s(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)]\!]^{\mathrm{fn}} \subset \Lambda^{r+s}(F_I) \otimes D^v(\mathcal{F}_I),$$
$$\big(\Lambda^r(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)\big) \lrcorner \big(\Lambda^s(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)\big) \subset \Lambda^{r+s-1}(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I).$$

(iii) *An element $\Omega \in \Lambda^*(\mathcal{F}_I) \otimes D(\mathcal{F}_I)$ lies in $\Lambda^*(\mathcal{F}) \otimes D^v(\mathcal{F}_I)$ if and only if*

$$\mathrm{i}_\Omega(U_I) = \Omega.$$

(iv) *The structural element is vertical: $U_I \in \Lambda^1(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$.*

From the last proposition it follows that complex (6.17) can be restricted up to

$$\begin{aligned} 0 \to D^v(\mathcal{F}_I) \to \Lambda^1(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I) \to \cdots \\ \cdots \to \Lambda^r(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I) \xrightarrow{\partial_I} \Lambda^{r+1}(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I) \to \cdots \end{aligned} \tag{6.18}$$

Cohomologies

$$H_I^r(\mathcal{E}) = \frac{\ker(\partial_I \colon \Lambda^r(\mathcal{F}_I) \otimes D^v(F_I) \to \Lambda^{r+1}(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I))}{\mathrm{im}(\partial_I \colon \Lambda^{r-1}(\mathcal{F}_I) \otimes D^v(F_I) \to \Lambda^r(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I))}$$

are called $\mathcal{C}$-*cohomologies* of a graded extension. The basic properties of the differential ∂_I in (6.18) are corollaries of Propositions 6.9 and 6.12:

PROPOSITION 6.13. *Let $(\mathcal{F}_I(\mathcal{E}), \mathcal{C}D_I(\mathcal{E}))$ be a $\mathcal{C}$-natural graded extension of the equation $\mathcal{E}$ and denote by L_I the operator L_{U_I}. Then for any $\Omega, \Theta \in \Lambda^*(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$ and $\rho \in \Lambda^*(\mathcal{F}_I)$ one has*

(i) $\partial_I(\rho \wedge \Omega) = (\mathrm{L}_I(\rho) - d\rho) \wedge \Omega + (-1)^{\rho_1} \cdot \rho \wedge \partial_I\Omega,$
(ii) $[\mathrm{L}_I, \mathrm{i}_\Omega] = \mathrm{i}_{\partial_I\Omega} + (-1)^{\Omega_1}\mathrm{L}_\Omega,$
(iii) $[\mathrm{i}_\Omega, \partial_I]\Theta = (-1)^{\Omega_1}(\partial_I\Omega) \lrcorner \Theta,$
(iv) $\partial_I[\![\Omega, \Theta]\!]^{\mathrm{fn}} = [\![\partial_I\Omega, \Theta]\!]^{\mathrm{fn}} + (-1)^{\Omega_1}[\![\Omega, \partial_I\Theta]\!]^{\mathrm{fn}}.$

Let $d_h = d - \mathrm{L}_I\colon \Lambda^*(\mathcal{F}_I) \to \Lambda^*(\mathcal{F}_I)$. From (6.13) and Proposition 6.6 (ii) it follows that $d_h \circ d_h = 0$. Similar to the nongraded case, we call d_h the *horizontal differential* of the extension $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$ and denote its cohomologies by $H_h^*(\mathcal{E}; I)$.

COROLLARY 6.14. *For any $\mathcal{C}$-natural graded extension one has*

(i) *The module $H_I^*(\mathcal{E}) = \sum_{r\geq 0} H_I^r(\mathcal{E})$ is a graded $H_h^*(\mathcal{E}; I)$-module.*
(ii) *$H_I^*(\mathcal{E})$ is a graded Lie algebra with respect to the Frölicher–Nijenhuis bracket inherited from $\Lambda^*(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$.*
(iii) *$H_I^*(\mathcal{E})$ inherits from $\Lambda^*(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$ the contraction operation*
$$H_I^r(\mathcal{E}) \lrcorner H_I^s(\mathcal{E}) \subset H_I^{r+s-1}(\mathcal{E}),$$
and $H_I^*(\mathcal{E})$, with the shifted grading, is a graded Lie algebra with respect to the inherited Richardson–Nijenhuis bracket.

2.6. Symmetries and deformations. Skipping standard reasoning, we define infinitesimal symmetries of a graded extension $(\mathcal{F}_I(\mathcal{E}), \mathcal{C}D_I(\mathcal{E}))$ as
$$D_{\mathcal{C}_I}(\mathcal{E}) = \{X \in D_I(\mathcal{E}) \mid [X, \mathcal{C}D_I(\mathcal{E})] \subset \mathcal{C}D_I(\mathcal{E})\};$$
$D_{\mathcal{C}_I}(\mathcal{E})$ forms an n-graded Lie algebra while $\mathcal{C}D_I(\mathcal{E})$ is its graded ideal consisting of trivial symmetries. Thus, a Lie algebra of *nontrivial symmetries* is
$$\mathrm{sym}_I\,\mathcal{E} = D_{\mathcal{C}_I}(\mathcal{E})/\mathcal{C}D_I(\mathcal{E}).$$
If the extension at hand is $\mathcal{C}$-natural, then, due to the connection $\mathcal{C}_I$, one has the direct sum decompositions
$$D(\mathcal{F}_I) = D^v(\mathcal{F}_I) \oplus \mathcal{C}D_I(\mathcal{E}), \quad D_{\mathcal{C}_I}(\mathcal{E}) = D^v_{\mathcal{C}_I}(\mathcal{E}) \oplus \mathcal{C}D_I(\mathcal{E}), \tag{6.19}$$
where
$$D^v_{\mathcal{C}_I}(\mathcal{E}) = \{X \in D^v_I(\mathcal{E}) \mid [X, \mathcal{C}D_I(\mathcal{E})] = 0\} = D^v(\mathcal{F}_I) \cap D_{\mathcal{C}_I}(\mathcal{E}),$$
and $\mathrm{sym}_I\,\mathcal{E}$ is identified with the first summand in (6.19).

Let $\varepsilon \in \mathbb{R}$ be a small parameter and $U_I(\varepsilon) \in \Lambda^1(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$ be a smooth family such that

(i) $U_I(0) = U_I$,
(ii) $[\![U_I(\varepsilon), U_I(\varepsilon)]\!]^{\mathrm{fn}} = 0$ for all ε.

Then $U_I(\epsilon)$ is a (vertical) *deformation* of a graded extension structure, and if
$$U_I(\varepsilon) = U_I + U_I^1 \cdot \varepsilon + o(\varepsilon),$$
then U_I^1 is called (vertical) *infinitesimal deformation* of U_I. Again, skipping motivations and literally repeating corresponding proof from Chapter 5, we have the following

THEOREM 6.15. *For any $\mathcal{C}$-natural graded extension $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$ of the equation $\mathcal{E}$ one has*

(i) *$H_I^0(\mathcal{E}) = \mathrm{sym}_I(\mathcal{E})$;*

(ii) *The module $H_I^1(\mathcal{E})$ consists of the classes of nontrivial infinitesimal vertical deformations of the graded extension structure U_I.*

The following result is an immediate consequence of the results of previous subsection:

THEOREM 6.16. *Let $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$ be a graded extension. Then*

(i) *The module $H_I^1(\mathcal{E})$ is an associative algebra with respect to contraction.*

(ii) *The mapping*

$$\mathcal{R}\colon H_I^1(\mathcal{E}) \to \mathrm{End}_{\mathbb{R}}(H_I^0(\mathcal{E})),$$

where

$$\mathcal{R}_\Omega(X) = X \rfloor \Omega, \quad X \in H_I^0(\mathcal{E}),\ \Omega \in H_I^1(\mathcal{E}),$$

is a representation of this algebra. And consequently,

(iii)

$$(\mathrm{sym}_I\,\mathcal{E}) \rfloor H_I^1(\mathcal{E}) \subset \mathrm{sym}_I\,\mathcal{E}.$$

2.7. Recursion operators. The first equality in (6.19) gives us the dual decomposition

$$\Lambda^1(\mathcal{F}_I) = \mathcal{C}\Lambda^1(\mathcal{F}_I) \oplus \Lambda_h^1(\mathcal{F}_I), \tag{6.20}$$

where

$$\mathcal{C}\Lambda^1(\mathcal{F}_I) = \{\omega \in \Lambda^1(\mathcal{F}_I) \mid \mathcal{C}D_I(\mathcal{E}) \rfloor \omega = 0\},$$
$$\Lambda_h^1(\mathcal{F}_I) = \{\omega \in \Lambda^1(\mathcal{F}_I) \mid D^v(\mathcal{F}_I) \rfloor \omega = 0\}.$$

In fact, let $\omega = \sum_\alpha f_\alpha\, dg_\alpha$, $f_\alpha, g_\alpha \in \mathcal{F}_I$, be a one-form. Then, since by definition $d = d_h + \mathrm{L}_I$, one has

$$\omega = \sum_\alpha f_\alpha(d_h g_\alpha + \mathrm{L}_I(g_\alpha)).$$

Let $X \in D^v(\mathcal{F}_I)$. Then from Proposition 6.13 (ii) it follows that

$$X \rfloor \mathrm{L}_I(g) = -\mathrm{L}_I(X \rfloor g) + \partial_I(X) \rfloor g + \mathrm{L}_X(g) = X(g), \quad g \in \mathcal{F}_I.$$

Hence,

$$X \rfloor d_h g = X \rfloor (d - \mathrm{L}_I)g = X(g) - X(g) = 0.$$

On the other hand,

$$\mathrm{L}_I(g) = U_I \rfloor dg,$$

and if $Y \in \mathcal{C}D_I(\mathcal{E})$, then

$$Y \rfloor \mathrm{L}_I(g) = Y \rfloor (U_I \rfloor dg) = (Y \rfloor U_I) \rfloor dg$$

due to Proposition 6.5 (v); but $Y \rfloor U_I = 0$ for any $Y \in \mathcal{C}D_I(\mathcal{E})$.

Thus, similar to the nongraded case, one has the decomposition

$$\Lambda^r(\mathcal{F}_I) = \sum_{p+q=r} \mathcal{C}^p\Lambda(\mathcal{F}_I) \wedge \Lambda_h^q(\mathcal{F}_I), \tag{6.21}$$

where

$$\mathcal{C}^p\Lambda(\mathcal{F}_I) = \underbrace{\mathcal{C}\Lambda^1(\mathcal{F}_I) \wedge \cdots \wedge \mathcal{C}\Lambda^1(\mathcal{F}_I)}_{p \text{ times}},$$

and

$$\Lambda_h^q(\mathcal{F}_I) = \underbrace{\Lambda_h^1(\mathcal{F}_I) \wedge \cdots \wedge \Lambda_h^1}_{q \text{ times}},$$

and the wedge product $\wedge$ is taken in the graded sense (see Subsection 1.2).

REMARK 6.6. The summands in (6.21) can also be described in the following way

$$\mathcal{C}^p\Lambda(\mathcal{F}_I) \wedge \Lambda_h^q(\mathcal{F}_I) = \{\omega \in \Lambda^{p+q}(\mathcal{F}_I) \mid X_1 \lrcorner \ldots \lrcorner X_{p+1} \lrcorner \omega = 0,$$
$$Y_1 \lrcorner \ldots \lrcorner Y_{q+1} \lrcorner \omega = 0 \text{ for all } X_\alpha \in D^v(\mathcal{F}_I), Y_\beta \in \mathcal{C}D_I(\mathcal{E})\}.$$

PROPOSITION 6.17. *Let $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$ be a $\mathcal{C}$-natural extension. Then one has*

$$\partial_I(\mathcal{C}^p\Lambda(\mathcal{F}_I) \wedge \Lambda_h^q(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)) \subset \mathcal{C}^p\Lambda(\mathcal{F}_I) \wedge \Lambda_h^{q+1}(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$$

for all $p, q \geq 0$.

The proof is based on two lemmas.

LEMMA 6.18. *$d_h\mathcal{C}^1\Lambda(\mathcal{F}_I) \subset \mathcal{C}^1\Lambda(\mathcal{F}_I) \wedge \Lambda_h^1(\mathcal{F}_I)$.*

PROOF OF LEMMA 6.18. Due to Remark 6.6, it is sufficient to show that

$$X^v \lrcorner Y^v \lrcorner d_h\omega = 0, \quad X^v, Y^v \in D^v(\mathcal{F}_I), \tag{6.22}$$

and

$$X^h \lrcorner Y^h \lrcorner d_h\omega = 0, \quad X^h, Y^h \in \mathcal{C}D_I(\mathcal{E}), \tag{6.23}$$

where $\omega \in \mathcal{C}^1\Lambda(F_I)$. Obviously, we can restrict ourselves to the case $\omega = \mathrm{L}_I(g)$, $g \in \mathcal{F}_I$:

$$Y^v \lrcorner d_h\omega = Y^v \lrcorner d_h\mathrm{L}_I(g) = -Y^v \lrcorner \mathrm{L}_I d_h g$$
$$= \mathrm{L}_I(Y^v \lrcorner d_h g) + \mathrm{L}_{Y^v}(d_h g) = d_h Y^v(g).$$

Hence,

$$X^v \lrcorner Y^v \lrcorner d_h\omega = X^v \lrcorner d_h Y^v(g) = 0,$$

which proves (6.22). Now,

$$Y^h \lrcorner d_h\omega = -Y^h \lrcorner \mathrm{L}_I d_h g = Y^h \lrcorner (d(U_I \lrcorner d_h g) - U_I \lrcorner d(d_h g)).$$

But U_I is a vertical element, i.e., $U_I \in \Lambda^1(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$. Therefore,

$$U_I \lrcorner d_h g = 0$$

and

$$Y^h \lrcorner d_h\omega = -Y^h \lrcorner U_I \lrcorner d(d_h g)$$

$$= -Y^h \lrcorner U_I) \lrcorner d(d_h g) - (Y^h \wedge U_I) \lrcorner d(d_h g).$$

The first summand in the right-hand side of the last equality vanishes, since, by definition, $Y^h \lrcorner U_I = 0$ for any $Y^h \in \mathcal{C}D_I(\mathcal{E})$. Hence,

$$\begin{aligned} X^h \lrcorner Y^h \lrcorner d_h\omega &= -X^h \lrcorner (Y^h \wedge U_I) \lrcorner d(d_h g) \\ &= -(X^h \lrcorner (Y^h \wedge U_I)) \lrcorner d(d_h g) - (X^h \wedge Y^h \wedge U_I) \lrcorner d(d_h g) \\ &= -(X^h \wedge Y^h \wedge U_I) \lrcorner d(d_h g). \end{aligned}$$

But $X^h \wedge Y^h \wedge U_I$ is a (form valued) 3-vector while $d(d_h g)$ is a 2-form; hence

$$X^h \lrcorner Y^h \lrcorner d_h\omega = 0,$$

which finishes the proof of Lemma 6.18. □

LEMMA 6.19. $\partial_I D^v(\mathcal{F}_I) \subset \Lambda_h^1 \otimes D^v(\mathcal{F}_I)$.

PROOF OF LEMMA 6.19. One can easily see that it immediately follows from Proposition 6.13 (iii). □

PROOF OF PROPOSITION 6.17. The result follows from previous lemmas and Proposition 6.13 (i) which can be rewritten as

$$\partial_I(\rho \wedge \Omega) = -d_h(\rho) \wedge \Omega + (-1)^{\rho_1} \rho \wedge \partial_I(\Omega).$$

□

Taking into account the last result, one has the following decomposition

$$H_I^r(\mathcal{E}) = \sum_{p+q=r} H_I^{p,q}(\mathcal{E}),$$

where

$$H_I^{p,q}(\mathcal{E}) = \ker(\partial_I^{p,q})/\mathrm{im}(\partial_I^{p,q-1}),$$

where $\partial^{i,j}\colon \mathcal{C}^i\Lambda(\mathcal{F}_I) \wedge \Lambda_k^j(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I) \to \mathcal{C}^i(\mathcal{F}_I) \wedge \Lambda_h^{j+1}(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$.

In particular,

$$H_I^1(\mathcal{E}) = H_I^{0,1}(\mathcal{E}) \oplus H_I^{1,0}(\mathcal{E}). \tag{6.24}$$

Note now that from the point of view of $H_I^1(\mathcal{E})$-action on $H_I^0(\mathcal{E}) = \mathrm{sym}_I\,\mathcal{E}$, the first summand in (6.24) is of no interest, since

$$D^v(\mathcal{F}_I) \lrcorner \Lambda_h^1(\mathcal{F}_I) = 0.$$

We call $H_I^{*,0}(\mathcal{E})$ the Cartan part of $H_I^*(\mathcal{E})$, while the elements of $H_I^{1,0}(\mathcal{E})$ are called *recursion operators* for the extension $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$. One has the following

PROPOSITION 6.20. $H_I^{p,0}(\mathcal{E}) = \ker \partial_I^{p,0}$.

PROOF. In fact, from Proposition 6.17 one has

$$\mathrm{im}(\partial_I) \cap (\mathcal{C}^*\Lambda(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)) = 0,$$

which proves the result. □

Note that $H_I^{*,0}(\mathcal{E})$ inherits an associative graded algebra structure with respect to contraction, $H_I^{1,0}(\mathcal{E})$ being its subalgebra.

2.8. Commutativity theorem. In this subsection we prove the following

THEOREM 6.21. $[\![H_I^{1,0}(\mathcal{E}), H_I^{1,0}(\mathcal{E})]\!]^{\mathrm{fn}} \subset H_I^{2,0}(\mathcal{E})$.

The proof is based on the following

LEMMA 6.22. *For any $\omega \in \mathcal{C}^1\Lambda(\mathcal{F}_I)$ one has*

$$U_I \lrcorner\, \omega = \omega. \tag{6.25}$$

PROOF OF LEMMA 6.22. It is sufficient to prove (6.25) for the generators of the module $\mathcal{C}^1\Lambda(\mathcal{F}_I)$ which are of the form

$$\omega = \mathrm{L}_I(g), \quad g \in \mathcal{F}_I.$$

From (6.10) one has

$$\mathrm{L}_I \circ \mathrm{i}_{U_I} - \mathrm{i}_{U_I} \circ \mathrm{L}_I + L_{U_I \lrcorner U_I} = \mathrm{i}_{[\![U_I, U_I]\!]^{\mathrm{fn}}},$$

or

$$\mathrm{L}_I \circ \mathrm{i}_{U_I} - \mathrm{i}_{U_I} \circ \mathrm{L}_I + \mathrm{L}_I = 0. \tag{6.26}$$

Applying (6.26) to some $g \in \mathcal{F}_I$, one sees that

$$U_I \lrcorner\, \mathrm{L}_I(g) = \mathrm{L}_I(g).$$

□

PROOF OF THEOREM 6.21. Let $\Omega, \Theta \in H_I^{1,0}(\mathcal{E})$, i.e., $\Omega, \Theta \in \mathcal{C}^1\Lambda(\mathcal{F}_I)$ and $\partial_I \Omega = \partial_I \Theta = 0$. Then from (6.11) it follows that

$$U_I \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}} = [\![U_I \lrcorner\, \Omega, \Theta]\!]^{\mathrm{fn}} + [\![\Omega, U_I \lrcorner\, \Theta]\!]^{\mathrm{fn}},$$

or, due to Lemma 6.22,

$$U_I \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}} = 2[\![\Omega, \Theta]\!]^{\mathrm{fn}}.$$

Hence,

$$\begin{aligned}
[\![\Omega, \Theta]\!]^{\mathrm{fn}} &= \frac{1}{2} U_I \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}} = \frac{1}{4} U_I \lrcorner (U_I \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}}) \\
&= \frac{1}{4}((U_I \lrcorner U_I) \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}} - (U_I \wedge U_I) \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}}) \\
&= \frac{1}{4}(U_I \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}} - (U_I \wedge U_I) \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}}) \\
&= \frac{1}{2}[\![\Omega, \Theta]\!]^{\mathrm{fn}} - \frac{1}{4}(U_I \wedge U_I) \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}},
\end{aligned}$$

or

$$[\![\Omega, \Theta]\!]^{\mathrm{fn}} = -\frac{1}{2}(U_I \wedge U_I) \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}}.$$

But $U_I \in \mathcal{C}^1\Lambda(\mathcal{F}_I) \otimes D^v(\mathcal{F}_I)$ which finishes the proof. □

COROLLARY 6.23. *The element U_I is a unit of the associative algebra $H_I^{1,0}(\mathcal{E})$.*

PROOF. The result follows from the definition of the element U_I and from Lemma 6.22. □

COROLLARY 6.24. *Under the assumption $H_I^{2,0}(\mathcal{E}) = 0$, all recursion operators for the graded extension $(\mathcal{F}_I, \mathcal{C}D_I(\mathcal{E}))$ commute with respect to the Frölicher–Nijenhuis bracket.*

Let $\Omega \in H_I^{1,0}(\mathcal{E})$ be a recursion operator. Denote its action on $H_I^0(\mathcal{E}) = \mathrm{sym}_I(\mathcal{E})$ by $\Omega(X) = X \lrcorner \Omega$, $X \in H_I^0(\mathcal{E})$. Then, from (6.11) it follows that

$$\begin{aligned} Y \lrcorner X \lrcorner [\![\Omega, \Theta]\!]^{\mathrm{fn}} = (-1)^{X\cdot Y}\Big(&(-1)^{Y\cdot\Omega}[\Omega(X), \Theta(Y)] \\ &+ (-1)^{(Y+\Omega)\cdot\Theta}[\Theta(X), \Omega(Y)] \\ &- (-1)^{\Omega\cdot\Theta}\Omega((-1)^{Y\cdot\Theta}[\Theta(X), Y] + [X, \Theta(Y)]) \\ &- \Theta((-1)^{Y\cdot\Omega}[\Omega(X), Y] + [X, \Omega(Y)]) \\ &+ ((-1)^{\Omega\cdot\Theta}\Omega \circ \Theta + \Theta \circ \Omega)[X, Y]\Big), \end{aligned} \tag{6.27}$$

for all $X, Y \in \mathrm{sym}_I(\mathcal{E})$, $\Omega, \Theta \in H_I^{1,0}(\mathcal{E})$.

COROLLARY 6.25. *If $H_I^{2,0}(\mathcal{E}) = 0$, then for any symmetries $X, Y \in \mathrm{sym}_I(\mathcal{E})$ and recursion operators $\Omega, \Theta \in H_I^{1,0}(\mathcal{E})$ one has*

$$\begin{aligned} &(-1)^{Y\cdot\Omega}[\Omega(X), \Theta(Y)] + (-1)^{(Y+\Omega)\cdot\Theta}[\Theta(X), \Omega(Y)] \\ &\quad = (-1)^{\Omega\cdot\Theta}\Omega((-1)^{Y\cdot\Omega}[\Theta(X), Y] + [X, \Theta(Y)]) + \Theta((-1)^{Y\cdot\Omega}[\Omega(X), Y] \\ &\qquad + [X, \Omega(Y)]) + ((-1)^{\Omega\circ\Theta}\Omega \circ \Theta + \Theta \circ \Omega)[X, Y]. \end{aligned} \tag{6.28}$$

In particular,

$$\begin{aligned} (1 + (-1)^{\Omega\cdot\Omega})\Big(&(-1)^{Y\cdot\Omega}[\Omega(X), \Omega(Y)] \\ &- (-1)^{Y\cdot\Omega}\Omega[\Omega(X), Y] - \Omega[X, \Omega(Y)] + \Omega^2[X, Y]\Big) = 0, \end{aligned}$$

and if $\Omega \cdot \Omega$ is even, then

$$[\Omega(X), \Omega(Y)] = \Omega([\Omega(X), Y] + (-1)^{Y\Omega}[X, \Omega(Y)] - (-1)^{Y\Omega}\Omega[X, Y]). \tag{6.29}$$

Using Corollary 6.25, one can describe a Lie algebra structure of $\mathrm{sym}_I \mathcal{E}$ in a way similar to Section 3 of Chapter 4.

3. Nonlocal theory and the case of evolution equations

Here we extend the theory of coverings and that of nonlocal symmetries (see Chapter 3 to the case of graded equations (cf. [**87**]). We confine ourselves to evolution equations though the results obtained, at least partially, are applicable to more general cases. For any graded equation the notion of its tangent covering (an add analog of the Cartan covering, see Example 3.2 on p. 100) is introduced which reduces computation of recursion operators to computations of special nonlocal symmetries. In this setting, we also solve the problem of extending "shadows" of recursion operators up to real ones.

3.1. The GDE(M) category. Let M be a smooth manifold and $A = C^\infty(M)$. We define the GDE(M) *category* of graded differential equations over M as follows. The objects of GDE(M) are pairs $(\mathcal{F}, \nabla_{\mathcal{F}})$, where $\mathcal{F}$ is a commutative n-graded A-algebra (the case $n = \infty$ is included) endowed with a filtration

$$A = \mathcal{F}_{-\infty} \subset \ldots \subset \mathcal{F}_i \subset \mathcal{F}_{i+1} \subset \ldots, \quad \bigcup_i \mathcal{F}_i = \mathcal{F}, \tag{6.30}$$

while $\nabla_{\mathcal{F}}$ is a flat $(A, \mathcal{F})$-connection (see Subsection 2.2), i.e.,

(i) $\nabla_{\mathcal{F}} \in \hom_{\mathcal{F}}(D(A, \mathcal{F}), D(\mathcal{F}))$,
(ii) $\nabla_{\mathcal{F}}(X)(a) = X(a)$, $X \in D(A, \mathcal{F})$, $a \in A$,
(iii) $[\nabla_{\mathcal{F}}(X), \nabla_{\mathcal{F}}(Y)] = \nabla_{\mathcal{F}}(\nabla_{\mathcal{F}}(X) \circ Y - \nabla_{\mathcal{F}}(Y) \circ X)$, $X, Y \in D(A, \mathcal{F})$.

From the definition it follows that the grading of $\nabla_{\mathcal{F}}$ is $\mathbf{0}$, and we also suppose that for any $X \in \mathcal{D}(A, \mathcal{F})$ the derivation $\nabla_{\mathcal{F}}(X)$ agrees with the filtration (6.30), i.e.,

$$\nabla_{\mathcal{F}}(X)(\mathcal{F}_i) \subset \mathcal{F}_{i+s}$$

for some $s = s(X)$ and all i large enough.

Let $(\mathcal{F}, \nabla_{\mathcal{F}})$ and $(\mathcal{G}, \nabla_{\mathcal{G}})$ be two objects and $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a graded filtered homomorphism. Then for any $X \in D(A, \mathcal{F})$ the composition $\varphi \circ X$ lies in $D(A, \mathcal{G})$. We say that it is a morphism of the object $(\mathcal{F}, \nabla_{\mathcal{F}})$ to $(\mathcal{G}, \nabla_{\mathcal{G}})$ if the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\nabla_{\mathcal{F}}(X)"'] & \mathcal{G} \arrow[d, "\nabla_{\mathcal{G}}(\varphi \circ X)"] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

is commutative for all $X \in D(A, \mathcal{F})$. If φ is a monomorphism, we say that it represents a *covering* of $(\mathcal{G}, \nabla_{\mathcal{G}})$ over $(\mathcal{F}, \nabla_{\mathcal{F}})$.

REMARK 6.7. Let $\mathcal{E}$ be an equation in some bundle over M. Then all graded extensions of $\mathcal{E}$ are obviously objects of GDE(M).

REMARK 6.8. The theory of the previous section can be literally applied to the objects of GDE(M) as well.

3.2. Local representation. In what follows, we shall deal with the following kinds of objects of the category GDE(M):

(i) infinite prolongations of differential equations;
(ii) their graded extensions;

(iii) coverings over (i) and (ii).

For particular applications local versions of these objects will be considered. It means the following:

(i) In a neighborhood $\mathcal{O} \subset M$ local coordinates $x = (x_1, \dots, x_n)$ are chosen (independent variables);
(ii) the bundle $\pi\colon E \to M$ in which $\mathcal{E}$ is defined is supposed to be a vector bundle, and it trivializes over $\mathcal{O}$. If $(e_1, \dots, e_m)$ is a basis of local sections of π over $\mathcal{O}$, then $f = u^1 e_1 + \dots + u^m e_m$ for any $f \in \Gamma(\pi|_{\mathcal{O}})$, and $u^1, \dots, u^m$ play the role of dependent variables for the equation $\mathcal{E}$;
(iii) the equation $\mathcal{E}$ is represented by a system of relations

$$\begin{cases} F_1(x, \dots, u^j_\sigma, \dots) = 0, \\ \dots\dots\dots\dots\dots\dots\dots \\ F_1(x, \dots, u^j_\sigma, \dots) = 0, \end{cases}$$

where $u^j_\sigma = \partial^{|\sigma|} u^j / \partial x_\sigma$, $\sigma = (i_1, \dots, i_n)$, $|\sigma| = i_1 + \dots + i_n \le k$, are coordinates in the manifold of k-jets $J^k(\pi)$, k being the order of $\mathcal{E}$;
(iv) a graded extension $\mathcal{F}$ of $\mathcal{F}(\mathcal{E})$ (see Subsection 2.3 is freely generated over $\mathcal{F}(\mathcal{E})$ by homogeneous elements $v^1, v^2, \dots$. It means that $\mathcal{F}_\infty$ is generated by v^j_τ, where $v^j_{\mathbf{0}} = v^j$ and

$$v^j_{(i_1, \dots, i_s + 1, \dots, i_n)} = [D_s, v^j_{(i_1, \dots, i_n)}],$$

D_s being the total derivative on $\mathcal{E}^\infty$ corresponding to $\partial/\partial x_s$. In this setting any graded extension of $\mathcal{E}$ can be represented as

$$\begin{cases} F_1(x, \dots, u^j_\sigma, \dots) + \phi_1(x, \dots, u^j_\sigma, \dots, v^j_\tau, \dots) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_r(x, \dots, u^j_\sigma, \dots) + \phi_r(x, \dots, u^j_\sigma, \dots, v^j_\tau, \dots) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \phi_{r+1}(x, \dots, u^j_\sigma, \dots v^j_\tau, \dots) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \phi_{r+l}(x, \dots, u^j_\sigma, \dots v^j_\tau, \dots) = 0, \end{cases}$$

where $\phi_1, \dots, \phi_r$ are functions such that $\phi_1 = 0, \dots, \phi_r = 0$ for $v^j_\tau = 0$.
(v) for any covering $\varphi\colon \mathcal{F} \to \mathcal{G}$ of the graded extension $\mathcal{F}$ by an object $(\mathcal{G}, \nabla_{\mathcal{G}})$ we assume that $\mathcal{G}$ is freely generated over $\mathcal{F}$ by homogeneous elements $w^1, w^2, \dots$ and

$$\nabla_{\mathcal{G}}\left(\frac{\partial}{\partial x_i}\right) = D_i + \sum_s X^s_i \frac{\partial}{\partial w^s} \stackrel{\text{def}}{=} \tilde{D}_i$$

with

$$[\tilde{D}_i, \tilde{D}_j] = [D_i, \sum_s X^s_j \frac{\partial}{\partial w^s}] + [\sum_s X^s_i \frac{\partial}{\partial w^s}, D_j]$$

$$+ [\sum_s X_i^s \frac{\partial}{\partial w^s}, \sum_s X_j^s \frac{\partial}{\partial w^s}] = 0,$$

where $i, j = 1, \dots, n$, $X_i^s \in \mathcal{G}$, and $D_1, \dots, D_n$ are total derivatives extended onto $\mathcal{F}$: $D_i = \nabla_{\mathcal{F}}(\partial/\partial x_i)$. Elements $w^1, w^2, \dots$ are called *nonlocal variables* related to the covering φ, the number of nonlocal variables being called the dimension of φ.

3.3. Evolution equations. Below we deal with super ($\mathbb{Z}_2$-graded) evolution equations $\mathcal{E}$ in two independent variables x and t:

$$\begin{cases} u_t^1 = f^1(x, t, u^1, \dots, u^m, \dots, u_k^1, \dots, u_k^m), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ u_t^m = f^m(x, t, u^1, \dots, u^m, \dots, u_k^1, \dots, u_k^m), \end{cases} \tag{6.31}$$

where $u^1, \dots, u^m$ are either of even or of odd grading, and u_s^j denotes $\partial u^j / \partial x^s$. We take $x, t, u_0^1, \dots, u_0^m, \dots, u_i^1, \dots u_i^j, \dots$ for the internal coordinates on $\mathcal{E}^\infty$. The total derivatives D_x and D_t restricted onto the infinite prolongation of (6.31) are of the form

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} \sum_{j=1}^{m} u_{i+1}^j \frac{\partial}{\partial u_i^j}, \\ D_t &= \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} \sum_{j=1}^{m} D_x^i(f^j) \frac{\partial}{\partial u_i^j}. \end{aligned} \tag{6.32}$$

In the chosen local coordinates, the structural element $U = U_{\mathcal{E}}$ of the equation $\mathcal{E}$ is represented as

$$U = \sum_{i=0}^{\infty} \sum_{j=1}^{m} (du_i^j - u_{i+1}^j \, dx - D^i(f^j) \, dt) \otimes \frac{\partial}{\partial u_i^j}. \tag{6.33}$$

Then for a basis of the module $\mathcal{C}^1\Lambda(\mathcal{E})$ one can choose the forms

$$\omega_i^j = \mathrm{L}_U(u_i^j) = du_i^j - u_{i+1}^j \, dx - D^i(f^j) \, dt,$$

while (6.33) is rewritten as

$$U = \sum_{i=0}^{\infty} \sum_{j=1}^{m} \omega_i^j \otimes \frac{\partial}{\partial u_i^j}. \tag{6.34}$$

Let $\Theta = \sum_{i=0}^{\infty} \sum_{j=1}^{m} \theta_i^j \otimes \partial/\partial u_i^j \in \Lambda^p(\mathcal{E}) \otimes D^v(\mathcal{E})$. Then from (6.6) one has

$$\begin{aligned} \partial_{\mathcal{E}}(\Theta) = [\![U, \Theta]\!]^{\mathrm{fn}} = \sum_{i=0}^{\infty} \sum_{j=1}^{m} \Big(& dx \wedge (\theta_{i+1}^j - D_x(\theta_i^j)) \\ & + dt \wedge \big(\sum_{\beta=0}^{\infty} \sum_{\alpha=1}^{m} \theta_\beta^\alpha \frac{\partial D_x^i f^j}{\partial u_\beta^\alpha} - D_t(\theta_i^j) \big) \Big) \otimes \frac{\partial}{\partial u_i^j}. \end{aligned} \tag{6.35}$$

From (6.35) one easily gets the following

THEOREM 6.26. *Let $\mathcal{E}$ be an equation of the form* (6.31). *Then $H_{\mathcal{C}}^{p,0}(\mathcal{E})$ consists of the elements*

$$\ni_\theta = \sum_{i=0}^{\infty}\sum_{j=1}^{m} D_x^i(\theta^j) \otimes \frac{\partial}{\partial u_i^j},$$

where $\theta = (\theta^1, \dots, \theta^m)$, $\theta^j \in \mathcal{C}^p\Lambda(\mathcal{E})$, is a vector-valued form satisfying the equations

$$\sum_{i=0}^{k}\sum_{j=1}^{m} D_x^i(\theta^j)\frac{\partial f^l}{\partial u_i^j} = 0, \quad l = 1, \dots, m, \tag{6.36}$$

or in short,

$$H_{\mathcal{C}}^{p,0}(\mathcal{E}) = \ker \ell_{\mathcal{E}}^{(p)},$$

where $\ell_{\mathcal{E}}^{(p)}$ is the extension of the operator of universal linearization operator onto the module $\mathcal{C}^p\Lambda(\mathcal{E}) \otimes_{\mathbb{R}} \mathbb{R}^m$:

$$(\ell_{\mathcal{E}}^{(p)}(\theta))^l = \sum_{i=0}^{k}\sum_{j=1}^{m} D_x^i(\theta^j)\frac{\partial f^l}{\partial u_i^j}, \quad l = 1, \dots, m. \tag{6.37}$$

3.4. Nonlocal setting and shadows. Let now φ be a covering of equation (6.31) determined by nonlocal variables $w^1, w^2, \dots$ with the extended total derivatives of the form

$$\tilde{D}_x = D_x + \sum_s X_s \frac{\partial}{\partial w_s},$$
$$\tilde{D}_t = D_t + \sum_s T_s \frac{\partial}{\partial w_s}, \tag{6.38}$$

satisfying the identity

$$[\tilde{D}_x, \tilde{D}_t] = 0. \tag{6.39}$$

Denote by $\mathcal{F}(\mathcal{E}_\varphi)$ the corresponding algebra of functions and by $\Lambda^*(\mathcal{E}_\varphi)$ and $D(\mathcal{E}_\varphi)$ the modules of differential forms and vector fields on $\mathcal{F}(\mathcal{E}_\varphi)$ respectively. Then the structural element of the covering object is

$$U_\varphi = U + \sum_s (d\omega_s - X_s\,dx - T_s\,dt) \otimes \frac{\partial}{\partial w_s}$$

and the identity

$$[\![U_\varphi, U_\varphi]\!]^{\mathrm{fn}} = 0$$

is fulfilled due to (6.39).

If now

$$\Theta = \sum_{i=0}^{\infty}\sum_{j=1}^{m} \theta_i^j \otimes \frac{\partial}{\partial u_i^j} + \sum_s \rho_s \otimes \frac{\partial}{\partial w_s}$$

is an element of the module $\Lambda^p(\mathcal{E}_\varphi) \otimes D^v(\mathcal{E}_\varphi)$, then one can easily see that

$$\begin{aligned}\partial_\varphi(\Theta) = [\![U_\varphi, \Theta]\!]^{\mathrm{fn}} = \sum_{i=0}^{\infty}\sum_{j=1}^{m}\Big(dx \wedge \big(\theta_{i+1}^j - \tilde{D}_x(\theta_i^j)\big)\\
+ dt \wedge \big(\sum_{\beta=0}^{\infty}\sum_{\alpha=1}^{m}\theta_\beta^\alpha \frac{\partial D_x^i f^j}{\partial u_\beta^\alpha} - \tilde{D}_t(\theta_i^j)\big)\Big) \otimes \frac{\partial}{\partial u_i^j}\\
+ \sum_s \Big(dx \wedge \big(\sum_{\beta=0}^{\infty}\sum_{\alpha=1}^{m}\theta_\beta^\alpha \frac{\partial X_s}{\partial u_\beta^\alpha} + \sum_\gamma \rho_\gamma \frac{\partial X_s}{\partial w_\gamma} - \tilde{D}_x(\rho_s)\big)\\
+ dt \wedge \big(\sum_{\beta=0}^{\infty}\sum_{\alpha=1}^{m}\theta_\beta^\alpha \frac{\partial X_s}{\partial u_\beta^\alpha} + \sum_\gamma \rho_\gamma \frac{\partial T_s}{\partial w_\gamma} - \tilde{D}_t(\rho_s)\big)\Big) \otimes \frac{\partial}{\partial w_s}.\end{aligned} \tag{6.40}$$

Again, confining oneself to the case $\Theta \in \mathcal{C}^p\Lambda(\mathcal{E}_\varphi) \otimes D^v(\mathcal{E}_\varphi)$, one gets the following

THEOREM 6.27. *Let $\mathcal{E}$ be an equation of the form* (6.31) *and φ be its covering with nonlocal variables $w_1, w_2, \dots$ and extended total derivatives given by* (6.38). *Then the module $H_{\mathcal{C}}^{p,0}(\mathcal{E}_\varphi)$ consists of the elements*

$$\Game_{\theta,\rho} = \sum_{i=0}^{\infty}\sum_{j=1}^{m}\tilde{D}_x^i(\theta^j) \otimes \frac{\partial}{\partial u_i^j} + \sum_s \rho_s \otimes \frac{\partial}{\partial w_s}, \tag{6.41}$$

where $\theta = (\theta^1, \dots, \theta^m)$ and $\rho = (\rho_1, \dots, \rho_s, \dots)$, $\theta^j, \rho_s \in \mathcal{C}^p\Lambda(\mathcal{E}_\varphi)$, are vector-valued forms satisfying the equations

$$\tilde{\ell}_{\mathcal{E}}^{(p)}(\theta) = 0, \tag{6.42}$$

and

$$\begin{aligned}\sum_{\beta=0}^{\infty}\sum_{\alpha=1}^{m}\tilde{D}_x^\beta(\theta^\alpha)\frac{\partial X_s}{\partial u_\beta^\alpha} + \sum_j \rho_j \frac{\partial X_s}{\partial w_j} = \tilde{D}_x(\rho_s),\\
\sum_{\beta=0}^{\infty}\sum_{\alpha=1}^{m}\tilde{D}_x^\beta(\theta^\alpha)\frac{\partial T_s}{\partial u_\beta^\alpha} + \sum_j \rho_j \frac{\partial T_s}{\partial w_j} = \tilde{D}_t(\rho_s),\end{aligned} \tag{6.43}$$

$s = 1, 2, \dots$, where $\tilde{\ell}_{\mathcal{E}}^{(p)}$ is the natural extension of $\ell_{\mathcal{E}}^{(p)}$ with D_x and D_t replaced by $\tilde{D}_x$ and $\tilde{D}_t$ in (6.37).

Similar to Chapter 5, we call (6.42) *shadow equations* and (6.42) *relation equations* for the element (θ, ρ); solutions of (6.42) are called *shadow solutions*, or simply *shadows*. Our main concern lies in reconstruction elements of the module $H_{\mathcal{C}}^{p,0}(\mathcal{E}_\varphi)$ from their shadows. Denote the set of such shadows by $SH_{\mathcal{C}}^{p,0}(\mathcal{E}_\varphi)$.

REMARK 6.9. Let φ be a covering. Consider horizontal one-forms

$$\omega_\varphi^s = d_h w_s = X_s\, dx + T_s\, dt, \quad s = 1, 2, \dots,$$

where d_h is the horizontal de Rham differential associated to φ. Then (6.42) can be rewritten as

$$\ni_{\theta,\rho}(\omega_\varphi^s) = d_h\rho_s, \quad s = 1, 2, \dots \tag{6.44}$$

REMARK 6.10. When X_s and T_s do not depend on nonlocal variables, the conditions of φ being a covering is equivalent to

$$d_h\omega_\varphi^s = 0, \quad s = 1, 2, \dots,$$

d_h being the horizontal differential on $\mathcal{E}$. In particular, one-dimensional coverings are identified with elements of $\ker(d_h)$. We say a one-dimensional covering φ to be *trivial* if corresponding form ω_φ is exact (for motivations see Chapter 3). Thus, the set of classes of nontrivial one-dimensional coverings φ with ω_φ independent of nonlocal variables is identified with the cohomology group $H_h^1(\mathcal{E})$, or with the group of nontrivial conservation laws for $\mathcal{E}$.

3.5. The functors K and T. Keeping in mind the problem of reconstructing recursion operators from their shadows, we introduce two functors in the category $\mathrm{GDE}(M)$. One of them is known from the classical (non-graded) theory (cf Chapter 3), the other is specific to graded equations and is a super counterpart of the Cartan even covering constructed in Chapter 3 (see also [**97**]).

Let $(\mathcal{F}, \nabla_{\mathcal{F}})$ be an object of the category $\mathrm{GDE}(M)$ and $H_h^1(\mathcal{F})$ be the $\mathbb{R}$-module of its first horizontal cohomology. Let $\{w_\alpha\}$ be a set of generators for $H_h^1(\mathcal{F})$, each w_α being the cohomology class of a form $\omega_\alpha \in \Lambda_h^1(\mathcal{F})$, $\omega_\alpha = \sum_{i=1}^m X_\alpha^i\,dx_i$. We define the functor $K\colon \mathrm{GDE}(M) \Rightarrow \mathrm{GDE}(M)$ of *killing* $H_h^1(\mathcal{F})$ as follows.

The algebra $K\mathcal{F}$ is a graded commutative algebra freely generated by $\{w_\alpha\}$ over $\mathcal{F}$ with $\mathrm{gr}(w_\alpha) = \mathrm{gr}(X_\alpha^i)$. The connection $\nabla_{K\mathcal{F}}$ looks as

$$\nabla_{K\mathcal{F}}\left(\frac{\partial}{\partial x_i}\right) = \nabla_{\mathcal{F}}\left(\frac{\partial}{\partial x_i}\right) + \sum_\alpha X_\alpha^i \frac{\partial}{\partial w_\alpha}.$$

From the fact that H_h^1 is a covariant functor from $\mathrm{GDE}(M)$ into the category of $\mathbb{R}$-modules it easily follows that K is a functor as well.

To define the functor $T\colon \mathrm{GDE}(M) \Rightarrow \mathrm{GDE}(M)$, let us set $T\mathcal{F} = \mathcal{C}^*\Lambda(\mathcal{F})$, where $\mathcal{C}^*\Lambda^*(\mathcal{F}) = \sum_{p\geq 0}\mathcal{C}^p\Lambda(\mathcal{F})$ is the module of all Cartan forms on $\mathcal{F}$ (see Subsection 2.7). If $\mathcal{F}$ is n-graded, then $T\mathcal{F}$ carries an obvious structure of $(n+1)$-graded algebra. The action of vector fields $\nabla_{\mathcal{F}}(X)$, $X \in D(M)$, on $\Lambda^*(\mathcal{F})$ by Lie derivatives preserves the submodule $\mathcal{C}^*\Lambda(\mathcal{F})$. Since $\mathcal{C}^*\Lambda(\mathcal{F})$, as a graded algebra, is generated by the elements χ and $d_{\mathcal{C}}\psi$, $\chi, \psi \in \mathcal{F}$, this action can be written down as

$$\begin{aligned}
\mathrm{L}_{\nabla_{\mathcal{F}}(X)}\chi &= \nabla_{\mathcal{F}}(X)\chi,\\
\mathrm{L}_{\nabla_{\mathcal{F}}(X)}d_{\mathcal{C}}\psi &= d_{\mathcal{C}}\nabla_{\mathcal{F}}(X)(\psi),\\
\mathrm{L}_{\nabla_{\mathcal{F}}(X)}(\chi d_{\mathcal{C}}\psi) &= \mathrm{L}_{\nabla_{\mathcal{F}}}(\chi)\cdot d_{\mathcal{C}}\psi + \chi\mathrm{L}_{\nabla_{\mathcal{F}}(X)}d_{\mathcal{C}}\psi.
\end{aligned}$$

Moreover, for any $X \in D(M)$ and $\omega \in \mathcal{C}^*\Lambda(\mathcal{F})$ one has

$$\nabla_{\mathcal{F}}(X) \lrcorner\, \omega = 0;$$

hence, for any $\theta \in \mathcal{C}^*\Lambda(\mathcal{F})$

$$(\theta \wedge \mathrm{L}_{\nabla_{\mathcal{F}}(X)})(\omega) \\ = \theta \wedge \mathrm{L}_{\nabla_{\mathcal{F}}(X)}(\omega) + (-1)^{\theta_1} d\theta \wedge (\nabla_{\mathcal{F}}(X) \lrcorner\, \omega) = \theta \wedge \mathrm{L}_{\nabla_{\mathcal{F}}(X)},(\omega),$$

which means that we have a natural extension of the connection $\nabla_{\mathcal{F}}$ in $\mathcal{F}$ up to a connection $\nabla_{T\mathcal{F}}$ in $T\mathcal{F}$. It is easy to see that the correspondence $T\colon (\mathcal{F}, \nabla_{\mathcal{F}}) \Rightarrow (T\mathcal{F}, \nabla_{T\mathcal{F}})$ is functorial. We call $(T\mathcal{F}, \Delta_{T\mathcal{F}})$ the (*odd*) *Cartan covering* of $(\mathcal{F}, \nabla_{\mathcal{F}})$.

In the case when $(\mathcal{F}, \nabla_{\mathcal{F}})$ is an evolution equation $\mathcal{E}$ of the form (6.31), $T(\mathcal{F}, \nabla_{\mathcal{F}})$ is again an evolution equation $T\mathcal{E}$ with additional dependent variables $v^1, \dots, v^m$ and additional relations

$$\begin{cases} v_t^1 = \sum_{i,j} \dfrac{\partial f^1}{\partial u_i^j} v_i^j, \\ \dots\dots\dots\dots\dots \\ v_t^m = \sum_{i,j} \dfrac{\partial f^m}{\partial u_i^j} v_i^j. \end{cases} \tag{6.45}$$

Note that if a variable u^j is of grading $(i_1, \dots, i_n)$, then the grading of v^j is $(i_1, \dots, i_n, 1)$.

3.6. Reconstructing shadows. Computerized computations on nonlocal objects, such as symmetries and recursion operators, can be effectively realized for shadows of these objects (see examples below). Here we describe a setting which guarantees the existence of symmetries and, in general, elements of $H_{\mathcal{C}}^{p,0}(\mathcal{E})$ corresponding to the shadows computed. Below we still consider evolution equations only.

PROPOSITION 6.28. *Let $\mathcal{E}$ be an evolution equation and φ be its covering. Let $\theta \in SH_{\mathcal{C}}^{p,0}(\mathcal{E}_\varphi)$. Then, if the coefficients X_s and T_s for the extensions of total derivatives do not depend on nonlocal variables for all s, then*

(i) *for any extension $\partial_{\theta,\rho}$ of θ up to a vector field on $\mathcal{E}_\varphi$ the forms*

$$\partial_{\theta,\rho}(\omega_\varphi^s) \overset{\text{def}}{=} \Omega^s$$

(see Remark 6.9 in Subsection 3.4) are d_h-closed on $\mathcal{E}_\varphi$;

(ii) *the element θ is extendable up to an element of $H_{\mathcal{C}}^{p,0}(\mathcal{E}_\varphi)$ if and only if all Ω^s are d_h-exact forms.*

PROOF. To prove the first statement, note that using Proposition 6.13 (i) one has

$$0 = \partial_\varphi^2(\partial_{\theta,\rho}) = \partial_\varphi \left(\sum_s (\partial_{\theta,\rho}(\omega_\varphi^s) + d_h\rho_s) \otimes \frac{\partial}{\partial \omega_s} \right)$$

$$= -\sum_s d_h\Omega^s \otimes \frac{\partial}{\partial \omega_s}. \quad (6.46)$$

The second statement immediately follows from (6.43). $\square$

REMARK 6.11. If X_s, T_s depend on $w_1, w_2, \dots$, then (6.46) transforms into

$$\sum_s \left(d_h\Omega^s - (\Omega^s + d_h\rho_s) \wedge \left(\frac{\partial X_s}{\partial w_s} dx + \frac{\partial T_s}{\partial w_s} dt \right) \right) \otimes \frac{\partial}{\partial w_s} = 0. \quad (6.47)$$

Let now $\theta \in SH_{\mathcal{C}}^{p,0}(\mathcal{E}_\varphi)$ and $\Phi \in H_{\mathcal{C}}^{q,0}(\mathcal{E}_\varphi)$. Then from Proposition 6.13 (iii) it follows that

$$[\mathrm{i}_\Phi, \partial_\varphi]\theta = (-1)^q(\partial_\varphi\Phi) \mathbin{\lrcorner} \theta = 0.$$

Hence, since by the definition of shadows $\partial_\varphi\theta$ is a φ-vertical element, $\mathrm{i}_\Phi\partial_\varphi\theta$ is vertical too. It means that $\partial_\varphi\mathrm{i}_\Phi\theta$ is a φ-vertical element, i.e., $\mathrm{i}_\varphi\theta \in SH_{\mathcal{C}}^{p+q-1}(\mathcal{E}_\varphi)$. It proves the following result (cf. similar results of Chapter 5):

PROPOSITION 6.29. *For any $\theta \in SH_{\mathcal{C}}^{p,0}(\mathcal{E}_\varphi)$ and $\Phi \in H_{\mathcal{C}}^{q,0}(\mathcal{E}_\varphi)$ the element $\Phi \mathbin{\lrcorner} \theta$ lies in $SH_{\mathcal{C}}^{p+q-1}(\mathcal{E}_\varphi)$. In particular, when applying a shadow of a recursion operator to a symmetry, one gets a shadow of a symmetry.*

The next result follows directly from the previous ones.

THEOREM 6.30. *Let $\mathcal{E}$ be an evolution equation of the form* (6.31) *and $\mathcal{E}_\varphi$ be its covering constructed by infinite application of the functor K: $\mathcal{E}_\varphi = K^{(\infty)}\mathcal{E}$, where*

$$K^{(\infty)}\mathcal{E} = \operatorname*{inj\,lim}_{n\to\infty}(K^n\mathcal{E}), \quad K^n\mathcal{E} = \underbrace{(K \circ \dots \circ K)}_{n \text{ times}}\mathcal{E}.$$

Then for any shadow $\mathcal{R}$ of a recursion operator in $\mathcal{E}_\varphi$ and a symmetry $\Phi \in \operatorname{sym}\mathcal{E}_\varphi$ the shadow $\mathcal{R}(\Phi)$ can be extended up to a symmetry of $\mathcal{E}_\varphi$. Thus, an action of $SH_{\mathcal{C}}^{1,0}(\mathcal{E}_\varphi)$ on $\operatorname{sym}(\mathcal{E}_\varphi)$ is defined modulo "shadowless" symmetries.

To be sure that elements of $SH_{\mathcal{C}}^{1,0}(\mathcal{E}_\varphi)$ can be extended up to recursion operators in an appropriate setting, we prove the following two results.

PROPOSITION 6.31. *Let $\mathcal{E}$ be an equation and $\mathcal{E}_\varphi$ be its covering by means of $T\mathcal{E}$. Then there exists a natural embedding*

$$T_{\mathrm{sym}} \colon H_{\mathcal{C}}^{*,0}(\mathcal{E}) \to \operatorname{sym}(T\mathcal{E})$$

of graded Lie algebras.

PROOF. Let $\Phi \in H_{\mathcal{C}}^{*,0}(\mathcal{E})$. Then L_Φ acts on $\Lambda^*(\mathcal{E})$ and this action preserves the submodule $\mathcal{C}^*\Lambda(\mathcal{E}) \subset \Lambda^*(\mathcal{E})$, since

$$[\mathrm{L}_\Phi, d_{\mathcal{C}}] = \mathrm{L}_{[\![\Phi, U_{\mathcal{E}}]\!]^{\mathrm{fn}}} = 0.$$

Let $X \in CD(\mathcal{E})$. Then, due to (6.11), $[\![X,\Phi]\!]^{\mathrm{fn}} \lrcorner U_{\mathcal{E}} = 0$. But, using (6.11) again, one can see that $[\![X,\Phi]\!]^{\mathrm{fn}}$ is a vertical element. Hence,

$$[\![X,\Phi]\!]^{\mathrm{fn}} \lrcorner U_{\mathcal{E}} = [\![X,\Phi]\!]^{\mathrm{fn}} = 0.$$

□

Proposition 6.31 allows one to compute elements of $H_{\mathcal{C}}^{*,0}(\mathcal{E})$ as nonlocal symmetries in $\mathcal{E}_\varphi = T\mathcal{E}$. This is the base of computational technology used in applications below.

The last result of this subsection follows from the previous ones.

THEOREM 6.32. *Let $\mathcal{E}$ be an evolution equation and $\mathcal{E}_\varphi$ be its covering constructed by infinite application of the functor $K \circ T$. Then any shadow $\Phi \in SH_{\mathcal{C}}^{*,0}(\mathcal{E}_\varphi)$ can be extended up to an element of $H_{\mathcal{C}}^{*,0}(\mathcal{E}_\varphi)$. In particular, to any shadow $SH_{\mathcal{C}}^{1,0}(\mathcal{E}_\varphi)$ a recursion operator corresponds in $\mathcal{E}_\varphi$.*

REMARK 6.12. For "fine obstructions" to shadows reconstruction one should use corresponding term of A.M. Vinogradov's $\mathcal{C}$-spectral sequence ([**102**], cf. [**58**]).

4. The Kupershmidt super KdV equation

As a first application of the graded calculus for symmetries of graded partial differential equations we discuss the symmetry structure of the so-called Kupershmidt super KdV equation, which is an extension of the classical KdV equation to the graded setting [**24**].

At this point we have already to make a remark. The equation under consideration will be a super equation but not a supersymmetric equation in the sense of Mathieu, Manin–Radul, where a supersymmetric equation is an equation admitting and odd, or supersymmetry [**74**], [**72**]. The super KdV equation is given as the following system of graded partial differential equations $\mathcal{E}$ for an *even* function u and an *odd* function φ in $J^3(\pi;\varphi)$, where $J^3(\pi;\varphi)$ is the space $J^3(\pi)$ for the bundle $\pi\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$, $(u,x,t)\mapsto(x,t)$, extended by the odd variable φ:

$$\begin{aligned} u_t &= 6uu_x - u_{xxx} + 3\varphi\varphi_{xx}, \\ \varphi_t &= 3u_x\varphi + 6u\varphi_x - 4\varphi_{xxx}, \end{aligned} \tag{6.48}$$

where subscripts denote partial derivatives with respect to x and t. As usual, t is the time variable and x is the space variable. Here u, x, t, u, u_x, u_t, u_{xx}, u_{xxx} are even (commuting) variables, while φ, φ_x, φ_{xx}, φ_{xxx} are odd (anticommuting) variables. In the sequel we shall often use the term "graded" instead of "super".

We introduce the total derivative operators D_x and D_t on the space $J^\infty(\pi;\varphi)$, by

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + \varphi_x\frac{\partial}{\partial \varphi} + u_{xx}\frac{\partial}{\partial u_x} + \varphi_{xx}\frac{\partial}{\partial \varphi_x} + \cdots,$$

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + \varphi_t\frac{\partial}{\partial \varphi} + u_{tx}\frac{\partial}{\partial u_x} + \varphi_{tx}\frac{\partial}{\partial \varphi_x} + \cdots \tag{6.49}$$

The infinite prolongation $\mathcal{E}^\infty$ is the submanifold of $J^\infty(\pi;\varphi)$ defined by the graded system of partial differential equations

$$\begin{aligned} &D_x^n D_t^m(u_t - 6uu_x + u_{xxx} - 3\varphi\varphi_{xx}) = 0,\\ &D_x^n D_t^m(\varphi_t - 3u_x\varphi - 6u\varphi_x + 4\varphi_{xxx}) = 0, \end{aligned}\tag{6.50}$$

where $n, m \in \mathbb{N}$.

We choose internal coordinates on $\mathcal{E}^\infty$ as $x, t, u, \varphi, u_1, \varphi_1, \ldots$, where we introduced a further notation

$$u_x = u_1, \quad \varphi_x = \varphi_1, \quad u_{xx} = u_2, \quad \varphi_{xx} = \varphi_2, \ldots \tag{6.51}$$

The restriction of the total derivative operators D_x and D_t to $\mathcal{E}^\infty$, again denoted by the same symbols, are then given by

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + \sum_{n\geq 0}(u_{n+1}\frac{\partial}{\partial u_n} + \varphi_{n+1}\frac{\partial}{\partial \varphi_n}),\\ D_t &= \frac{\partial}{\partial t} + \sum_{n\geq 0}((u_n)_t\frac{\partial}{\partial u_n} + (\varphi_n)_t\frac{\partial}{\partial \varphi_n}). \end{aligned}\tag{6.52}$$

We note that (6.48) admits a scaling symmetry, which leads to the introduction of a degree to each variable,

$$\begin{aligned} &\deg(x) = -1, && \deg(t) = -3,\\ &\deg(u) = 2, && \deg(u_1) = 3, \ldots,\\ &\deg(\varphi) = \frac{3}{2}, && \deg(\varphi_1) = \frac{5}{2}, \ldots \end{aligned}\tag{6.53}$$

From this we see that each term in (6.48) is of degree 5 and $4\frac{1}{2}$ respectively.

4.1. Higher symmetries. We start the discussion of searching for (higher) symmetries at the representation of vertical vector fields,

$$\partial_\Phi = \Phi^u\frac{\partial}{\partial u} + \Phi^\varphi\frac{\partial}{\partial \varphi} + \sum_{n>0}\left(D_x^n(\Phi^u)\frac{\partial}{\partial u_n} + D_x^n(\Phi^\varphi)\frac{\partial}{\partial \varphi_n}\right), \tag{6.54}$$

where $\Phi = (\Phi^u, \Phi^\varphi)$ is the generating function of the vertical vector field ∂_Φ. We restrict our search for higher symmetries to even vector fields, meaning that Φ^u is even, while Φ^φ is odd.

Moreover we restrict our search for higher symmetries to vector fields ∂_Φ whose generating function $\Phi = (\Phi^u, \Phi^\varphi)$ depends on the variables x, t, u, $\varphi, \ldots, u_5$, φ_5. These requirements lead to a representation of the function $\Phi = (\Phi^u, \Phi^\varphi)$, Φ^u, $\Phi^\varphi \in C^\infty(x, t, u, u_1, \ldots, u_5) \otimes \Lambda(\varphi, \ldots, \varphi_5)$ in the following form

$$\begin{aligned} \Phi^u = f_0 &+ f_1\varphi\varphi_1 + f_2\varphi\varphi_2 + f_3\varphi\varphi_3 + f_4\varphi\varphi_4 + f_5\varphi\varphi_5 + f_6\varphi_1\varphi_2\\ &+ f_7\varphi_1\varphi_3 + f_8\varphi_1\varphi_4 + f_9\varphi_1\varphi_5 + f_{10}\varphi_2\varphi_3 + f_{11}\varphi_2\varphi_4 + f_{12}\varphi_2\varphi_5\\ &+ f_{13}\varphi_3\varphi_4 + f_{14}\varphi_3\varphi_5 + f_{15}\varphi_4\varphi_5 + f_{16}\varphi\varphi_1\varphi_2\varphi_3 + f_{17}\varphi\varphi_1\varphi_2\varphi_4 \end{aligned}$$

$$
\begin{aligned}
&+ f_{18}\varphi\varphi_1\varphi_2\varphi_5 + f_{19}\varphi\varphi_1\varphi_3\varphi_4 + f_{20}\varphi\varphi_1\varphi_3\varphi_5 + f_{21}\varphi\varphi_1\varphi_4\varphi_5\\
&+ f_{22}\varphi\varphi_2\varphi_3\varphi_4 + f_{23}\varphi\varphi_2\varphi_3\varphi_5 + f_{24}\varphi\varphi_2\varphi_4\varphi_5 + f_{25}\varphi\varphi_3\varphi_4\varphi_5\\
&+ f_{26}\varphi_1\varphi_2\varphi_3\varphi_4 + f_{27}\varphi_1\varphi_2\varphi_3\varphi_5 + f_{28}\varphi_2\varphi_3\varphi_4\varphi_5\\
&+ f_{29}\varphi\varphi_1\varphi_2\varphi_3\varphi_4\varphi_5,
\end{aligned}
$$

$$
\begin{aligned}
\Phi^\varphi &= g_1\varphi + g_2\varphi_1 + g_3\varphi_2 + g_4\varphi_3 + g_5\varphi_4 + g_6\varphi_5\\
&+ g_7\varphi\varphi_1\varphi_2 + g_8\varphi\varphi_1\varphi_3 + g_9\varphi\varphi_1\varphi_4 + g_{10}\varphi\varphi_1\varphi_5 + g_{11}\varphi\varphi_2\varphi_3\\
&+ g_{12}\varphi\varphi_2\varphi_4 + g_{13}\varphi\varphi_2\varphi_5 + g_{14}\varphi\varphi_3\varphi_4 + g_{15}\varphi\varphi_3\varphi_5 + g_{16}\varphi\varphi_4\varphi_5\\
&+ g_{17}\varphi_1\varphi_2\varphi_3 + g_{18}\varphi_1\varphi_2\varphi_4 + g_{19}\varphi_1\varphi_2\varphi_5 + g_{20}\varphi_1\varphi_3\varphi_4 + g_{21}\varphi_1\varphi_3\varphi_5\\
&+ g_{22}\varphi_1\varphi_4\varphi_5 + g_{23}\varphi_2\varphi_3\varphi_4 + g_{24}\varphi_2\varphi_3\varphi_5 + g_{25}\varphi_2\varphi_4\varphi_5 + g_{26}\varphi_3\varphi_4\varphi_5\\
&+ g_{27}\varphi\varphi_1\varphi_2\varphi_3\varphi_4 + g_{28}\varphi\varphi_1\varphi_2\varphi_3\varphi_5 + g_{29}\varphi\varphi_1\varphi_2\varphi_4\varphi_5 + g_{30}\varphi\varphi_1\varphi_3\varphi_4\varphi_5\\
&+ g_{31}\varphi\varphi_2\varphi_3\varphi_4\varphi_5 + g_{32}\varphi_1\varphi_2\varphi_3\varphi_4\varphi_5,
\end{aligned}
\tag{6.55}
$$

where $f_0, \dots, f_{29}$, $g_1, \dots, g_{32}$ are functions depending on the even variables x, t, u, $u_1, \dots, u_5$. We have to mention here that we are constructing generic elements, even and odd explicitly, of the following exterior algebra $C^\infty(x, t, u, \dots, u_5) \otimes \Lambda(\varphi, \dots, \varphi_5)$, where $\Lambda(\varphi, \dots, \varphi_5)$ is the (exterior) algebra generated by $\varphi, \dots, \varphi_5$ The symmetry condition (6.37) for $p = 0$ reads in this case to the system

$$
\begin{aligned}
D_t(\Phi^u) &= \partial_\Phi(6uu_1 - u_3 + 3\varphi\varphi_2),\\
D_t(\Phi^\varphi) &= \partial_\Phi(3u_1\varphi + 6u\varphi_1 - 4\varphi_3),
\end{aligned}
\tag{6.56}
$$

which results in equations

$$
\begin{aligned}
&D_t(\Phi^u) - 6\Phi^u u_1 - 6uD_x(\Phi^u) + D_x^3(\Phi^u) - 3\Phi^\varphi\varphi_2 - 3\varphi D_x^2(\Phi^\varphi) = 0,\\
&D_t(\Phi^\varphi) - 3D_x(\Phi^u)\varphi - 3u_1\Phi^\varphi - 6\Phi^u\varphi_1 - 6uD_x(\Phi^\varphi) + 4D_x^3(\Phi^\varphi) = 0.
\end{aligned}
\tag{6.57}
$$

Substitution of the representation(6.55) of $\Phi = (\Phi^u, \Phi^\varphi))$, leads to an overdetermined system of classical partial differential equations for the coefficients $f_0, \dots, f_{26}$, $g_1, \dots, g_{32}$, which are, as mentioned above, functions depending on the variables x, t, u, $u_1, \dots, u_5$.

The general solution of equations (6.57) and (6.55) is generated by the functions

$$
\begin{aligned}
&\Phi_1 = (u_1, \varphi_1);\\
&\Phi_2 = (6uu_1 - u_3 + 3\varphi\varphi_2, 3u_1\varphi + 6u\varphi_1 - 4\varphi_3);\\
&\Phi_3 = (6tu_1 + 1, 6t\varphi_1);\\
&\Phi_4 = (3t(6uu_1 - u_3 + 3\varphi\varphi_2) + x(u_1) + 2u,\\
&\qquad 3t(3u_1\varphi + 6u\varphi_1 - 4\varphi_3) + x\varphi_1 + \frac{3}{2}\varphi);\\
&\Phi_5 = (u_5 - 10u_3u - 20u_2u_1 + 30u_1u^2 - 15\varphi\varphi_4 - 10\varphi_1\varphi_3\\
&\qquad + 30u_1\varphi\varphi_1 + 30u\varphi\varphi_2,
\end{aligned}
$$

$$16\varphi_5 - 40u\varphi_3 - 60u_1\varphi_2 - 50u_2\varphi_1 + 30u^2\varphi_1 + 30u_1u\varphi - 15u_3\varphi). \tag{6.58}$$

We note that the vector fields $\mathcal{D}_{\Phi_1}$, $\mathcal{D}_{\Phi_2}$, $\mathcal{D}_{\Phi_3}$, $\mathcal{D}_{\Phi_4}$ are equivalent to the classical symmetries

$$\begin{aligned} S_1 &= \frac{\partial}{\partial x}, \\ S_2 &= \frac{\partial}{\partial t}, \\ S_3 &= t\frac{\partial}{\partial x} - \frac{1}{6}\frac{\partial}{\partial u}, \\ S_4 &= -x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + \frac{3}{2}\varphi\frac{\partial}{\partial \varphi}. \end{aligned} \tag{6.59}$$

In (6.59) S_1, S_2 reflect space and time translation, S_3 reflects Galilean invariance, while S_4 reflects the scaling as mentioned already. In (6.50), the evolutionary vector field $\mathcal{D}_{\Phi_5}$ is the first higher symmetry of the super KdV equation and reduces to

$$(u_5 - 10u_3u - 20u_2u_1 + 30u_1u^2)\frac{\partial}{\partial u} + \dots, \tag{6.60}$$

in the absence of odd variables φ, $\varphi_1, \dots$, being then just the classical first higher symmetry of the KdV equation

$$u_t = 6uu_1 - u_3. \tag{6.61}$$

4.2. A nonlocal symmetry. In this subsection we demonstrate the existence and construction of nonlocal higher symmetries for the super KdV equation (6.48). The construction runs exactly along the same lines as it is for the classical equations.

So we start at the construction of conservation laws, conserved densities and conserved quantities as discussed in Section 2. According to this construction we arrive, amongst others, at the following two conservation laws, i.e.,

$$\begin{aligned} D_t(u) &= D_x(3u^2 - u_2 + 3\varphi\varphi_1), \\ D_t(u^2 + 3\varphi\varphi_1) &= D_x(4u^3 + u_1^2 - 2uu_2 + 12u\varphi\varphi_1 + 8\varphi_1\varphi_2 - 4\varphi\varphi_3), \end{aligned} \tag{6.62}$$

from which we obtain the nonlocal variables

$$\begin{aligned} p_1 &= \int_{-\infty}^{x} u\,dx, \\ p_3 &= \int_{-\infty}^{x} (u^2 + 3\varphi\varphi_1)\,dx. \end{aligned} \tag{6.63}$$

Now using these new nonlocal variables p_1, p_3, we define the augmented system $\mathcal{E}'$ of partial differential equations for the variables u, p_1, p_3, φ, where u, p_1, p_3 are *even* and φ is *odd*,

$$u_t = 6uu_x - u_{xxx} + 3\varphi\varphi_{xx},$$

$$\begin{aligned}
\varphi_t &= 3u_x\varphi + 6u\varphi_x - 4\varphi_{xxx},\\
(p_1)_x &= u,\\
(p_1)_t &= 3u^2 - u_2 + 3\varphi\varphi_1,\\
(p_3)_x &= u^2 + 3\varphi\varphi_1,\\
(p_3)_t &= 4u^3 + u_1^2 - 2uu_2 + 12u\varphi\varphi_1 + 8\varphi_1\varphi_2 - 4\varphi\varphi_3.
\end{aligned} \tag{6.64}$$

Internal coordinates for the infinite prolongation $\mathcal{E}'^\infty$ of this augmented system (6.64) are given as x, t, u, p_1, p_3, φ, u_1, $\varphi_1, \dots$. The total derivative operators $\widetilde{D}_x$ and $\widetilde{D}_t$ on $\mathcal{E}'^\infty$ are given by

$$\begin{aligned}
\widetilde{D}_x &= D_x + u\frac{\partial}{\partial p_1} + (u^2 + 3\varphi\varphi_1)\frac{\partial}{\partial p_3},\\
\widetilde{D}_t &= D_t + (3u^2 - u_2 + 3\varphi\varphi_1)\frac{\partial}{\partial p_1}\\
&\quad + (4u^3 + u_1^2 - 2uu_2 + 12u\varphi\varphi_1 + 8\varphi_1\varphi_2 - 4\varphi\varphi_3)\frac{\partial}{\partial p_3}.
\end{aligned} \tag{6.65}$$

We are motivated by the result for the classical KdV equation (see Section 5 of Chapter 3) and our search is for a nonlocal vector field ∂_Φ of the following form

$$\Phi = C_1 t\Phi_4 + C_2 x\Phi_2 + C_3 p_1\Phi_1 + p_3\Phi^* + \Phi^{**}, \tag{6.66}$$

whereas in (6.66) C_1, C_2, C_3 are constants and $\Phi^* = (\Phi^{*u}, \Phi^{*\varphi})$, $\Phi^{**} = (\Phi^{**u}, \Phi^{**\varphi})$ are functions to be determined.

We now apply the symmetry condition resulting from the augmented system (6.64), compare with (6.57)

$$\begin{aligned}
&\widetilde{D}_t(\Phi^u) - 6\Phi^u u_1 - 6u\widetilde{D}_x(\Phi^u) + \widetilde{D}_x^3(\Phi^u) - 3\Phi^\varphi\varphi_2 - 3\varphi\widetilde{D}_x^2(\Phi^\varphi) = 0,\\
&\widetilde{D}_t(\Phi^\varphi) - 3\widetilde{D}_x(\Phi^u)\varphi - 3u_1\Phi^\varphi - 6\Phi^u\varphi_1 - 6u\widetilde{D}_x(\Phi^\varphi) + 4\widetilde{D}_x^3(\Phi^\varphi) = 0.
\end{aligned} \tag{6.67}$$

Condition (6.67) leads to an overdetermined system of partial differential equations for the functions Φ^{*u}, $\Phi^{*\varphi}$, Φ^{**u}, $\Phi^{**\varphi}$, whose dependency on the internal variables is induced by the scaling of the super KdV equation, which means that we are in effect searching for a vector field ∂_Φ, where Φ^u, Φ^φ are of degree 4 and $3\frac{1}{2}$ respectively. Solving the overdetermined system of equations leads to the following result.

The vector field ∂_Φ with Φ defined by

$$\Phi = -\frac{3}{4}t\Phi_5 - \frac{1}{4}x\Phi_2 - \frac{1}{2}p_1\Phi_1 + \Phi^{**}, \tag{6.68}$$

where Φ_5, Φ_2, Φ_1 are defined by (6.58) and

$$\Phi^{**} = (u_2 - 2u^2 - \frac{3}{2}\varphi\varphi_1, \frac{7}{2}\varphi_2 - 3u\varphi), \tag{6.69}$$

is a *nonlocal higher symmetry* of the super KdV equation (6.48). In effect, the function Φ is the shadow of the associated symmetry of (6.64).

The $\partial/\partial p_1$- and $\partial/\partial p_3$-components of the symmetry $\mathcal{Э}_\Phi$ can be computed from the invariance of the equations (6.70),

$$(p_1)_x = u,$$
$$(p_3)_x = u^2 + 3\varphi\varphi_1, \tag{6.70}$$

but considered in a once more augmented setting. The reader is referred to the construction of nonlocal symmetries for the classical KdV equation for the details of this calculation.

It would be possible to describe the recursion here, but we prefer to postpone it to the chapter devoted to the deformations of the equation structure (see Chapter 7), from which the recursion operator can be obtained rather easily and straightforwardly.

5. The Kupershmidt super mKdV equation

As a second application of the graded calculus for symmetries of graded partial differential equations, we discuss the symmetry structure of the so-called Kupershmidt super mKdV equation, which is an extension of the classical mKdV equation to the graded setting [**24**].

The super mKdV equation is given as the following system of graded partial differential equations $\mathcal{E}$ for an *even* function v and an *odd* function ψ on $J^3(\pi;\psi)$ (see the notation in the previous section),

$$v_t = 6v^2 v_x - v_{xxx} + \frac{3}{4}\psi_x\psi_{xx} + \frac{3}{4}\psi\psi_{xxx} + \frac{3}{2}v_x\psi\psi_x + \frac{3}{2}v\psi\psi_{xx},$$
$$\psi_t = (6v^2 - 6v_x)\psi_x + (6vv_x - 3v_{xx})\psi - 4\psi_{xxx}, \tag{6.71}$$

where subscripts denote partial derivatives with respect to x, t. Here t is the time variable and x is the space variable, v, x, t, v, v_x, v_t, v_{xx}, v_{xxx} are even (commuting) variables, while ψ, ψ_x, ψ_{xx}, ψ_{xxx} are odd (anticommuting) variables.

We introduce the total derivative operators D_x, D_t on $J^\infty(\pi;\psi)$ by

$$D_x = \frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v} + \psi_x\frac{\partial}{\partial \psi} + v_{xx}\frac{\partial}{\partial v_x} + \psi_{xx}\frac{\partial}{\partial \psi_x} + \cdots,$$
$$D_t = \frac{\partial}{\partial t} + v_t\frac{\partial}{\partial v} + \psi_t\frac{\partial}{\partial \psi} + v_{tx}\frac{\partial}{\partial v_x} + \psi_{tx}\frac{\partial}{\partial \psi_x} + \cdots \tag{6.72}$$

The infinite prolongation $\mathcal{E}^\infty$ is the submanifold of $J^\infty(\pi;\psi)$ defined by the graded system of partial differential equations

$$D_x^n D_t^m\Big(v_t - 6v^2 v_x + v_{xxx} - \frac{3}{4}\psi_x\psi_{xx} - \frac{3}{4}\psi\psi_{xxx} - \frac{3}{2}v_x\psi\psi_x - \frac{3}{2}v\psi\psi_{xx}\Big) = 0,$$
$$D_x^n D_t^m\big(\psi_t - (6v^2 - 6v_x)\psi_x - (6vv_x - 3v_{xx})\psi + 4\psi_{xxx}\big) = 0, \tag{6.73}$$

where $n, m \in \mathbb{N}$.

We choose internal coordinates on $\mathcal{E}^\infty$ as x, t, v, ψ, v_1, $\psi_1, \ldots$, where we use a notation

$$v_x = v_1, \quad \psi_x = \psi_1, \quad v_{xx} = v_2, \quad \psi_{xx} = \psi_2, \ldots \tag{6.74}$$

The restriction of the total derivative operators D_x, D_t to $\mathcal{E}^\infty$, again denoted by the same symbols, are then given by

$$D_x = \frac{\partial}{\partial x} + \sum_{n\geq 0}\left(v_{n+1}\frac{\partial}{\partial v_n} + \psi_{n+1}\frac{\partial}{\partial \psi_n}\right),$$
$$D_t = \frac{\partial}{\partial t} + \sum_{n\geq 0}\left((v_n)_t\frac{\partial}{\partial v_n} + (\psi_n)_t\frac{\partial}{\partial \psi_n}\right). \tag{6.75}$$

We note that (6.71) admits a scaling symmetry, which leads to the assigning a degree to each variable,

$$\begin{aligned} \deg(x) &= -1, & \deg(t) &= -3,\\ \deg(v) &= 1, & \deg(v_1) &= 2,\dots,\\ \deg(\psi) &= \frac{1}{2}, & \deg(\psi_1) &= \frac{3}{2},\dots \end{aligned} \tag{6.76}$$

From this we see that each term in (6.71) is of degree 4 and $3\frac{1}{2}$ respectively.

5.1. Higher symmetries. We start the discussion of searching for (higher) symmetries at the representation of vertical vector fields,

$$\mathcal{Э}_\Phi = \Phi^v\frac{\partial}{\partial v} + \Phi^\psi\frac{\partial}{\partial \psi} + \sum_{n>0}\left(D_x^n(\Phi^v)\frac{\partial}{\partial v_n} + D_x^n(\Phi^\psi)\frac{\partial}{\partial \psi_n}\right), \tag{6.77}$$

where $\Phi = (\Phi^v, \Phi^\psi)$ is the generating function of the vertical vector field $Э_\Phi$. We restrict our search for higher symmetries to even vector fields, meaning that Φ^v is even, while Φ^ψ is odd. Moreover we restrict our search for higher symmetries to vector fields $Э_\Phi$ whose generating function $\Phi = (\Phi^v, \Phi^\psi)$ depends on the variables x, t, v, $\psi, \dots, v_5$, ψ_5. The above mentioned requirements lead to a representation of the function $\Phi = (\Phi^v, \Phi^\psi)$ in the following form

$$\begin{aligned} \Phi^v = {} & f_0 + f_1\psi\psi_1 + f_2\psi\psi_2 + f_3\psi\psi_3 + f_4\psi\psi_4 + f_5\psi\psi_5 + f_6\psi_1\psi_2\\ & + f_7\psi_1\psi_3 + f_8\psi_1\psi_4 + f_9\psi_1\psi_5 + f_{10}\psi_2\psi_3 + f_{11}\psi_2\psi_4 + f_{12}\psi_2\psi_5\\ & + f_{13}\psi_3\psi_4 + f_{14}\psi_3\psi_5 + f_{15}\psi_4\psi_5 + f_{16}\psi\psi_1\psi_2\psi_3 + f_{17}\psi\psi_1\psi_2\psi_4\\ & + f_{18}\psi\psi_1\psi_2\psi_5 + f_{19}\psi\psi_1\psi_3\psi_4 + f_{20}\psi\psi_1\psi_3\psi_5 + f_{21}\psi\psi_1\psi_4\psi_5\\ & + f_{22}\psi\psi_2\psi_3\psi_4 + f_{23}\psi\psi_2\psi_3\psi_5 + f_{24}\psi\psi_2\psi_4\psi_5 + f_{25}\psi\psi_3\psi_4\psi_5\\ & + f_{26}\psi_1\psi_2\psi_3\psi_4 + f_{27}\psi_1\psi_2\psi_3\psi_5 + f_{28}\psi_2\psi_3\psi_4\psi_5\\ & + f_{29}\psi\psi_1\psi_2\psi_3\psi_4\psi_5, \end{aligned}$$

$$\begin{aligned} \Phi^\psi = {} & g_1\psi + g_2\psi_1 + g_3\psi_2 + g_4\psi_3 + g_5\psi_4 + g_6\psi_5\\ & + g_7\psi\psi_1\psi_2 + g_8\psi\psi_1\psi_3 + g_9\psi\psi_1\psi_4 + g_{10}\psi\psi_1\psi_5 + g_{11}\psi\psi_2\psi_3\\ & + g_{12}\psi\psi_2\psi_4 + g_{13}\psi\psi_2\psi_5 + g_{14}\psi\psi_3\psi_4 + g_{15}\psi\psi_3\psi_5 + g_{16}\psi\psi_4\psi_5\\ & + g_{17}\psi_1\psi_2\psi_3 + g_{18}\psi_1\psi_2\psi_4 + g_{19}\psi_1\psi_2\psi_5 + g_{20}\psi_1\psi_3\psi_4 + g_{21}\psi_1\psi_3\psi_5\\ & + g_{22}\psi_1\psi_4\psi_5 + g_{23}\psi_2\psi_3\psi_4 + g_{24}\psi_2\psi_3\psi_5 + g_{25}\psi_2\psi_4\psi_5 + g_{26}\psi_3\psi_4\psi_5 \end{aligned}$$

$$+ g_{27}\psi\psi_1\psi_2\psi_3\psi_4 + g_{28}\psi\psi_1\psi_2\psi_3\psi_5 + g_{29}\psi\psi_1\psi_2\psi_4\psi_5 + g_{30}\psi\psi_1\psi_3\psi_4\psi_5$$
$$+ g_{31}\psi\psi_2\psi_3\psi_4\psi_5 + g_{32}\psi_1\psi_2\psi_3\psi_4\psi_5, \tag{6.78}$$

where $f_0, \ldots, f_{29}$, $g_1, \ldots, g_{32}$ are functions depending on the even variables x, t, v, $v_1, \ldots, v_5$. We have to mention here that we are constructing generic elements, even and odd explicitly, of the exterior algebra $C^\infty(x,t,v,\ldots,v_5)\otimes \Lambda(\psi,\ldots,\psi_5)$, where $\Lambda(\psi,\ldots,\psi_5)$ is the exterior algebra generated by the elements $\psi,\ldots,\psi_5$. The symmetry condition (6.37) reads in this case

$$D_t(\Phi^v) = \partial_\Phi(6v^2v_1 - v_3 + \frac{3}{4}\psi_1\psi_2 + \frac{3}{4}\psi\psi_3 + \frac{3}{2}v_1\psi\psi_1 + \frac{3}{2}v\psi\psi_2),$$
$$D_t(\Phi^\psi) = \partial_\Phi((6v^2 - 6v_1)\psi_1 + (6vv_1 - 3v_2)\psi - 4\psi_3), \tag{6.79}$$

which results in equations

$$\begin{aligned}
&D_t(\Phi^v) - 12\Phi^v vv_1 - 6v^2D_x(\Phi^v) + D_x^3(\Phi^v) - \frac{3}{4}D_x(\Phi^\psi)\psi_2\\
&\quad - \frac{3}{4}\psi_1 D_x^2(\Phi^\psi) - \frac{3}{4}\Phi^\psi\psi_3 - \frac{3}{4}\psi D_x^3(\Phi^\psi)\\
&\quad - \frac{3}{2}D_x(\Phi^v)\psi\psi_1 - \frac{3}{2}v_1\Phi^\psi\psi_1 - \frac{3}{2}v_1\psi D_x(\Phi^\psi)\\
&\quad - \frac{3}{2}\Phi^v\psi\psi_2 - \frac{3}{2}v\Phi^\psi\psi_2 - \frac{3}{2}v\psi D_x^2(\Phi^\psi) = 0,\\
&D_t(\Phi^\psi) - (12v\Phi^v - 6D_x(\Phi^v))\psi_1 - (6v^2 - 6v_1)D_x(\Phi^\psi)\\
&\quad - (6\Phi^v v_1 + 6vD_x(\Phi^v) - 3D_x^2(\Phi^v))\psi\\
&\quad - (6vv_1 - 3v_2)\Phi^\psi + 4D_x^3(\Phi^\psi) = 0.
\end{aligned} \tag{6.80}$$

Substitution of the representation (6.78) for $\Phi = (\Phi^v, \Phi^\psi))$, leads to an overdetermined system of classical partial differential equations for the coefficients $f_0, \ldots, f_{26}$, $g_1, \ldots, g_{32}$ which are as mentioned above functions depending on the variables x, t, v, $v_1, \ldots, v_5$.

The general solution of equations (6.80) and (6.78) is generated by the functions

$$\Phi_1 = (v_1, \psi_1),$$
$$\Phi_2 = (-v_3 + 6v^2v_1 + \frac{3}{2}v_1\psi\psi_1 + \frac{3}{2}v\psi\psi_2 + \frac{3}{4}\psi\psi_3 + \frac{3}{4}\psi_1\psi_2,$$
$$- 4\psi_3 + (6v^2 - 6v_1)\psi_1 + (6vv_1 - 3v_2)\psi),$$
$$\Phi_3 = -2x\Phi_1 - 6t\Phi_2 + (-2v, -\psi),$$
$$\Phi_4^v = v_5 - 10v_3v^2 - 40v_2v_1v - 10v_1^3 + 30v_1v^4 - \frac{15}{4}\psi\psi_5 - \frac{25}{4}\psi_1\psi_4 - \frac{5}{2}\psi_2\psi_3$$
$$- \frac{15}{2}v\psi\psi_4 - 5v\psi_1\psi_3 + (\frac{15}{2}v^2 - 15v_1)\psi\psi_3 + (\frac{15}{2}v^2 - 5v_1)\psi_1\psi_2$$
$$+ (15v^3 + 15v_1v - 15v_2)\psi\psi_2 + (45v_1v^2 - \frac{15}{2}v_3)\psi\psi_1,$$
$$\Phi_4^\psi = 16\psi_5 + (40v_1 - 40v^2)\psi_3 + (60v_2 - 120v_1v)\psi_2$$

$$+ (50v_3 - 100v_2v - 60v_1v^2 - 70v_1^2 + 30v^4)\psi_1$$
$$+ (15v_4 - 30v_3v - 30v_2v^2 - 60v_2v_1 - 60v_1^2v + 60v_1v^3)\psi. \tag{6.81}$$

We note that the vector fields ∂_{Φ_1}, ∂_{Φ_2}, ∂_{Φ_3} are equivalent to the classical symmetries

$$\begin{aligned} S_1 &= \frac{\partial}{\partial x}, \\ S_2 &= \frac{\partial}{\partial t}, \\ S_4 &= 2x\frac{\partial}{\partial x} + 6t\frac{\partial}{\partial t} - 2v\frac{\partial}{\partial v} - \psi\frac{\partial}{\partial \psi}. \end{aligned} \tag{6.82}$$

In (6.82), S_1, S_2 reflect space and time translation, while S_3 reflects the scaling as mentioned already, (6.73). The field ∂_{Φ_4} is the first higher symmetry of the super mKdV equation and reduces to the evolutionary vector field

$$(v_5 - 10v_3v^2 - 40v_2v_1v - 10v_1^3 + 30v_1v^4)\frac{\partial}{\partial v} + \dots, \tag{6.83}$$

in the absence of odd variables ψ, $\psi_1, \dots$, being then just the classical first higher symmetry of the mKdV equation.

$$v_t = 6v^2v_x - v_{xxx}. \tag{6.84}$$

REMARK 6.13. It should be noted that this section is just a copy of the previous one concerning the Kupershmidt super KdV equation, except for the specific results! This demonstrates the algorithmic structure of the symmetry computations.

5.2. A nonlocal symmetry. In this subsection we demonstrate the existence and construction of nonlocal higher symmetries for the super mKdV equation (6.71). The construction runs exactly along the same lines as it is for the classical equations.

So we start at the construction of conservation laws, conserved densities and conserved quantities as discussed in Section 2. According to this construction, we arrive, amongst others, at the following two conservation laws, i.e.,

$$\begin{aligned} D_t(v) &= D_x(2v^3 - v_2 + \frac{3}{4}\psi\psi_2 + \frac{3}{2}v\psi\psi_1), \\ D_t(v^2 + \frac{1}{4}\psi\psi_1) &= D_x(3v^4 - 2v_2v + v_1^2 - \psi\psi_3 + 2\psi_1\psi_2 \\ &\quad + \frac{3}{2}v\psi\psi_2 - 3v_1\psi\psi_1 + \frac{9}{2}v^2\psi\psi_1), \end{aligned} \tag{6.85}$$

from which we obtain the nonlocal variables

$$\begin{aligned} p_0 &= \int_{-\infty}^{x} v\,dx, \\ p_1 &= \int_{-\infty}^{x} (v^2 + \frac{1}{4}\psi\psi_1)\,dx. \end{aligned} \tag{6.86}$$

Now using these new nonlocal variables p_0, p_1 we define the augmented system $\mathcal{E}'$ of partial differential equations for the variables v, p_0, p_1, ψ, where v, p_0, p_1 are *even* and ψ is *odd*,

$$\begin{aligned}
v_t &= 6v^2 v_x - v_{xxx} + \frac{3}{4}\psi_x\psi_{xx} + \frac{3}{4}\psi\psi_{xxx} + \frac{3}{2}v_x\psi\psi_x + \frac{3}{2}v\psi\psi_{xx},\\
\psi_t &= (6v^2 - 6v_x)\psi_x + (6vv_x - 3v_{xx})\psi - 4\psi_{xxx},\\
(p_0)_x &= v,\\
(p_0)_t &= 2v^3 - v_2 + \frac{3}{4}\psi\psi_2 + \frac{3}{2}v\psi\psi_1,\\
(p_1)_x &= v^2 + \frac{1}{4}\psi\psi_1,\\
(p_1)_t &= 3v^4 - 2v_2v + v_1^2 - \psi\psi_3 + 2\psi_1\psi_2 + \frac{3}{2}v\psi\psi_2 - 3v_1\psi\psi_1 + \frac{9}{2}v^2\psi\psi_1.
\end{aligned} \tag{6.87}$$

Internal coordinates for the infinite prolongation $\mathcal{E}'^\infty$ of this augmented system (6.87)) are given as x, t, v, p_0, p_1, ψ, v_1, $\psi_1, \dots$. The total derivative operators $\widetilde{D}_x$ and $\widetilde{D}_t$ on $\mathcal{E}'^\infty$ are given by

$$\begin{aligned}
\widetilde{D}_x &= D_x + v\frac{\partial}{\partial p_0} + (v^2 + \frac{1}{4}\psi\psi_1)\frac{\partial}{\partial p_1},\\
\widetilde{D}_t &= D_t + (2v^3 - v_2 + \frac{3}{4}\psi\psi_2 + \frac{3}{2}v\psi\psi_1)\frac{\partial}{\partial p_0}\\
&\quad + (3v^4 - 2v_2v + v_1^2 - \psi\psi_3 + 2\psi_1\psi_2 + \frac{3}{2}v\psi\psi_2 - 3v_1\psi\psi_1 + \frac{9}{2}v^2\psi\psi_1)\frac{\partial}{\partial p_1}.
\end{aligned} \tag{6.88}$$

We are motivated by the result for the classical KdV equation (see Section 5 of Chapter 3) and our search is for a nonlocal vector field ∂_Φ of the following form

$$\Phi = C_1t\Phi_4 + C_2x\Phi_2 + C_3p_1\Phi_1 + \Phi^*, \tag{6.89}$$

whereas in (6.89) C_1, C_2, C_3 are constants and Φ^* is a two-component function to be determined.

We now apply the symmetry condition resulting from the augmented system (6.87) (compare with (6.80)):

$$\begin{aligned}
&\widetilde{D}_t(\Phi^v) - 12\Phi^v vv_1 - 6v^2\widetilde{D}_x(\Phi^v) + \widetilde{D}_x^3(\Phi^v) - \frac{3}{4}\widetilde{D}_x(\Phi^\psi)\psi_2\\
&- \frac{3}{4}\psi_1\widetilde{D}_x^2(\Phi^\psi) - \frac{3}{4}\Phi^\psi\psi_3 - \frac{3}{4}\psi\widetilde{D}_x^3(\Phi^\psi) - \frac{3}{2}\widetilde{D}_x(\Phi^v)\psi\psi_1 - \frac{3}{2}v_1\Phi^\psi\psi_1\\
&- \frac{3}{2}v_1\psi\widetilde{D}_x(\Phi^\psi) - \frac{3}{2}\Phi^v\psi\psi_2 - \frac{3}{2}v\Phi^\psi\psi_2 - \frac{3}{2}v\psi\widetilde{D}_x^2(\Phi^\psi) = 0,
\end{aligned}$$

$$\widetilde{D}_t(\Phi^\psi) - (12v\Phi^v - 6\widetilde{D}_x(\Phi^v))\psi_1 - (6v^2 - 6v_1)\widetilde{D}_x(\Phi^\psi)$$

$$- (6\Phi^v v_1 + 6v\widetilde{D}_x(\Phi^v) - 3\widetilde{D}_x^2(\Phi^v))\psi - (6vv_1 - 3v_2)\Phi^\psi + 4\widetilde{D}_x^3(\Phi^\psi) = 0. \tag{6.90}$$

Condition (6.90) leads to an overdetermined system of partial differential equations for the functions Φ^{*u}, $\Phi^{*\psi}$, whose dependency on the internal variables is induced by the scaling of the super mKdV equation, which means that we are in effect searching for a vector field $\mathcal{Э}_\Phi$, where Φ^v, Φ^ψ are of degree 3 and $2\frac{1}{2}$ respectively.

Solving the overdetermined system of equations leads to the following result.

The vector field $\mathcal{Э}_\Phi$ with Φ defined by

$$\Phi = -\frac{3}{2}t\Phi_4 - \frac{1}{2}x\Phi_2 + p_1\Phi_1 + \Phi^*, \tag{6.91}$$

where Φ_4, Φ_2, Φ_1 are defined by (6.81) and

$$\Phi^* = (-\frac{3}{2}v_2 + 2v^3 + v\psi\psi_1 + \frac{7}{8}\psi\psi_2, -5\psi_2 - v\psi + 4v^2\psi - 4v_1\psi), \tag{6.92}$$

is a *nonlocal higher symmetry* of the super mKdV equation (6.71). In effect, the function Φ is the shadow of the associated symmetry of (6.87).

The $\partial/\partial p_0$- and $\partial/\partial p_1$-components of the symmetry $\mathcal{Э}_\Phi$ can be computed from the invariance of the equations

$$\begin{aligned} (p_0)_x &= v, \\ (p_1)_x &= v^2 + \frac{1}{4}\psi\psi_1, \end{aligned} \tag{6.93}$$

but considered in a once more augmented setting. The reader is referred to the construction of nonlocal symmetries for the classical KdV equation for the details of this calculation.

6. Supersymmetric KdV equation

In this section we shall discuss symmetries and conservation laws of the supersymmetric extension of the KdV equation as it was proposed by several authors [**68, 74, 87**].

We shall construct a supersymmetry transforming odd variables into even variables and vice versa. We shall also construct a nonlocal symmetry of the supersymmetric KdV equation, which together with the already known supersymmetry generates a graded Lie algebra of symmetries, comprising a hierarchy of *bosonic* higher symmetries and a hierarchy of *nonlocal higher fermionic* (or super) symmetries. The well-known supersymmetry is just the first term in this hierarchy.

Moreover, higher even and odd conservation laws and conserved quantities arise in a natural and elegant way in the construction of the infinite dimensional graded Lie algebra of symmetries. The construction of higher even symmetries is given in Subsection 6.1, while the construction of the above mentioned nonlocal symmetry together with the graded Lie algebra structure is given in Subsection 6.2.

6.1. Higher symmetries. The existence of higher even symmetries of the supersymmetric extension of KdV equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \tag{6.94}$$

shall be discussed here. We start at the supersymmetric extension given by Mathieu [**74**], i.e.,

$$\begin{aligned} u_t &= -u_3 + 6uu_1 - a\varphi\varphi_2, \\ \varphi_t &= -\varphi_3 + (6-a)\varphi_1 u + a\varphi u_1. \end{aligned} \tag{6.95}$$

In (6.95), integer indices refer to differentiation with respect to x, i.e., $u_3 = \partial^3 u/\partial x^3$; x, t, u are *even*, while φ is *odd*; the parameter a is real. Taking $\varphi \equiv 0$, we get (6.94).

For internal local coordinates on the infinite jet bundle $J^\infty(\pi;\varphi)$ we choose the functions x, t, u, φ, u_1, $\varphi_1, \dots$ The total derivative operators D_x, D_t are defined by

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + \varphi_1\frac{\partial}{\partial \varphi} + u_2\frac{\partial}{\partial u_1} + \varphi_2\frac{\partial}{\partial \varphi_1} + \cdots, \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + \varphi_t\frac{\partial}{\partial \varphi} + D_x(u_t)\frac{\partial}{\partial u_1} + D_x(\varphi_t)\frac{\partial}{\partial \varphi_1} + \cdots \end{aligned} \tag{6.96}$$

The vertical vector field V, the representation of which is given by

$$V = \sum_{i=0}^{\infty} D_x^i(\Phi^u)\frac{\partial}{\partial u_i} + \sum_{i=0}^{\infty} D_x^i(\Phi^\varphi)\frac{\partial}{\partial \varphi_i}, \tag{6.97}$$

with generating function $\Phi = (\Phi^u, \Phi^\varphi)$, is a symmetry of (6.95), if the following conditions are satisfied

$$\begin{aligned} D_t(\Phi^u) &= -D_x^3(\Phi^u) + \Phi^u 6u_1 + D_x(\Phi^u)6u - a\Phi^\varphi\varphi_2 + aD_x^2(\Phi^\varphi)\varphi, \\ D_t(\Phi^\varphi) &= -D_x^3(\Phi^\varphi) + (6-a)D_x(\Phi^\varphi)u + (6-a)\Phi^u\varphi_1 + a\Phi^\varphi u_1 \\ &\quad + aD_x(\Phi^u)\varphi. \end{aligned} \tag{6.98}$$

In (6.98), Φ^u, Φ^φ are functions depending on a finite number of jet variables.

We restrict our search for higher symmetries at this moment to *even* vector fields, moreover our search is for vector fields, whose generating function $\Phi = (\Phi^u, \Phi^\varphi)$ depends on x, t, u, φ, u_1, $\varphi_1, \dots, u_5$, φ_5. More specifically,

$$\begin{aligned} \Phi^u &= f_1 + f_2\varphi\varphi_1 + f_3\varphi\varphi_2 + f_4\varphi\varphi_3 + f_5\varphi\varphi_4 + f_6\varphi_1\varphi_2 + f_7\varphi_1\varphi_3, \\ \Phi^\varphi &= g_1\varphi + g_2\varphi_1 + g_3\varphi_2 + g_4\varphi_3 + g_5\varphi_4 + g_6\varphi_5, \end{aligned} \tag{6.99}$$

whereas in (6.99) $f_1, \dots, f_7$, $g_1, \dots, g_6$ are dependent on the *even* variables x, t, $u, \dots, u_5$. Formula (6.99) is motivated by the standard grading in the classical case of (6.94),

$$\deg(x) = -1, \quad \deg(t) = -3, \quad \deg(u) = 2, \quad \deg(\varphi) = \frac{3}{2}. \tag{6.100}$$

and results for other problems.

In effect, this means that we are not only searching for Φ^u and Φ^φ in the appropriate jet bundle but also restricted to a certain maximal degree. In this case we assume the vector field to be of degree less than or equal to 5, which means that Φ^u, Φ^φ are of degree at most 7 and $6\frac{1}{2}$ respectively.

Substitution of (6.99) into (6.98) does lead to an overdetermined system of partial differential equations for the functions $f_1, \ldots, f_7$, $g_1, \ldots, g_6$. The solution of this overdetermined system of equations leads to the following result

THEOREM 6.33. *For $a = 3$, there are four vector fields $\partial_{\Phi_1}, \ldots, \partial_{\Phi_4}$ satisfying the higher symmetry condition* (6.98), *i.e.*,

$$\begin{aligned}
\Phi_1 &= (u_1, \varphi_1), \\
\Phi_2 &= (u_3 - 6u_1u + 3\varphi\varphi_2, \varphi_3 - 3\varphi_1 u - 3\varphi u_1), \\
\Phi_3 &= -(u_5 - 10u_3u - 20u_2u_1 + 30u_1u^2 + 5\varphi\varphi_4 + 5\varphi_1\varphi_3 \\
&\quad - 20u\varphi\varphi_2 - 20u_1\varphi\varphi_1, \varphi_5 - 5u\varphi_3 - 10u_1\varphi_2 - 10u_2\varphi_1 + 10u^2\varphi_1 \\
&\quad + 20u_1u\varphi - 5u_3\varphi), \\
\Phi_4 &= -3t\Phi_2 + x\Phi_1 + (2u, \frac{3}{2}\varphi).
\end{aligned} \tag{6.101}$$

If $a \neq 3$, then ∂_{Φ_3} is not a symmetry of (6.95).

Next, our search is for *odd* vector fields (6.97) satisfying (6.98); the assumption on the generating function $\Phi = (\Phi^u, \Phi^\varphi)$ is

$$\begin{aligned}
\Phi^u &= f_1\varphi + f_2\varphi_1 + f_3\varphi_2 + f_4\varphi_3 + f_5\varphi_4 + f_6\varphi_5 + f_7\varphi_6, \\
\Phi^\varphi &= g_1 + g_2\varphi\varphi_1 + g_3\varphi\varphi_2 + g_4\varphi\varphi_3 + g_5\varphi\varphi_4 + g_6\varphi\varphi_5 + g_7\varphi_1\varphi_2 \\
&\qquad + g_8\varphi_1\varphi_3 + g_9\varphi_1\varphi_4 + g_{10}\varphi_2\varphi_3,
\end{aligned} \tag{6.102}$$

where $f_1, \ldots, f_7$, $g_1, \ldots, g_{10}$ are dependent on x, t, $u, \ldots, u_5$.

Solving the resulting overdetermined system of partial differential equations leads to:

THEOREM 6.34. *There exists only one odd symmetry $Y_{\frac{1}{2}}$ of* (6.95), *i.e.*,

$$\Phi_{Y_{\frac{1}{2}}} = (\varphi_1, u). \tag{6.103}$$

In order to obtain the Lenard recursion operator we did proceed in a way similar to that discussed in Section 5 of Chapter 3, but unfortunately we were not successful. We shall discuss a recursion for higher symmetries, resulting from the graded Lie algebra structure in the next subsection, while the construction of the recursion operator for the supersymmetric KdV equation is discussed in Chapter 7.

6.2. Nonlocal symmetries and conserved quantities. By the introduction of nonlocal variables, we derive here a nonlocal *even* symmetry for the supersymmetric KdV equation in the case $a = 3$

$$u_t = -u_3 + 6u_1u - 3\varphi\varphi_2,$$

$$\varphi_t = -\varphi_3 + 3\varphi_1 u + 3\varphi u_1, \tag{6.104}$$

which together with the supersymmetry $\ni_{Y_{\frac{1}{2}}}$ generates two infinite hierarchies of higher symmetries. The even and odd nonlocal variables and conserved quantities arise in a natural way.

We start with the observation that

$$D_t(\varphi) = D_x(-\varphi_2 + 3\varphi u) \tag{6.105}$$

is a conservation law for (6.104), or equivalently,

$$q_{\frac{1}{2}} = \int_{-\infty}^{x} \varphi \, dx, \tag{6.106}$$

is a potential of (6.104), i.e.,

$$(q_{\frac{1}{2}})_x = \varphi, (q_{\frac{1}{2}})_t = -\varphi_2 + 3\varphi u. \tag{6.107}$$

The quantity $Q_{\frac{1}{2}}$ defined by

$$Q_{\frac{1}{2}} = \int_{-\infty}^{\infty} \varphi \, dx \tag{6.108}$$

is a conserved quantity of the supersymmetric KdV equation (6.104).

We now make the following observation:

THEOREM 6.35. *The nonlocal vector field $\ni_{Z_1}$, whose generating function Φ_{Z_1} is*

$$\Phi_{Z_1} = (q_{\frac{1}{2}}\varphi_1, q_{\frac{1}{2}}u - \varphi_1) \tag{6.109}$$

is a nonlocal symmetry of the KdV equation (6.104). *Moreover, there is no nonlocal symmetry linear with respect to $q_{\frac{1}{2}}$ which satisfies* (6.95) *with $a \neq 3$.*

The function Φ_{Z_1} is in effect the shadow of a nonlocal symmetry of the augmented system of equations

$$\begin{aligned}
u_t &= -u_3 + 6u_1 u - 3\varphi\varphi_2, \\
\varphi_t &= -\varphi_3 + 3\varphi_1 u + 3\varphi u_1, \\
(q_{\frac{1}{2}})_x &= \varphi, \\
(q_{\frac{1}{2}})_t &= -\varphi_2 + 3\varphi u.
\end{aligned} \tag{6.110}$$

Total partial derivative operators $\widetilde{D}_x$ and $\widetilde{D}_t$ are given here by

$$\begin{aligned}
\widetilde{D}_x &= D_x + \varphi \frac{\partial}{\partial q_{\frac{1}{2}}}, \\
\widetilde{D}_t &= D_t + (-\varphi_2 + 3\varphi u) \frac{\partial}{\partial q_{\frac{1}{2}}},
\end{aligned}$$

and the generating function Φ_{Z_1} satisfies the invariance of the first and the second equation in (6.110), i.e.,

$$\widetilde{D}_t(\Phi^u_{Z_1}) + \widetilde{D}^3_x(\Phi^u_{Z_1}) - \Phi^u_{Z_1}6u_1 - \widetilde{D}_x(\Phi^u_{Z_1})6u + 3\Phi^\varphi_{Z_1}\varphi_2 - 3\widetilde{D}^2_x(\Phi^\varphi_{Z_1})\varphi = 0,$$
$$\widetilde{D}_t(\Phi^\varphi_{Z_1}) + \widetilde{D}^3_x(\Phi^\varphi_{Z_1}) - 3\widetilde{D}_x(\Phi^\varphi_{Z_1})u - 3\Phi^u_{Z_1}\varphi_1 - 3\Phi^\varphi_{Z_1}u_1 - 3\widetilde{D}_x(\Phi^u_{Z_1})\varphi = 0.$$

The vector field $\ni_{Z_1}$ together with the vector field $\ni_{Y_{\frac{1}{2}}}$ play a fundamental role in the construction of the graded Lie algebra of symmetries of (6.104).

From now on, for obvious reasons, we shall restrict ourselves to (6.104), i.e., to the case $a = 3$.

REMARK 6.14. All odd variables $\varphi_0, \varphi_1, \dots, q_{\frac{1}{2}}$ are, with respect to the grading (6.100), of degree $n/2$, where n is odd. The vector field $\ni_{Z_1}$ is even, while $\ni_{Y_{\frac{1}{2}}}$ is odd.

We now want to compute the graded Lie algebra with $\ni_{Z_1}$ and $\ni_{Y_{\frac{1}{2}}}$ as "seed elements".

In order to do so, we have to prolong the vector field $\ni_{Y_{\frac{1}{2}}}$ towards the nonlocal variable $q_{\frac{1}{2}}$, or by just writing $\ni_{Y_{\frac{1}{2}}}$ for this prolongation, we have to calculate the component $\partial/\partial q_{\frac{1}{2}}$, in the augmented setting (6.110)).

The calculation is as follows. The coefficient $Y_{\frac{1}{2}}^{q_{\frac{1}{2}}}$ has to be such that the vector field $\ni_{Y_{\frac{1}{2}}}$ leaves invariant (6.105), i.e., the Lie derivative of (6.105) with respect to $\ni_{Y_{\frac{1}{2}}}$ is to be zero.

Since

$$\begin{aligned}
Y_{\frac{1}{2}}^{u_1} &= D_x(Y_{\frac{1}{2}}^u) = D_x(\varphi_1) = \varphi_2,\\
Y_{\frac{1}{2}}^{\varphi_1} &= D_x(Y_{\frac{1}{2}}^\varphi) = D_x(u) = u_1,\\
Y_{\frac{1}{2}}^{\varphi_2} &= D_x(Y_{\frac{1}{2}}^{\varphi_1}) = D_x(u_1) = u_2,
\end{aligned}\tag{6.111}$$

the invariance of the third and fourth equation in (6.110) leads to

$$\begin{aligned}
&\widetilde{D_x}(Y_{\frac{1}{2}}^{q_{\frac{1}{2}}}) - u = 0,\\
&\widetilde{D_t}(Y_{\frac{1}{2}}^{q_{\frac{1}{2}}}) + \widetilde{D_x}^2(Y_{\frac{1}{2}}^\varphi) - 3Y_{\frac{1}{2}}^\varphi u + 3\varphi Y_{\frac{1}{2}}^u = 0,
\end{aligned}\tag{6.112}$$

from which we have

$$\begin{aligned}
&\widetilde{D}_x(Y_{\frac{1}{2}}^{q_{\frac{1}{2}}}) - u = 0,\\
&\widetilde{D}_t(Y_{\frac{1}{2}}^{q_{\frac{1}{2}}}) + u_2 - 3u^2 + 3\varphi\varphi_1 = 0.
\end{aligned}\tag{6.113}$$

By (6.109), (6.111), (6.113) we are led in a natural and elegant way to the introduction of a new nonlocal *even* variable p_1, defined by

$$p_1 = \int_{-\infty}^x u\, dx \tag{6.114}$$

and satisfying the system of equations

$$
\begin{aligned}
(p_1)_x &= u, \\
(p_1)_t &= -u_2 + 3u^2 - 3\varphi\varphi_1,
\end{aligned}
\tag{6.115}
$$

i.e., p_1 is a potential of the supersymmetric KdV equation (6.104); the compatibility conditions being satisfied, while the associated conserved quantity is P_1.

Now the vector field $\partial_{Y_{\frac{1}{2}}}$ is given in the setting (6.110) by

$$
\partial_{Y_{\frac{1}{2}}} = \varphi_1 \frac{\partial}{\partial u} + u \frac{\partial}{\partial \varphi} + p_1 \frac{\partial}{\partial q_{\frac{1}{2}}} + \dots \tag{6.116}
$$

Computation of the graded commutator $[\partial_{Z_1}, \partial_{Y_{\frac{1}{2}}}]$ of ∂_{Z_1} and $\partial_{Y_{\frac{1}{2}}}$ leads us to a new symmetry of the KdV equation, given by

$$
\partial_{Y_{\frac{3}{2}}} = [\partial_{Z_1}, \partial_{Y_{\frac{1}{2}}}], \tag{6.117}
$$

where the generating function is given by

$$
\Phi_{Y_{\frac{3}{2}}} = (2q_{\frac{1}{2}} u_1 - p_1\varphi_1 + u\varphi - \varphi_2, 2q_{\frac{1}{2}}\varphi_1 - p_1 u + u_1). \tag{6.118}
$$

This symmetry is a new nonlocal *odd* symmetry of (6.104) and is of degree $\frac{3}{2}$.

Note that as polynomials in $q_{\frac{1}{2}}$ and p_1, the coefficients in (6.118) just constitute the generating functions of the symmetries $2\partial_{X_1}$ and $-\partial_{Y_{\frac{1}{2}}}$ respectively, i.e.,

$$
\Phi_{Y_{\frac{3}{2}}} = 2q_{\frac{1}{2}}\Phi_1 - p_1\Phi_{Y_{\frac{1}{2}}} + (u\varphi - \varphi_2, u_1). \tag{6.119}
$$

We now proceed by induction.

In order to compute the graded Lie bracket $[\partial_{Z_1}, \partial_{Y_{\frac{3}{2}}}]$, we first have to compute the prolongation of ∂_{Z_1} towards the nonlocal variables p_1 and $q_{\frac{1}{2}}$, which is equivalent to the computation of the $\partial/\partial p_1$- and the $\partial/\partial q_{\frac{1}{2}}$-components of the vector field ∂_{Z_1}, again denoted by the same symbol ∂_{Z_1}.

It is perhaps illustrative to mention at this stage that we are in effect considering the following augmented system of graded partial differential equations

$$
\begin{aligned}
u_t &= -u_3 + 6u_1u - 3\varphi\varphi_2, \\
\varphi_t &= \varphi_3 + 3\varphi_1 u + 3\varphi u_1, \\
(q_{\frac{1}{2}})_x &= \varphi, \\
(q_{\frac{1}{2}})_t &= -\varphi_2 + 3\varphi u, \\
(p_1)_x &= u, \\
(p_1)_t &= -u_2 + 3u^2 - 3\varphi\varphi_1,
\end{aligned}
\tag{6.120}
$$

We now consider the invariance of the fifth equation in (6.120), i.e., of $(p_1)_x = u$, by the vector field ∂_{Z_1} which leads to the condition

$$\widetilde{D}_x(Z_1^{p_1}) = q_{\frac{1}{2}}\varphi_1, \tag{6.121}$$

from which we have

$$Z_1^{p_1} = q_{\frac{1}{2}}\varphi. \tag{6.122}$$

The $\partial/\partial q_{\frac{1}{2}}$-component of ∂_{Z_1}, i.e., $Z_1^{q_{\frac{1}{2}}}$ has to satisfy the invariance of the third equation in (6.120), i.e., $(q_{\frac{1}{2}})_x = \varphi$ by the vector field ∂_{Z_1} which leads to the condition

$$\widetilde{D_x}(Z_1^{q_{\frac{1}{2}}}) - Z_1^{\varphi} = 0,$$

i.e.,

$$\widetilde{D_x}(Z_1^{q_{\frac{1}{2}}}) - q_{\frac{1}{2}}u + \varphi_1 = 0, \tag{6.123}$$

from which we derive

$$Z_1^{q_{\frac{1}{2}}} = q_{\frac{1}{2}}p_1 - \varphi - \int_{-\infty}^{x} p_1\varphi\, dx. \tag{6.124}$$

So prolongation of ∂_{Z_1} towards the nonlocal variable $q_{\frac{1}{2}}$, or equivalently, computation of the $\partial/\partial q_{\frac{1}{2}}$-component of the vector field ∂_{Z_1}, requires formal introduction of a new *odd* nonlocal variable $q_{\frac{3}{2}}$ defined by

$$q_{\frac{3}{2}} = \int_{-\infty}^{x} p_1\varphi\, dx, \tag{6.125}$$

where

$$\begin{aligned} (q_{\frac{3}{2}})_x &= p_1\varphi, \\ (q_{\frac{3}{2}})_t &= p_1(-\varphi_2 + 3u\varphi) - u_1\varphi + u\varphi_1, \end{aligned} \tag{6.126}$$

while the compatibility condition on (6.126) is satisfied; so $q_{\frac{3}{2}}$ is a new *odd* potential, $Q_{\frac{3}{2}}$ being the new odd conserved quantity.

The vector field ∂_{Z_1} is now given in the augmented setting (6.120) by

$$\partial_{Z_1} = q_{\frac{1}{2}}\varphi_1\frac{\partial}{\partial u} + (q_{\frac{1}{2}}u - \varphi_1)\frac{\partial}{\partial \varphi} + (q_{\frac{1}{2}}p_1 - \varphi - q_{\frac{3}{2}})\frac{\partial}{\partial q_{\frac{1}{2}}} + q_{\frac{1}{2}}\varphi\frac{\partial}{\partial p_1}. \tag{6.127}$$

The system of graded partial differential equations under consideration is now the once more augmented system (6.120):

$$\begin{aligned} u_t &= -u_3 + 6u_1u - 3\varphi\varphi_2, \\ \varphi_t &= -\varphi_3 + 3\varphi_1 u + 3\varphi u_1, \\ (q_{\frac{1}{2}})_x &= \varphi, \\ (q_{\frac{1}{2}})_t &= -\varphi_2 + 3\varphi u, \end{aligned}$$

$$
\begin{aligned}
(p_1)_x &= u,\\
(p_1)_t &= -u_2 + 3u^2 - 3\varphi\varphi_1,\\
(q_{\frac{3}{2}})_x &= p_1\varphi,\\
(q_{\frac{3}{2}})_t &= p_1(-\varphi_2 + 3u\varphi) - u_1\varphi + u\varphi_1.
\end{aligned} \tag{6.128}
$$

The prolongation of the vector field $\partial_{Y_{\frac{3}{2}}}$ towards the nonlocal variables $q_{\frac{1}{2}}$, p_1 is now constructed from the respective equations for $(q_{\frac{1}{2}})_x$ and $(p_1)_x$, (6.128) resulting in

$$
\begin{aligned}
Y_{\frac{3}{2}}^{q_{\frac{1}{2}}} &= 2q_{\frac{1}{2}}\varphi - \frac{1}{2}p_1^2 + u,\\
Y_{\frac{3}{2}}^{p_1} &= 2q_{\frac{1}{2}}u - p_1\varphi - \varphi_1.
\end{aligned} \tag{6.129}
$$

Computation of the graded Lie bracket $[\partial_{Z_1}, \partial_{Y_{\frac{3}{2}}}]$ leads to

$$
\begin{aligned}
\partial_{Y_{\frac{5}{2}}} = {} & (-2q_{\frac{3}{2}}u_1 + \frac{1}{2}p_1^2\varphi_1 + p_1(\varphi_2 - u\varphi) - 4u_1\varphi - 3u\varphi_1 + \varphi_3)\frac{\partial}{\partial u}\\
& + (-2q_{\frac{3}{2}}\varphi_1 + \frac{1}{2}p_1^2 u - p_1u_1 + u_2 - 2u^2 + \varphi\varphi_1)\frac{\partial}{\partial \varphi}\\
& + (-2q_{\frac{3}{2}}u + \frac{1}{2}p_1^2\varphi + p_1\varphi_1 - 4u\varphi + \varphi_2)\frac{\partial}{\partial p_1}\\
& + (-2q_{\frac{3}{2}}p_1\varphi + \frac{1}{8}p_1^4 - p_1^2u + p_1u_1 - \frac{1}{2}u^2 - \varphi\varphi_1)\frac{\partial}{\partial q_{\frac{3}{2}}} + \cdots,
\end{aligned} \tag{6.130}
$$

whereas the $\partial/\partial p_1$- and $\partial/\partial q_{\frac{3}{2}}$-components of $\partial_{Y_{\frac{5}{2}}}$ are obtained by the invariance of the associated differential equations for these variables in (6.128).

In order to obtain the $\partial/\partial q_{\frac{1}{2}}$-component of $\partial_{Y_{\frac{5}{2}}}$, we have to require the invariance of the equation $(q_{\frac{1}{2}})_x - \varphi = 0$, which results in the following condition

$$
D_x(Y_{\frac{5}{2}}^{q_{\frac{1}{2}}}) = -2q_{\frac{3}{2}}\varphi_1 + \frac{1}{2}p_1^2u - p_1u_1 + u_2 - 2u^2 + \varphi\varphi_1, \tag{6.131}
$$

from which we have

$$
\begin{aligned}
Y_{\frac{5}{2}}^{q_{\frac{1}{2}}} &= \frac{1}{6}p_1^3 - p_1u + u_1 - 2q_{\frac{3}{2}}\varphi + \int_{-\infty}^{x}(u^2 + 2(p_1\varphi)\varphi + \varphi\varphi_1 - 2u^2)\,dx\\
&= \frac{1}{6}p_1^3 - p_1u + u_1 - 2q_{\frac{3}{2}}\varphi - \int_{-\infty}^{x}(u^2 - \varphi\varphi_1)\,dx.
\end{aligned} \tag{6.132}
$$

So expression (6.132) requires in a natural way the introduction of the *even* nonlocal variable p_3, defined by

$$
p_3 = \int_{-\infty}^{x}(u^2 - \varphi\varphi_1)\,dx, \tag{6.133}
$$

where

$$
(p_3)_x = u^2 - \varphi\varphi_1,
$$

$$(p_3)_t = 4u^3 - 2u_2u + u_1^2 - 9u\varphi\varphi_1 + \varphi\varphi_3 - 2\varphi_1\varphi_2. \tag{6.134}$$

Here p_3 is a well-known potential, P_3 being the associated conserved quantity.

Finally, the commutator $\vartheta_{Y_{\frac{7}{2}}} = [\vartheta_{Z_1}, \vartheta_{Y_{\frac{5}{2}}}]$ requires the prolongation of the vector field ϑ_{Z_1} towards the nonlocal variable $q_{\frac{3}{2}}$, obtained by the invariance of the condition $(q_{\frac{3}{2}})_x - p_1\varphi = 0$ by ϑ_{Z_1}, so

$$D_x(Z_1^{q_{\frac{3}{2}}}) = \vartheta_{Z_1}(p_1\varphi) = (q_{\frac{1}{2}}\varphi)\varphi + p_1(q_{\frac{1}{2}}u - \varphi_1). \tag{6.135}$$

Integration of (6.135) leads to

$$Z_1^{q_{\frac{3}{2}}} = \frac{1}{2}p_1^2 q_{\frac{1}{2}} - p_1\varphi - \int_{-\infty}^{x} (\frac{1}{2}p_1^2\varphi - u\varphi)\, dx. \tag{6.136}$$

The new *odd* nonlocal variable $q_{\frac{5}{2}}$ is, due to (6.136), formally defined by

$$q_{\frac{5}{2}} = \int_{-\infty}^{x} (\frac{1}{2}p_1^2\varphi - u\varphi)\, dx. \tag{6.137}$$

Here $q_{\frac{5}{2}}$ is a nonlocal *odd* potential of the supersymmetric KdV equation (6.104),

$$\begin{aligned}(q_{\frac{5}{2}})_x &= \frac{1}{2}p_1^2\varphi - u\varphi,\\ (q_{\frac{5}{2}})_t &= \frac{1}{2}p_1^2(-\varphi_2 + 3u\varphi) + p_1(-u_1\varphi + u\varphi_1) + u_2\varphi - u_1\varphi_1 - 4u^2\varphi + u\varphi_2.\end{aligned} \tag{6.138}$$

Proceeding in this way, we obtain a hierarchy of nonlocal higher supersymmetries by induction,

$$\vartheta_{Y_{n+\frac{1}{2}}} = [\vartheta_{Z_1}, \vartheta_{Y_{n-\frac{1}{2}}}], \quad n \in \mathbb{N}. \tag{6.139}$$

The higher *even* potentials $p_1, p_3, \ldots$ arise in a natural way in the prolongation of the vector fields $\vartheta_{Y_{2n+\frac{1}{2}}}$ towards the nonlocal variable $q_{\frac{1}{2}}$, whereas the higher *nonlocal odd* potentials $q_{\frac{1}{2}}$, $q_{\frac{3}{2}}$, $q_{\frac{5}{2}}, \ldots$ are obtained in the prolongation of the recursion symmetry ϑ_{Z_1}.

To obtain the graded Lie algebra structure of symmetries we calculate the graded Lie bracket of the vector fields derived so far. The result is remarkable and fascinating:

$$\begin{aligned}[\vartheta_{Y_{\frac{1}{2}}}, \vartheta_{Y_{\frac{1}{2}}}] &= 2\vartheta_{X_1},\\ [\vartheta_{Y_{\frac{3}{2}}}, \vartheta_{Y_{\frac{3}{2}}}] &= 2\vartheta_{X_3},\\ [\vartheta_{Y_{\frac{5}{2}}}, \vartheta_{Y_{\frac{5}{2}}}] &= 2\vartheta_{X_5},\end{aligned} \tag{6.140}$$

so the "squares" of the supersymmetries $\vartheta_{Y_{\frac{1}{2}}}$, $\vartheta_{Y_{\frac{3}{2}}}$, $\vartheta_{Y_{\frac{5}{2}}}$ are just the "classical" symmetries $2\vartheta_{X_1}$, $2\vartheta_{X_3}$, $2\vartheta_{X_5}$ obtained previously (see (6.101)). The other commutators are

$$
\begin{aligned}
[\partial_{Y_{\frac{1}{2}}}, \partial_{Y_{\frac{3}{2}}}] &= 0,\\
[\partial_{Y_{\frac{1}{2}}}, \partial_{Y_{\frac{5}{2}}}] &= -2\partial_{X_3},\\
[\partial_{Y_{\frac{3}{2}}}, \partial_{Y_{\frac{5}{2}}}] &= 0,\\
[\partial_{X_1}, \partial_{X_3}] &= [\partial_{X_1}, \partial_{X_5}] = [\partial_{X_3}, \partial_{X_5}] = 0,\\
[\partial_{Z_1}, \partial_{X_1}] &= [\partial_{Z_1}, \partial_{X_3}] = [\partial_{Z_1}, \partial_{X_5}] = 0,\\
[\partial_{Y_{n+\frac{1}{2}}}, \partial_{X_{2m+1}}] &= 0,
\end{aligned}
\tag{6.141}
$$

where $n = 0, 1, 2$, $m = 0, 1, 2$. We conjecture that in this way we obtain an infinite hierarchy of *nonlocal odd symmetries* $Y_{n+\frac{1}{2}}$, $n \in \mathbb{N}$, and an infinite hierarchy of ordinary even higher symmetries X_{2n+1}, $n \in \mathbb{N}$, while the even and odd nonlocal variables p_{2n+1}, $q_{n+\frac{1}{2}}$ and the associated conserved quantities P_{2n+1}, $Q_{n+\frac{1}{2}}$ are obtained by the prolongation of the vector fields $Y_{n+\frac{1}{2}}$ and Z_1 respectively.

We finish this section with a lemma concerning the Lie algebra structure of the symmetries.

LEMMA 6.36. *Let X_{2n+1}, $n \in \mathbb{N}$, be defined by*

$$X_{2n+1} = \frac{1}{2}[Y_{n+\frac{1}{2}}, Y_{n+\frac{1}{2}}], \tag{6.142}$$

and assume that

$$[Z_1, X_{2n+1}] = 0, \quad n \in \mathbb{N}. \tag{6.143}$$

Then

1. $[Y_{n+\frac{1}{2}}, Y_{m+\frac{1}{2}}] = \begin{cases} (-1)^{m-n} 2X_{n+m+1} & m-n \textit{ is even},\\ 0 & m-n \textit{ is odd}. \end{cases}$
2. $[Y_{n+\frac{1}{2}}, X_{2m+1}] = 0$, $n, m \in \mathbb{N}$.
3. $[X_{2n+1}, X_{2m+1}] = 0$, $n, m \in \mathbb{N}$.

PROOF. The proof of (1) is by induction on $k = m - n$. First consider the cases $k = 1$ and $k = 2$:

$$
\begin{aligned}
0 &= [Z_1, [Y_{n+\frac{1}{2}}, Y_{n+\frac{1}{2}}]] = [Y_{n+1+\frac{1}{2}}, Y_{n+\frac{1}{2}}] + [Y_{n+\frac{1}{2}}, Y_{n+1+\frac{1}{2}}]\\
&= 2[Y_{n+\frac{1}{2}}, Y_{n+1+\frac{1}{2}}],\\
0 &= [Z_1, [Y_{n+\frac{1}{2}}, Y_{n+1+\frac{1}{2}}]] = [Y_{n+1+\frac{1}{2}}, Y_{n+1+\frac{1}{2}}] + [Y_{n+\frac{1}{2}}, Y_{n+2+\frac{1}{2}}],
\end{aligned}
\tag{6.144}
$$

so

$$[Y_{n+\frac{1}{2}}, Y_{n+2+\frac{1}{2}}] = -2X_{2n+3}. \tag{6.145}$$

For general k, the result is obtained from the identity

$$0 = [Z_1, [Y_{n+\frac{1}{2}}, Y_{n+k+\frac{1}{2}}]] = [Y_{n+1+\frac{1}{2}}, Y_{n+k+\frac{1}{2}}] + [Y_{n+\frac{1}{2}}, Y_{n+k+1+\frac{1}{2}}], \tag{6.146}$$

i.e.,

$$[Y_{n+\frac{1}{2}}, Y_{n+k+1+\frac{1}{2}}] = -[Y_{n+1+\frac{1}{2}}, Y_{n+k+\frac{1}{2}}]. \tag{6.147}$$

The proof of (3) is a consequence of (2) by

$$[X_{2n+1}, X_{2m+1}] = [[Y_{n+\frac{1}{2}}, Y_{n+\frac{1}{2}}], X_{2m+1}] = 2[Y_{n+\frac{1}{2}}, [Y_{n+\frac{1}{2}}, X_{2m+1}]] = 0. \tag{6.148}$$

So we are left with the proof of statement (2), the proof of which is by induction too. Let us prove the following statement:

$$E(n):\quad \text{for all } i \leq n,\ j \leq n \text{ one has } [Y_{i+\frac{1}{2}}, X_{2j+1}] = 0.$$

One can see that $E(0)$ is true for obvious reasons: $[Y_{\frac{1}{2}}, X_1] = 0$. The induction step is in three parts,

(b1): $[Y_{n+1+\frac{1}{2}}, X_{2n+3}] = 0$;
(b2): $[Y_{n+1+\frac{1}{2}}, X_{2j+1}] = 0$, $j \leq n$;
(b3): $[Y_{i+\frac{1}{2}}, X_{2n+3}] = 0$, $i \leq n$.

The proof of (b1) is obvious by means of the definition of X_{2n+3}.

The proof of (b2) follows from

$$\begin{aligned}[Y_{n+1+\frac{1}{2}}, X_{2j+1}]] &= [[Z_1, Y_{n+\frac{1}{2}}], X_{2j+1}] \\ &= [Z_1, [Y_{n+\frac{1}{2}}, X_{2j+1}]] + [[Z_1, X_{2j+1}], Y_{n+\frac{1}{2}}] = 0, \end{aligned} \tag{6.149}$$

while both terms in the right-hand side are equal to zero by assumption and (6.144) respectively.

Finally, the proof of (b3) follows from

$$[Y_{i+\frac{1}{2}}, X_{2n+3}] = \frac{1}{2}[Y_{i+\frac{1}{2}}, [Y_{n+1+\frac{1}{2}}, Y_{n+1+\frac{1}{2}}]] = [[Y_{i+\frac{1}{2}}, Y_{n+1+\frac{1}{2}}], Y_{n+1+\frac{1}{2}}] = 0, \tag{6.150}$$

by statement 1 of Lemma 6.36, which completes the proof of this lemma. □

7. Supersymmetric mKdV equation

Since constructions and computations in this section are completely similar to those carried through in the previous section, we shall here present just the results for the supersymmetric mKdV equation (6.151)

$$\begin{aligned} v_t &= -v_3 + 6v^2 v_1 - 3\varphi(v\varphi_1)_1, \\ \varphi_t &= -\varphi_3 + 3v(v\varphi)_1. \end{aligned} \tag{6.151}$$

Note that the supersymmetric mKdV equation (6.151) is graded

$$\begin{aligned} &\deg(x) = -1, && \deg(t) = -3, \\ &\deg(v) = 1, && \deg(\varphi) = \frac{1}{2}. \end{aligned} \tag{6.152}$$

The supersymmetry $\overline{Y}_{\frac{1}{2}}$ of (6.151) is given by

$$\overline{Y}_{\frac{1}{2}} = \varphi_1 \frac{\partial}{\partial v} + v \frac{\partial}{\partial \varphi}. \tag{6.153}$$

The associated nonlocal variable $\overline{q}_{\frac{1}{2}}$ and the conserved quantity $\overline{Q}_{\frac{1}{2}}$ are given by

$$\overline{q}_{\frac{1}{2}} = \int_{-\infty}^{x} (v\varphi)\, dx, \quad \overline{Q}_{\frac{1}{2}} = \int_{-\infty}^{\infty} (v\varphi)\, dx, \tag{6.154}$$

where

$$\begin{aligned} (\overline{q}_{\frac{1}{2}})_x &= v\varphi, \\ (\overline{q}_{\frac{1}{2}})_t &= -v_2\varphi + v_1\varphi_1 - v\varphi_2 + 3v^3\varphi. \end{aligned} \tag{6.155}$$

The nonlocal symmetry $\overline{Z}_1$ is given by

$$\overline{Z}_1 = (\overline{q}_{\frac{1}{2}}\varphi_1) \frac{\partial}{\partial v} + (\overline{q}_{\frac{1}{2}} v - \varphi_1) \frac{\partial}{\partial \varphi}. \tag{6.156}$$

We now present the *even* nonlocal variables $\overline{p}_1$, $\overline{p}_3$ and the *odd* nonlocal variables $\overline{q}_{\frac{1}{2}}$, $\overline{q}_{\frac{3}{2}}$, $\overline{q}_{\frac{5}{2}}$, where

$$\begin{aligned} \overline{p}_1 &= \int_{-\infty}^{x} (v^2 - \varphi\varphi_1)\, dx, \\ \overline{p}_3 &= \int_{-\infty}^{x} (-v^4 - v_1^2 + 3\varphi\varphi_1 v^2 + \varphi_1\varphi_2)\, dx, \\ \overline{q}_{\frac{1}{2}} &= \int_{-\infty}^{x} (v\varphi)\, dx, \\ \overline{q}_{\frac{3}{2}} &= \int_{-\infty}^{x} (\overline{p}_1 v\varphi + v\varphi_1)\, dx, \\ \overline{q}_{\frac{5}{2}} &= \int_{-\infty}^{x} (-\overline{p}_1^2 v\varphi - 2\overline{p}_1 v\varphi_1 + v^3\varphi - 2v\varphi_2)\, dx. \end{aligned} \tag{6.157}$$

The x-derivatives of these nonlocal variables are just the integrands in (6.157), while the t-derivatives are given by

$$\begin{aligned} (\overline{p}_1)_t &= 3v^4 + v_1^2 - 2vv_2 - 9v^2\varphi\varphi_1 + \varphi\varphi_3 - 2\varphi_1\varphi_2, \\ (\overline{p}_3)_t &= -4v^6 + 4v_2v^3 - v_2^2 + 2v_1v_3 - 12v^2v_1^2 + 21v^4\varphi\varphi_1 \\ &\quad - 9vv_2\varphi\varphi_1 + 3v_1^2\varphi\varphi_1 + 12vv_1\varphi\varphi_2 - 3v^2\varphi\varphi_3 \\ &\quad + 9v^2\varphi_1\varphi_2 - \varphi_1\varphi_4 + 2\varphi_2\varphi_3, \\ (\overline{q}_{\frac{1}{2}})_t &= -v_2\varphi + v_1\varphi_1 - v\varphi_2 + 3v^3\varphi, \\ (\overline{q}_{\frac{3}{2}})_t &= \overline{p}_1(-v_2\varphi + v_1\varphi_1 - v\varphi_2 + 3v^3\varphi) + 2v^2v_1\varphi \\ &\quad - v_2\varphi_1 + 4v^3\varphi_1 + v_1\varphi_2 - v\varphi_3, \\ (\overline{q}_{\frac{5}{2}})_t &= -\overline{p}_1^2(-v_2\varphi + v_1\varphi_1 - v\varphi_2 + 3v^3\varphi) \end{aligned}$$

$$
\begin{aligned}
&+ \overline{p}_1(2v\varphi_3 - 2v_1\varphi_2 - 8v^3\varphi_1 + 2v_2\varphi_1 - 4v^2v_1\varphi) \\
&+ 2v\varphi_4 - 2v_1\varphi_3 - 9v^3\varphi_2 + 2v_2\varphi_2 - 13v^2v_1\varphi_1 \\
&+ 4v\varphi\varphi_1\varphi_2 + 5v^5\varphi - 9v^2v_2\varphi.
\end{aligned} \tag{6.158}
$$

The resulting symmetries are given here by

$$
\begin{aligned}
\overline{Z}_1 &= (\overline{q}_{\frac{1}{2}}\varphi_1)\frac{\partial}{\partial v} + (\overline{q}_{\frac{1}{2}}v - \varphi_1)\frac{\partial}{\partial \varphi} + (\overline{q}_{\frac{1}{2}}v\varphi + \varphi\varphi_1)\frac{\partial}{\partial \overline{p}_1} \\
&\quad + (\overline{q}_{\frac{1}{2}}\overline{p}_1 - \overline{q}_{\frac{3}{2}})\frac{\partial}{\partial \overline{q}_{\frac{1}{2}}}, \\
\overline{Y}_{\frac{1}{2}} &= \varphi_1\frac{\partial}{\partial v} + v\frac{\partial}{\partial \varphi} + v\varphi\frac{\partial}{\partial \overline{p}_1} + \overline{p}_1\frac{\partial}{\partial \overline{q}_{\frac{1}{2}}}, \\
\overline{Y}_{\frac{3}{2}} &= (2\overline{q}_{\frac{1}{2}}v_1 - \overline{p}_1\varphi_1 + v^2\varphi - \varphi_2)\frac{\partial}{\partial v} + (2\overline{q}_{\frac{1}{2}}\varphi_1 - \overline{p}_1 v + v_1)\frac{\partial}{\partial \varphi} \\
&\quad + (2\overline{q}_{\frac{1}{2}}(v^2 - \varphi\varphi_1) - \overline{p}_1 v\varphi - 2v\varphi_1 + v_1\varphi)\frac{\partial}{\partial \overline{p}_1} \\
&\quad + (2\overline{q}_{\frac{1}{2}}v\varphi - \frac{1}{2}\overline{p}_1^2 + \frac{1}{2}v^2 + \varphi\varphi_1)\frac{\partial}{\partial \overline{q}_{\frac{1}{2}}}, \\
\overline{Y}_{\frac{5}{2}} &= (\frac{1}{2}\overline{p}_1^2\varphi_1 + \overline{p}_1(\varphi_2 - v^2\varphi) - 2\overline{q}_{\frac{3}{2}}v_1 + \varphi_3 - \frac{5}{2}v^2\varphi_1 - 3vv_1\varphi)\frac{\partial}{\partial v} \\
&\quad + (\frac{1}{2}\overline{p}_1^2 v - \overline{p}_1 v_1 - 2\overline{q}_{\frac{3}{2}}\varphi_1 + v_2 - \frac{3}{2}v^3 + 2v\varphi\varphi_1)\frac{\partial}{\partial \varphi} \\
&\quad + (\frac{1}{2}\overline{p}_1^2 v\varphi + \overline{p}_1(2v\varphi_1 - v_1\varphi) - 2\overline{q}_{\frac{3}{2}}(v^2 - \varphi\varphi_1) + 2v\varphi_2 - 2v_1\varphi_1 \\
&\quad + v_2\varphi - \frac{7}{2}v^3\varphi)\frac{\partial}{\partial \overline{p}_1} \\
&\quad + (\frac{1}{6}\overline{p}_1^3 - 2\overline{q}_{\frac{3}{2}}v\varphi - \overline{p}_1(\varphi\varphi_1 + \frac{1}{2}v^2) - \varphi\varphi_2 + vv_1 + \overline{p}_3)\frac{\partial}{\partial \overline{q}_{\frac{1}{2}}} \\
&\quad + (\frac{1}{8}\overline{p}_1^4 - \frac{1}{4}\overline{p}_1^2(v^2 + 4\varphi\varphi_1) - 2\overline{p}_1\overline{q}_{\frac{3}{2}}v\varphi - \overline{p}_1\varphi\varphi_2 \\
&\quad - 2\overline{q}_{\frac{3}{2}}v\varphi_1 - \varphi_1\varphi_2 + v^2\varphi\varphi_1 - \frac{11}{8}v^4 + vv_2 - \frac{1}{2}v_1^2)\frac{\partial}{\partial \overline{q}_{\frac{3}{2}}}, \\
\overline{X}_1 &= v_1\frac{\partial}{\partial v} + \varphi_1\frac{\partial}{\partial \varphi}, \\
\overline{X}_3 &= (-v_3 + 6v^2v_1 - 3v\varphi\varphi_2 - 3v_1\varphi\varphi_1)\frac{\partial}{\partial v} + (-\varphi_3 + 3v^2\varphi_1 + 3vv_1\varphi)\frac{\partial}{\partial \varphi}, \\
\overline{X}_5 &= (v_5 - 10v_3v^2 - 40v_2v_1v - 10v_1^3 + 30v_1v^4 \\
&\quad + 5v\varphi\varphi_4 + 10v_1\varphi\varphi_3 + 5v\varphi\varphi_3 + 5v_1\varphi_1\varphi_2 - 20v^3\varphi\varphi_2 \\
&\quad + 10v_2\varphi\varphi_2 + 5v_3\varphi\varphi_1 - 60v_1v^2\varphi\varphi_1)\frac{\partial}{\partial v}
\end{aligned}
$$

$$
\begin{aligned}
&+ (\varphi_5 - 5v^2\varphi_3 - 15v_1 v\varphi_2 - 15v_2 v\varphi_1 - 10v_1^2\varphi_1 + 10v^4\varphi_1 \\
&- 5v_3 v\varphi - 10v_2 v_1\varphi + 20v_1 v^3\varphi)\frac{\partial}{\partial\varphi}.
\end{aligned}
\tag{6.159}
$$

The graded Lie algebra structure of the symmetries is similar to the structure of that for the supersymmetric extension of the KdV equation considered in the previous section.

8. Supersymmetric extensions of the NLS

Symmetries, conservation laws, and prolongation structures of the supersymmetric extensions of the KdV and mKdV equation, constructed by Manin–Radul, Mathieu [**72, 74**], have already been investigated in previous sections.

A supersymmetric extension of the cubic Schrödinger equation has been constructed by Kulish [**15**] and has been discussed by Roy Chowdhury [**89**], who applied the Painlevé criterion to it. A simple calculation shows however that the system does not admit a nontrivial prolongation structure. Moreover, as it can readily be seen, the resulting system of equations does not inherit the grading of the classical NLS equation.

We shall now discuss a formal construction of supersymmetric extensions of the classical integrable systems, the cubic Schrödinger equation being just a very interesting application of this construction, which does inherit its grading, based on considerations along the lines of Mathieu [**74**]. This construction leads to *two supersymmetric extensions*, one of which contains a free parameter. The resulting systems are proven to admit infinite series of local and nonlocal symmetries and conservation laws.

8.1. Construction of supersymmetric extensions. We shall discuss supersymmetric extensions of the nonlinear Schrödinger equation

$$
iq_t = -q_{xx} + k(q^*q)q, \tag{6.160}
$$

where q is a complex valued function. If we put $q = u + iv$ then (6.160) reduces to a system of two nonlinear equations

$$
\begin{aligned}
u_t &= -v_{xx} + kv(u^2 + v^2), \\
v_t &= u_{xx} - ku(u^2 + v^2).
\end{aligned}
\tag{6.161}
$$

Symmetries, conservation laws and coverings for this system were discussed by several authors, see [**88**] and references therein.

Now we want to construct a supersymmetric extension of (6.161). This construction is based on two main principles:

1. *The existence of a supersymmetry* $Y_{\frac{1}{2}}$, whose "square"

$$
[Y_{\frac{1}{2}}, Y_{\frac{1}{2}}] \doteq \frac{\partial}{\partial x}, \tag{6.162}
$$

where in (6.162)) "$\doteq$" refers to equivalence classes of symmetries[2].

[2] Recall that by the definition of a higher

2. *The existence of a higher (third) order even symmetry* X_3, which reduces to the classical symmetry of (6.162) in the absence of odd variables.

The technical construction heavily relies on the grading of equations (6.161) and (6.162),

$$\begin{aligned}
&\deg(x) = -1,\ \deg(t) = -2,\ \deg(u) = 1,\ \deg(v) = 1,\\
&\deg(u_x) = \deg(v_x) = 2,\ \deg(u_t) = \deg(v_t) = 3,\\
&\deg(u_{xx}) = \deg(v_{xx}) = 3, \dots
\end{aligned} \tag{6.163}$$

Condition 1, together with the assumption that the odd variables φ, ψ to be introduced are of degree ≥ 0, immediately leads to two possible choices for the degree of φ, ψ and the supersymmetry $Y_{\frac{1}{2}}$, namely,

$$\begin{aligned}
&\deg(\varphi) = \deg(\psi) = \frac{1}{2},\\
&Y_{\frac{1}{2}} = \varphi_1 \frac{\partial}{\partial u} + \psi_1 \frac{\partial}{\partial v} + \frac{u}{2}\frac{\partial}{\partial \varphi} + \frac{v}{2}\frac{\partial}{\partial \psi},
\end{aligned} \tag{6.164}$$

or

$$\begin{aligned}
&\deg(\varphi) = \deg(\psi) = \frac{3}{2},\\
&Y_{\frac{1}{2}} = \varphi \frac{\partial}{\partial u} + \psi \frac{\partial}{\partial v} + \frac{u_1}{2}\frac{\partial}{\partial \varphi} + \frac{v_1}{2}\frac{\partial}{\partial \psi},
\end{aligned} \tag{6.165}$$

where it should be noted that the presentations (6.164), (6.165) for $Y_{\frac{1}{2}}$ are not unique, but can always be achieved by simple linear transformations $(\varphi, \psi) \mapsto (\varphi', \psi')$. The choice (6.165) leads to just one possible extension of (6.161), namely,

$$\begin{aligned}
u_t &= -v_{xx} + kv(u^2 + v^2) + \alpha\varphi\psi,\\
v_t &= u_{xx} - ku(u^2 + v^2) + \beta\varphi\psi,\\
\varphi_t &= f_1[u, v, \varphi, \psi],\\
\psi_t &= f_2[u, v, \varphi, \psi],
\end{aligned} \tag{6.166}$$

where f_1, f_2 are functions of degree $7/2$ depending on u, v, φ, ψ and their derivatives with respect to x.

A straightforward computer computation, however, shows that there does not exist a supersymmetric extension of (6.162) satisfying the two basic principles and (6.164) in this case. Therefore we can restrict ourselves to the case (6.164) from now on.

For reasons of convenience, we shall use subscripts to denote differentiation with respect to x in the sequel, i.e., $u_1 = u_x$, $u_2 = u_{xx}$, etc. In the

symmetry (see Chapter 2), it a coset in the quotient $D_{\mathcal{C}}(\mathcal{E})/\mathcal{C}D$. Usually, we choose a canonical representative of this coset — the vertical derivation which was proved to be an evolutionary one. But in some cases it is more convenient to choose other representatives.

case of (6.164), a supersymmetric extension of (6.161) by two odd variables φ, ψ is given by

$$\begin{aligned} u_t &= -v_2 + kv(u^2+v^2) + f_1[u,v,\varphi,\psi], \\ v_t &= u_2 + ku(u^2+v^2) + f_2[u,v,\varphi,\psi], \\ \varphi_t &= f_3[u,v,\varphi,\psi], \\ \psi_t &= f_4[u,v,\varphi,\psi], \end{aligned} \tag{6.167}$$

where f_1, f_2 are functions in of degree 3 and f_3, f_4 are functions of degree 5/2. Expressing these functions into all possible terms of appropriate degree requires the introduction of **72** constants.

Moreover, basic Principle 2 requires the existence of a vector field

$$X_3 = g_1[u,v,\varphi,\psi]\frac{\partial}{\partial u} + g_2[u,v,\varphi,\psi]\frac{\partial}{\partial v} + g_3[u,v,\varphi,\psi]\frac{\partial}{\partial \varphi} + g_4[u,v,\varphi,\psi]\frac{\partial}{\partial \psi} \tag{6.168}$$

of degree 3 (i.e., g_1, g_2 and g_3, g_4 have to be functions of degree 4 and 7/2, respectively) which is a symmetry of (6.167) and, in the absence of odd variables, reduces to

$$\bar{X}_3 = (u_3 - 3k(u^2+v^2)u_1)\frac{\partial}{\partial u} + (v_3 - 3k(u^2+v^2)v_1)\frac{\partial}{\partial v}, \tag{6.169}$$

the classical third order symmetry of (6.161). The condition that (6.168) is a higher order symmetry of (6.167) gives rise to a large number of equations for both the **72** constants determining (6.167) and the **186** constants determining (6.168). Solving this system of equations leads to the following theorem.

THEOREM 6.37. *The NLS equation* (6.161) *admits two supersymmetric extensions satisfying the basic Principles* 1 *and* 2. *These systems are*:

Case A. *The supersymmetric equation in this case is given by*

$$\begin{aligned} u_t &= -v_2 + kv(u^2+v^2) + 4ku_1\varphi\psi - 4kv(\varphi\varphi_1 + \psi\psi_1), \\ v_t &= u_2 - ku(u^2+v^2) + 4kv_1\varphi\psi + 4ku(\varphi\varphi_1 + \psi\psi_1), \\ \varphi_t &= -\psi_2 + k(u^2+v^2)\psi + 4k\varphi\psi\varphi_1, \\ \psi_t &= \varphi_2 - k(u^2+v^2)\varphi + 4k\varphi\psi\psi_1 \end{aligned} \tag{6.170}$$

with a third order symmetry

$$\begin{aligned} X_3 = \Big(&u_3 - 3ku_1(u^2+v^2) + 6kv_2\varphi\psi + 3ku_1(\varphi\varphi_1 + \psi\psi_1) \\ &+ 3kv_1(\varphi\psi_1 + \varphi_1\psi) + 3ku(\varphi\varphi_2 + \psi\psi_2) + 3kv(\psi\varphi_2 - \varphi\psi_2) + 6kv\varphi_1\psi_1\Big)\frac{\partial}{\partial u} \\ &+ \Big(v_3 - 3kv_1(u^2+v^2) - 6ku_2\varphi\psi + 3kv_1(\varphi\varphi_1 + \psi\psi_1) - 3ku_1(\varphi\psi_1 + \varphi_1\psi) \\ &+ 3kv(\varphi\varphi_2 + \psi\psi_2) - 3ku(\psi\varphi_2 - \varphi\psi_2) - 6ku\varphi_1\psi_1\Big)\frac{\partial}{\partial v} \end{aligned}$$

$$+\Big(\varphi_3+6k\varphi\psi\psi_2-\frac{3}{2}k(u^2+v^2)\varphi_1+\frac{3}{2}k(uv_1-u_1v)\psi-\frac{3}{2}k(uu_1+vv_1)\varphi\Big)\frac{\partial}{\partial\varphi}$$
$$+\Big(\psi_3-6k\varphi\psi\varphi_2-\frac{3}{2}k(u^2+v^2)\psi_1-\frac{3}{2}k(uv_1-u_1v)\varphi-\frac{3}{2}k(uu_1+vv_1)\psi\Big)\frac{\partial}{\partial\psi}. \tag{6.171}$$

Case B. *The supersymmetric equation in this case is given by*

$$\begin{aligned}
u_t &= -v_2+kv(u^2+v^2)-(c-4k)u_1\varphi\psi-4kv\psi\psi_1-(c+8k)u\psi\varphi_1\\
&\quad+4ku\varphi\psi_1+cv\varphi\varphi_1,\\
v_t &= u_2-ku(u^2+v^2)-(c-4k)v_1\varphi\psi+4ku\varphi\varphi_1+(c+8k)v\varphi\psi_1\\
&\quad-4kv\psi\varphi_1-cu\psi\psi_1,\\
\varphi_t &= -\psi_2+k(3u^2+v^2)\psi-2kuv\varphi+(c-4k)\varphi\psi\varphi_1,\\
\psi_t &= \varphi_2-k(u^2+3v^2)\varphi+2kuv\psi-(c-4k)\varphi\psi\psi_1,
\end{aligned} \tag{6.172}$$

where c is an arbitrary real constant. This system has a third order symmetry

$$\begin{aligned}
X_3 = \Big(&u_3-3ku_1(u^2+v^2)-\frac{3}{2}(c-4k)v_2\varphi\psi+12kv_1(\varphi\psi_1+\varphi_1\psi)\\
&-\frac{3}{2}(c+4k)u\psi\psi_2+\frac{3}{2}(c+4k)v\varphi\psi_2+12kv\varphi_1\psi_1\Big)\frac{\partial}{\partial u}\\
+\Big(&v_3-3kv_1(u^2+v^2)+\frac{3}{2}(c-4k)u_2\varphi\psi-12ku_1(\varphi\psi_1+\varphi_1\psi)\\
&+\frac{3}{2}(c+4k)u\psi\varphi_2-\frac{3}{2}(c+4k)v\varphi\varphi_2-12ku\varphi_1\psi_1\Big)\frac{\partial}{\partial v}\\
+\Big(&\varphi_3-\frac{3}{2}(c-4k)\varphi\psi\psi_2-3k(u^2+v^2)\varphi_1+6kv_1(u\psi-v\varphi)\Big)\frac{\partial}{\partial\varphi}\\
+\Big(&\psi_3+\frac{3}{2}(c-4k)\varphi\psi\varphi_2-3k(u^2+v^2)\psi_1-6ku_1(u\psi-v\varphi)\Big)\frac{\partial}{\partial\psi}.
\end{aligned} \tag{6.173}$$

Equations (6.170) and (6.172) may also be written in complex form. Namely, if we put $q=u+iv$ and $\omega=\varphi+i\psi$, equations (6.170) and (6.172) are easily seen to originate from the complex equation

$$\begin{aligned}
iq_t &= -q_2+k(q^*q)q-2kq(\omega^*\omega_1+\omega\omega_1^*)+c_2q(\omega^*\omega_1-\omega\omega_1^*)\\
&\quad+(c_1+2k)(q\omega^*-q^*\omega)\omega_1+(c_1-c_2)q_1\omega\omega^*,\\
i\omega_t &= -\omega_2+k(q^*q)\omega+\frac{1}{2}c_2q(q^*\omega-q\omega^*)+(c_1-c_2)\omega\omega^*\omega_1,
\end{aligned} \tag{6.174}$$

where c_1, c_2 are arbitrary complex constants.

Now from (6.174), equation (6.170) can be obtained by putting $c_1=-4k$ and $c_2=0$, while equation (6.172) can be obtained by putting $c_1=c$, $c_2=4k$.

Hence we have found two supersymmetric extensions of the classical NLS equation, one of them containing a free parameter. We shall discuss symmetries of these systems in subsequent subsections.

8.2. Symmetries and conserved quantities. Let us now describe symmetries and conserved densities of equation (6.174).

8.2.1. *Case A.* In this section we shall discuss symmetries, supersymmetries, recursion symmetries and conservation laws for case A, i.e., the supersymmetric extension of the NLS given by equation (6.170).

We searched for higher or generalized local symmetries of this system and obtained the following result.

THEOREM 6.38. *The local generalized (x,t)-independent symmetries of degree ≤ 3 of equation* (6.170) *are given by*

$$
\begin{aligned}
X_0 &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},\\
\bar{X}_0 &= \psi\frac{\partial}{\partial \varphi} - \varphi\frac{\partial}{\partial \psi},\\
Y_{\frac{1}{2}} &= -\psi_1\frac{\partial}{\partial u} + \varphi_1\frac{\partial}{\partial v} + \frac{1}{2}v\frac{\partial}{\partial \varphi} - \frac{1}{2}u\frac{\partial}{\partial \psi},\\
\bar{Y}_{\frac{1}{2}} &= \varphi_1\frac{\partial}{\partial u} + \psi_1\frac{\partial}{\partial v} + \frac{1}{2}u\frac{\partial}{\partial \varphi} + \frac{1}{2}v\frac{\partial}{\partial \psi},\\
X_1 &= u_1\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v} + \varphi_1\frac{\partial}{\partial \varphi} + \psi_1\frac{\partial}{\partial \psi},\\
X_2 &\doteq \frac{\partial}{\partial t},
\end{aligned}
\tag{6.175}
$$

together with X_3 as given by (6.171).

Similarly we obtained the following conserved quantities and conservation laws

THEOREM 6.39. *All local conserved quantities of degree ≤ 2 of system* (6.170) *are given by*

$$
\begin{aligned}
P_0 &= \int_{-\infty}^{\infty} \varphi\psi\,dx,\\
Q_{\frac{1}{2}} &= \int_{-\infty}^{\infty} (u\psi - v\varphi)\,dx,\\
\bar{Q}_{\frac{1}{2}} &= \int_{-\infty}^{\infty} (u\varphi + v\psi)\,dx,\\
P_1 &= \int_{-\infty}^{\infty} (u^2 + v^2 - 2\varphi\varphi_1 - 2\psi\psi_1)\,dx,\\
P_2 &= \int_{-\infty}^{\infty} (uv_1 + 2\varphi_1\psi_1)\,dx,
\end{aligned}
\tag{6.176}
$$

with the associated conservation laws

$$
\begin{aligned}
p_{0,x} &= \varphi\psi,\\
p_{0,t} &= \varphi\varphi_1 + \psi\psi_1,
\end{aligned}
$$

$$
\begin{aligned}
q_{\frac{1}{2},x} &= u\psi - v\varphi,\\
q_{\frac{1}{2},t} &= u\varphi_1 + v\psi_1 - u_1\varphi - v_1\psi,\\
\bar{q}_{\frac{1}{2},x} &= u\varphi + v\psi,\\
\bar{q}_{\frac{1}{2},t} &= -u\psi_1 + v\varphi_1 + u_1\psi - v_1\varphi,\\
p_{1,x} &= u^2 + v^2 - 2\varphi\varphi_1 - 2\psi\psi_1,\\
p_{1,t} &= 2u_1v - 2uv_1 - 4\varphi_1\psi_1 + 2\varphi_2\psi + 2\varphi\psi_2,\\
p_{2,x} &= uv_1 + 2\varphi_1\psi_1,\\
p_{2,t} &= u_2u - \frac{1}{2}(u_1^2 + v_1^2) + \frac{1}{4}k(v^4 - 2u^2v^2 - 3u^4) + 2(\psi_1\psi_2 + \varphi_1\varphi_2)\\
&\quad + 4ku^2(\varphi\varphi_1 + \psi\psi_1) + 4kuv_1\varphi\psi + 8k\varphi\psi\varphi_1\psi_1. \qquad (6.177)
\end{aligned}
$$

From the conservation laws given in Theorem 6.39 we can introduce nonlocal variables by formally defining

$$
\begin{aligned}
p_0 &= D_x^{-1}p_{0,x},\\
q_{\frac{1}{2}} &= D_x^{-1}q_{\frac{1}{2},x},\\
\bar{q}_{\frac{1}{2}} &= D_x^{-1}\bar{q}_{\frac{1}{2},x},\\
p_1 &= D_x^{-1}p_{1,x},\\
p_2 &= D_x^{-1}p_{2,x}. \qquad (6.178)
\end{aligned}
$$

Using these nonlocal variables one can try to find a nonlocal generalized symmetry, which might be used in the construction of an infinite hierarchy of symmetries and conserved quantities for (6.170). From the associated computations we arrive at the following theorem.

THEOREM 6.40. *The supersymmetric NLS equation given by* (6.170) *admits a nonlocal symmetry of degree* 1 *of the form*

$$
\begin{aligned}
Z_1 &= q_{\frac{1}{2}}\left(-\psi_1\frac{\partial}{\partial u} + \varphi_1\frac{\partial}{\partial v} + \frac{1}{2}\frac{\partial}{\partial\varphi} - \frac{1}{2}\frac{\partial}{\partial\psi}\right)\\
&\quad + \bar{q}_{\frac{1}{2}}\left(\varphi_1\frac{\partial}{\partial u} + \psi_1\frac{\partial}{\partial v} + \frac{1}{2}\frac{\partial}{\partial\varphi} + \frac{1}{2}\frac{\partial}{\partial\psi}\right)\\
&\quad - 2v\varphi\psi\frac{\partial}{\partial u} + 2u\varphi\psi\frac{\partial}{\partial v} + k^{-1}\varphi_1\frac{\partial}{\partial\varphi} + k^{-1}\psi_1\frac{\partial}{\partial\psi}\\
&= q_{\frac{1}{2}}Y_{\frac{1}{2}} - \bar{q}_{\frac{1}{2}}\bar{Y}_{\frac{1}{2}} + B \qquad (6.179)
\end{aligned}
$$

where B is given by

$$
B = -2v\varphi\psi\frac{\partial}{\partial u} + 2u\varphi\psi\frac{\partial}{\partial v} + k^{-1}\varphi_1\frac{\partial}{\partial\varphi} + k^{-1}\psi_1\frac{\partial}{\partial\psi}. \qquad (6.180)
$$

The existence of the symmetry Z_1 of the form (6.179) should be compared with the existence of a similar symmetry for the supersymmetric KdV equation, considered in the previous Sections 4.2 and 6. It should be noted that relation (6.179) just holds for the $\partial/\partial u$-, $\partial/\partial v$-, $\partial/\partial\varphi$- and $\partial/\partial\psi$-components. Starting from (6.175) and (6.179), we can construct new symmetries of (6.170) by using the graded commutator of vector fields

$$[X,Y]=X\circ Y-(-1)^{|X|\cdot|Y|}Y\circ X.$$

Computing the commutators of (6.175) we get the identities

$$\begin{aligned}
&[Y_{\frac12},Y_{\frac12}]=X_1, && [\bar Y_{\frac12},\bar Y_{\frac12}]=X_1,\\
&[X_0,Y_{\frac12}]=-\bar Y_{\frac12}, && [X_0,\bar Y_{\frac12}]=Y_{\frac12},\\
&[\bar X_0,Y_{\frac12}]=\bar Y_{\frac12}, && [\bar X_0,\bar Y_{\frac12}]=-Y_{\frac12},
\end{aligned}\tag{6.181}$$

all other commutators of (6.175) being zero.

In order to compute the commutators $[Z_1,Y_{\frac12}]$ and $[Z_1,\bar Y_{\frac12}]$, we are forced to compute the prolongations of the vector fields $Y_{\frac12},\bar Y_{\frac12}$ towards the nonlocal variables $q_{\frac12}$ and $\bar q_{\frac12}$. In other words we have to compute the $\partial/\partial q_{\frac12}$- and $\partial/\partial\bar q_{\frac12}$-components of the vector field $Y_{\frac12}$ and $\bar Y_{\frac12}$. These components can be obtained by requiring the invariance of $q_{\frac12,x}$ and $\bar q_{\frac12,x}$, i.e.,

$$\begin{aligned}
D_x(Y_{\frac12}^{q_{\frac12}})&=Y_{\frac12}(q_{\frac12,x})=Y_{\frac12}(u\psi-v\varphi)=Y_{\frac12}^u\psi+uY_{\frac12}^\psi-Y_{\frac12}^v\varphi-vY_{\frac12}^\varphi\\
D_x(Y_{\frac12}^{\bar q_{\frac12}})&=Y_{\frac12}(\bar q_{\frac12,x})=Y_{\frac12}(u\varphi+v\psi)=Y_{\frac12}^u\varphi+uY_{\frac12}^\varphi+Y_{\frac12}^v\psi+vY_{\frac12}^\psi
\end{aligned}\tag{6.182}$$

where $Y_{\frac12}^{q_{\frac12}}$ and $Y_{\frac12}^{\bar q_{\frac12}}$ are the $\partial/\partial q_{\frac12}$- and $\partial/\partial\bar q_{\frac12}$-components of $Y_{\frac12}$. Similar relations hold for $\bar Y_{\frac12}^{q_{\frac12}}$ and $\bar Y_{\frac12}^{\bar q_{\frac12}}$.

A straightforward computation yields

$$\begin{aligned}
&Y_{\frac12}^{q_{\frac12}}=-\frac12 p_1, && \bar Y_{\frac12}^{q_{\frac12}}=\varphi\psi,\\
&Y_{\frac12}^{\bar q_{\frac12}}=\varphi\psi, && \bar Y_{\frac12}^{\bar q_{\frac12}}=\frac12 p_1,\\
&Y_{\frac12}^{p_1}=-(u\psi-v\varphi), && \bar Y_{\frac12}^{p_1}=u\varphi+v\psi.
\end{aligned}\tag{6.183}$$

Now the commutators $[Z_1,Y_{\frac12}]$ and $[Z_1,\bar Y_{\frac12}]$ give the following results:

$$\begin{aligned}
Y_{\frac32}=[Z_1,Y_{\frac12}]&=q_{\frac12}X_1+\frac12 p_1Y_{\frac12}\\
&+\Big(-k^{-1}\psi_2+\frac12(u^2+3v^2)\psi+3\varphi\psi\varphi_1+uv\varphi\Big)\frac{\partial}{\partial u}\\
&+\Big(k^{-1}\varphi_2+\frac12(3u^2+v^2)\varphi+3\varphi\psi\psi_1-uv\psi\Big)\frac{\partial}{\partial v}\\
&+\Big(-\frac12k^{-1}v_1+\frac32u\varphi\psi\Big)\frac{\partial}{\partial\varphi}+\Big(\frac12k^{-1}u_1+\frac32v\varphi\psi\Big)\frac{\partial}{\partial\psi},\\
\bar Y_{\frac32}=[Z_1,\bar Y_{\frac12}]&=-\bar q_{\frac12}X_1+\frac12 p_1\bar Y_{\frac12}
\end{aligned}$$

$$+\Big(k^{-1}\psi_2 - \frac{1}{2}(u^2+3v^2)\varphi + 3\varphi\psi\psi_1 + uv\psi\Big)\frac{\partial}{\partial u}$$
$$+\Big(k^{-1}\psi_2 - \frac{1}{2}(3u^2+v^2)\varphi - 3\varphi\psi\varphi_1 - uv\varphi\Big)\frac{\partial}{\partial v}$$
$$+\Big(-\frac{1}{2}k^{-1}u_1 - \frac{3}{2}v\varphi\psi\Big)\frac{\partial}{\partial\varphi} + \Big(\frac{1}{2}k^{-1}v_1 + \frac{3}{2}u\varphi\psi\Big)\frac{\partial}{\partial\psi}, \tag{6.184}$$

i.e., $Y_{\frac{3}{2}}$ and $\bar{Y}_{\frac{3}{2}}$ are two new higher order supersymmetries of (6.170). In effect, we are here considering the supersymmetric NLS equation in the graded Abelian covering by the variables $p_1, q_{\frac{1}{2}}, \bar{q}_{\frac{1}{2}}$, where the following system of differential equations holds

$$\begin{aligned}
u_t &= -v_2 + kv(u^2+v^2) + 4ku_1\varphi\psi - 4kv(\varphi\varphi_1 + \psi\psi_1),\\
v_t &= u_2 - ku(u^2+v^2) + 4kv_1\varphi\psi + 4ku(\varphi\varphi_1 + \psi\psi_1),\\
\varphi_t &= -\psi_2 + k(u^2+v^2)\psi + 4k\varphi\psi\varphi_1,\\
\psi_t &= \varphi_2 - k(u^2+v^2)\varphi + 4k\varphi\psi\psi_1,\\
q_{\frac{1}{2},x} &= u\psi - v\varphi,\\
q_{\frac{1}{2},t} &= u\varphi_1 + v\psi_1 - u_1\varphi - v_1\psi,\\
\bar{q}_{\frac{1}{2},x} &= u\varphi + v\psi,\\
\bar{q}_{\frac{1}{2},t} &= -u\psi_1 + v\varphi_1 + u_1\psi - v_1\varphi,\\
p_{1,x} &= u^2 + v^2 - 2\varphi\varphi_1 - 2\psi\psi_1,\\
p_{1,t} &= 2u_1v - 2uv_1 - 4\varphi_1\psi_1 + 2\varphi_2\psi + 2\varphi\psi_2.
\end{aligned} \tag{6.185}$$

We are now able to prove the following lemma.

LEMMA 6.41. *By defining*

$$\begin{aligned}
Y_{n+\frac{1}{2}} &= [Z_1, Y_{n-\frac{1}{2}}],\\
\bar{Y}_{n+\frac{1}{2}} &= [Z_1, \bar{Y}_{n-\frac{1}{2}}],
\end{aligned} \tag{6.186}$$

$n = 1, 2, \ldots,$ *we obtain two infinite hierarchies of nonlocal supersymmetries of equation* (6.170).

PROOF. First of all note, that the vector field $\partial/\partial p_1$ is a nonlocal symmetry of (6.170) and an easy computation shows that

$$\begin{aligned}
[\frac{\partial}{\partial p_1}, Z_1] &= 0,\\
[\frac{\partial}{\partial p_1}, Y_{\frac{3}{2}}] &= \frac{1}{2}Y_{\frac{1}{2}},\\
[\frac{\partial}{\partial p_1}, \bar{Y}_{\frac{3}{2}}] &= \frac{1}{2}\bar{Y}_{\frac{1}{2}}.
\end{aligned} \tag{6.187}$$

Secondly, note that by an induction argument and using the Jacobi identity and (6.187) it is easy to prove that

$$[\frac{\partial}{\partial p_1}, Y_{n+\frac{1}{2}}] = [\frac{\partial}{\partial p_1}, [Z_1, Y_{n-\frac{1}{2}}]] = [Z_1, \frac{1}{2}Y_{n-\frac{3}{2}}] = \frac{1}{2}Y_{n-\frac{1}{2}},$$
$$[\frac{\partial}{\partial p_1}, \bar{Y}_{n+\frac{1}{2}}] = \frac{1}{2}\bar{Y}_{n-\frac{1}{2}}. \tag{6.188}$$

From (6.188) it immediately follows that the assumption $Y_{n+\frac{1}{2}} = 0$ leads to the conclusion that also $Y_{n-\frac{1}{2}} = 2[\partial/\partial p_1, Y_{n+\frac{1}{2}}] = 0$, which proves that the hierarchies $\{Y_{n+\frac{1}{2}}\}_{n\in\mathbb{N}}$, $\{\bar{Y}_{n+\frac{1}{2}}\}_{n\in\mathbb{N}}$ are infinite. □

Higher order conservation laws arise in the construction of prolongation of the vector fields $Y_{\frac{1}{2}}$, $\bar{Y}_{\frac{1}{2}}$ and Z_1 towards nonlocal variables, the first of which resulted in (6.183).

In order to compute the $Z_1^{q_{\frac{1}{2}}}$ component of the vector field Z_1 we have to require the invariance of the equation $q_{\frac{1}{2},x} = u\psi - v\varphi$, i.e.,

$$\begin{aligned} D_x(Z_1^{q_{\frac{1}{2}}}) = Z_1(q_{\frac{1}{2},x}) &= q_{\frac{1}{2}} Y_{\frac{1}{2}}(q_{\frac{1}{2},x}) - \bar{q}_{\frac{1}{2}} \bar{Y}_{\frac{1}{2}}(q_{\frac{1}{2},x}) + B(q_{\frac{1}{2},x}) \\ &= q_{\frac{1}{2}}(-\frac{1}{2}p_{1,x}) - \bar{q}_{\frac{1}{2}}(\varphi\psi)_1 + B(q_{\frac{1}{2},x}) \end{aligned}$$

due to (6.177) and (6.179), from which we obtain

$$\begin{aligned} Z_1^{q_{\frac{1}{2}}} &= -\frac{1}{2}p_1 q_{\frac{1}{2}} - \bar{q}_{\frac{1}{2}}(\varphi\psi) + k^{-1}(u\psi - v\varphi) \\ &\quad + \int_{-\infty}^{x} \Big(\frac{1}{2}p_1(u\psi - v\varphi) - k^{-1}(u_1\psi - v_1\varphi)\Big)\, dx \end{aligned} \tag{6.189}$$

and in a similar way

$$\begin{aligned} D_x(Z_1^{\bar{q}_{\frac{1}{2}}}) = Z_1(\bar{q}_{\frac{1}{2},x}) &= q_{\frac{1}{2}} Y_{\frac{1}{2}}(\bar{q}_{\frac{1}{2},x}) - \bar{q}_{\frac{1}{2}} \bar{Y}_{\frac{1}{2}}(\bar{q}_{\frac{1}{2},x}) + B(\bar{q}_{\frac{1}{2},x}) \\ &= q_{\frac{1}{2}}(\varphi\psi)_1 - \bar{q}_{\frac{1}{2}}(\frac{1}{2}p_{1,x}) + B(\bar{q}_{\frac{1}{2},x}), \end{aligned}$$

yielding

$$\begin{aligned} Z_1^{\bar{q}_{\frac{1}{2}}} &= q_{\frac{1}{2}}(\varphi\psi) - \frac{1}{2}p_1\bar{q}_{\frac{1}{2}} + k^{-1}(u\varphi + v\psi) \\ &\quad + \int_{-\infty}^{x} \Big(\frac{1}{2}p_1(u\varphi + v\psi) - k^{-1}(u_1\varphi + v_1\psi))\Big)\, dx. \end{aligned} \tag{6.190}$$

So the prolongation of Z_1 towards the nonlocal variables $q_{\frac{1}{2}}$, $\bar{q}_{\frac{1}{2}}$ requires the introduction of two additional nonlocal variables

$$\begin{aligned} q_{\frac{3}{2}} &= \int_{-\infty}^{x} \Big(\frac{1}{2}p_1(u\psi - v\varphi) - k^{-1}(u_1\psi - v_1\varphi)\Big)\, dx, \\ \bar{q}_{\frac{3}{2}} &= \int_{-\infty}^{x} \Big(\frac{1}{2}p_1(u\varphi + v\psi) - k^{-1}(u_1\varphi + v_1\psi)\Big)\, dx. \end{aligned} \tag{6.191}$$

It is a straightforward check that $q_{\frac{3}{2}}$, $\bar{q}_{\frac{3}{2}}$ are associated to nonlocal conserved quantities $Q_{\frac{3}{2}}$, $\bar{Q}_{\frac{3}{2}}$. Thus we have found two new nonlocal variables $q_{\frac{3}{2}}$ and $\bar{q}_{\frac{3}{2}}$ with

$$\begin{aligned}
q_{\frac{3}{2},x} &= \frac{1}{2}p_1(u\psi - v\varphi) - k^{-1}(u_1\psi - v_1\varphi),\\
\bar{q}_{\frac{3}{2},x} &= \frac{1}{2}p_1(u\varphi + v\psi) - k^{-1}(u_1\varphi + v_1\psi).
\end{aligned} \tag{6.192}$$

From this we proceed to construct the nonlocal components of $Y_{\frac{1}{2}}$ and $\bar{Y}_{\frac{1}{2}}$ with respect to $q_{\frac{3}{2}}$, $\bar{q}_{\frac{3}{2}}$, which can be obtained by requiring the invariance of $q_{\frac{3}{2},x}$ and $\bar{q}_{\frac{3}{2},x}$.

In this way we find

$$\begin{aligned}
D_x(Y_{\frac{1}{2}}^{q_{\frac{3}{2}}}) &= Y_{\frac{1}{2}}(q_{\frac{3}{2},x})\\
&= \frac{1}{2}Y_{\frac{1}{2}}(p_1)(u\psi - v\varphi) + \frac{1}{2}p_1 Y_{\frac{1}{2}}(u\psi - v\varphi)\\
&\quad - k^{-1}Y_{\frac{1}{2}}(u_1\psi - v_1\varphi)\\
&= \frac{1}{2}p_1(-\psi_1\psi - \frac{1}{2}u^2 - \varphi_1\varphi - \frac{1}{2}v^2)\\
&\quad - k^{-1}(-\psi_2\psi - \frac{1}{2}uu_1 - \varphi_2\varphi - \frac{1}{2}vv_1)
\end{aligned} \tag{6.193}$$

yielding

$$Y_{\frac{1}{2}}^{q_{\frac{3}{2}}} = \frac{1}{8}p_1^2 - \frac{1}{4}k^{-1}(u^2 + v^2) + 4(\varphi\varphi_1 + \psi\psi_1) \tag{6.194}$$

In similar way we find

$$\begin{aligned}
Y_{\frac{1}{2}}^{\bar{q}_{\frac{3}{2}}} &= \frac{1}{2}p_1\varphi\psi - k^{-1}(\varphi_1\psi + \varphi\psi_1 + \frac{1}{2}uv) - k^{-1}\int_{-\infty}^{x}(uv_1 + 2\varphi_1\psi_1)\,dx,\\
\bar{Y}_{\frac{1}{2}}^{q_{\frac{3}{2}}} &= \frac{1}{2}p_1\varphi\psi - k^{-1}(\varphi_1\psi + \varphi\psi_1 + \frac{1}{2}uv) - k^{-1}\int_{-\infty}^{x}(uv_1 + 2\varphi_1\psi_1)\,dx,\\
\bar{Y}_{\frac{1}{2}}^{\bar{q}_{\frac{3}{2}}} &= \frac{1}{8}p_1^2 - \frac{1}{4}k^{-1}(u^2 + v^2) - 4(\varphi\varphi_1 + \psi\psi_1).
\end{aligned} \tag{6.195}$$

Hence we see from (6.193) that the computation of the nonlocal components $Y_{\frac{1}{2}}^{q_{\frac{3}{2}}}$ and $\bar{Y}_{\frac{1}{2}}^{\bar{q}_{\frac{3}{2}}}$ requires the introduction of a new nonlocal variable

$$p_2 = \int_{-\infty}^{x}(uv_1 + 2\varphi_1\psi_1)\,dx. \tag{6.196}$$

It is easily verified that p_2 is associated to a conserved quantity P_2. In arriving at the previous results, (6.195), we are working in a covering of the supersymmetric NLS equation with nonlocal variables p_1, $q_{\frac{1}{2}}$, $\bar{q}_{\frac{1}{2}}$, $q_{\frac{3}{2}}$, $\bar{q}_{\frac{3}{2}}$, p_2; i.e., we consider system (6.185), together with the differential equations, defining $q_{\frac{3}{2}}$, $\bar{q}_{\frac{3}{2}}$, p_2.

Summarizing the results obtained so far, we see that the odd potentials $Q_{\frac{1}{2}}$, $\bar{Q}_{\frac{1}{2}}$, $Q_{\frac{3}{2}}$ and $\bar{Q}_{\frac{3}{2}}$ enter in a natural way in the prolongation of Z_1, whereas the even potentials P_1 and P_2 enter in the prolongation of $Y_{\frac{1}{2}}$ and $\bar{Y}_{\frac{1}{2}}$. This situation is similar to that arising in the supersymmetric KdV equation treated in Section 6.

8.2.2. *Case B.* In order to gain insight in the structure of the supersymmetric NLS equation (6.172), we start with the computation of (x,t)-independent conserved quantities of degree ≤ 3. We arrive at the following result.

THEOREM 6.42. *The supersymmetric NLS equation* (6.172) *admits the following set of local even and odd conserved quantities of degree* ≤ 3:

$$
\begin{aligned}
P_0 &= \int_{-\infty}^{\infty} \varphi\psi \, dx,\\
Q_{\frac{1}{2}} &= \int_{-\infty}^{\infty} (u\psi - v\varphi)\, dx,\\
P_1 &= \int_{-\infty}^{\infty} \frac{1}{2}k^{-1}\Big((c+4k)(\varphi\varphi_1 + \psi\psi_1) + 2k(u^2+v^2)\Big)\, dx,\\
Q_{\frac{3}{2}} &= \int_{-\infty}^{\infty} (u\varphi_1 + v\psi_1)\, dx,\\
P_2 &= \int_{-\infty}^{\infty} \frac{1}{4}k^{-1}\Big((c+4k)\varphi_1\psi_1 + (c+12k)k(u^2+v^2)\varphi\psi - 4kuv_1\Big)\, dx,\\
Q_{\frac{5}{2}} &= \int_{-\infty}^{\infty} -k^{-1}\Big(u\psi_2 - v\varphi_2\\
&\quad - k(u^2+v^2)(u\psi - v\varphi) - 4k\varphi\psi(u\varphi_1 + v\psi_1)\Big)\, dx.
\end{aligned}
\tag{6.197}
$$

Moreover, in the case where $c = -4k$ *we have an additional local conserved quantity of degree* 3 *given by*

$$
\begin{aligned}
P_3 = \int_{-\infty}^{\infty} \Big(&16uv(\varphi\psi_1 - \psi\varphi_1)\\
&+ 32uv\varphi\psi + (u^2+v^2)^2 - 2k^{-1}(uu_2 + vv_2)\Big)\, dx.
\end{aligned}
\tag{6.198}
$$

Motivated by the nonlocal results in case A, we introduce the nonlocal variables p_0, $q_{\frac{1}{2}}$, p_1, $q_{\frac{3}{2}}$, p_2 and $q_{\frac{5}{2}}$ as formal integrals associated to the conserved quantities given in (6.197).

Including these new nonlocal variables in our computations, we get an additional set of *nonlocal* conserved quantities

$$
\begin{aligned}
\bar{Q}_{\frac{1}{2}} &= \int_{-\infty}^{\infty} \frac{1}{2}\Big(2q_{\frac{3}{2}} + (c+4k)\varphi\psi q_{\frac{1}{2}}\Big)\, dx,\\
\bar{P}_1 &= \int_{-\infty}^{\infty} \frac{1}{2}k^{-1}\Big(2k(u\psi - v\varphi)q_{\frac{1}{2}} - (\varphi\varphi_1 + \psi\psi_1)\Big)\, dx,
\end{aligned}
$$

$$\bar{P}_2 = \int_{-\infty}^{\infty} -\frac{1}{2}k^{-1}\Big(2k(u\varphi_1 + v\psi_1)q_{\frac{1}{2}} - \varphi_1\psi_1\Big)\,dx, \tag{6.199}$$

as well as an additional conserved quantity in the case $c = -4k$, namely

$$\bar{P}_0 = \int_{-\infty}^{\infty} p_1\,dx.$$

This situation can be described as higher nonlocalities, or covering of a covering, and as it will be shown lead to new interesting results.

REMARK 6.15. The results (6.197), (6.199) indicate the existence of a double hierarchy of odd conserved quantities $\{Q_{n+\frac{1}{2}}\}_{n\in\mathbb{N}}$ as well as a double hierarchy of even conserved quantities $\{P_n\}_{n\in\mathbb{N}}$.

In order to obtain any further results, we also need the conserved quantity $Q_{\frac{7}{2}}$ of degree $7/2$ which is given by

$$\begin{aligned} Q_{\frac{7}{2}} = \int_{-\infty}^{\infty} \frac{1}{6}k^{-1}\Big(&2u\varphi_3 + 2v\psi_3 - 2kv^3\psi_1 - 2ku^3\varphi_1 - 6kuv^2\varphi_1 + 6ku^2v_1\psi \\ &- 12kuvv_1\varphi + 2c(u\psi - v\varphi)\varphi_1\psi_1 - (c-12k)\varphi\psi(u\psi_2 - v\varphi_2)\Big)\,dx. \end{aligned} \tag{6.200}$$

Let us stress that now, *by the introduction of the nonlocal variables* $\bar{q}_{\frac{1}{2}}$, $\bar{p}_1$, $\bar{p}_2$ *and* $q_{\frac{7}{2}}$, *associated to the appropriate conserved quantities, we are able to remove the condition* $c = -4k$ *on the existence of the conserved quantities* P_3 *and* $\bar{P}_0$. *By also including* $\bar{q}_{\frac{1}{2}}$, $\bar{p}_1$, $\bar{p}_2$ *and* $q_{\frac{7}{2}}$ *in our computations, we find four additional conserved quantities given by*

$$\begin{aligned} \bar{P}_0 &= \int_{-\infty}^{\infty} \Big(p_1 + (c+4k)\bar{p}_1\Big)\,dx, \\ \bar{Q}_{\frac{3}{2}} &= \int_{-\infty}^{\infty} -\frac{1}{2}k^{-1}\Big(2kq_{\frac{5}{2}} + 2k(u^2+v^2)q_{\frac{1}{2}} + (\varphi\varphi_1 + \psi\psi_1)q_{\frac{1}{2}}\Big)\,dx, \\ P_3 &= \int_{-\infty}^{\infty} k^{-1}\Big(2k(c+4k)(u\psi - v\varphi)q_{\frac{5}{2}} + 2(c+4k)(u^2\psi\psi_1 + v^2\varphi\varphi_1) \\ &\quad - 2(c+12k)uv\psi\varphi_1 - 2(c-4k)uv\varphi\psi_1 \\ &\quad + 32ku_1v\varphi\psi + k(u^2+v^2)^2 - 2(uu_2 + vv_2)\Big)\,dx, \\ \bar{P}_3 &= \int_{-\infty}^{\infty} \frac{1}{2}k^{-1}\Big(4k^2(u\psi - v\varphi)q_{\frac{5}{2}} + 2k(u\varphi_1 + v\psi_1)q_{\frac{3}{2}} \\ &\quad - (\varphi_1\varphi_2 + \psi_1\psi_2) + (c-4k)\varphi\psi\varphi_1\psi_1\Big)\,dx. \end{aligned} \tag{6.201}$$

Note that the first equation in (6.201) and the third one in (6.201) reduce to the second equations in (6.199) and (6.197) respectively under the condition $c = -4k$. Furthermore, from the computation of the t-component of the conservation law $\bar{q}_{\frac{3}{2}}$ associated to $\bar{Q}_{\frac{3}{2}}$, it becomes apparent why the introduction of the nonlocal variable $q_{\frac{7}{2}}$ is required in its construction.

So P_3 is just an ordinary conserved quantity of this supersymmetric extension; for $c=-4k$ it is just a local conserved quantity, while for other values of c it is a nonlocal one.

We now turn to the construction of the Lie algebra of even and odd symmetries for the supersymmetric NLS equation (6.172). According to the introduction of the nonlocal variables associated to the conserved quantities obtained earlier in this section we find the following result.

THEOREM 6.43. *The supersymmetric NLS equation* (6.172) *admits the following set of even and odd symmetries of degree ≤ 2. The symmetries of degree* 0 *are given by*

$$\begin{aligned} X_0 &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} + \psi\frac{\partial}{\partial \varphi} - \varphi\frac{\partial}{\partial \psi}, \\ \bar{X}_0 &= \psi q_{\frac{1}{2}}\frac{\partial}{\partial u} - \varphi q_{\frac{1}{2}}\frac{\partial}{\partial v} - \frac{1}{8}k^{-1}\varphi\frac{\partial}{\partial \varphi} - \frac{1}{8}k^{-1}\psi\frac{\partial}{\partial \varphi}; \end{aligned} \tag{6.202}$$

the symmetries of degree $1/2$ *by*

$$\begin{aligned} Y_{\frac{1}{2}} &= \varphi_1\frac{\partial}{\partial u} + \psi_1\frac{\partial}{\partial v} + \frac{1}{2}u\frac{\partial}{\partial \varphi} + \frac{1}{2}v\frac{\partial}{\partial \psi}, \\ \bar{Y}_{\frac{1}{2}} &= vq_{\frac{1}{2}}\frac{\partial}{\partial u} - uq_{\frac{1}{2}}\frac{\partial}{\partial v} + (q_{\frac{1}{2}}\psi - \frac{1}{4}k^{-1}u)\frac{\partial}{\partial \varphi} \\ &\quad + (-q_{\frac{1}{2}}\varphi - \frac{1}{4}k^{-1}v)\frac{\partial}{\partial \psi} \end{aligned} \tag{6.203}$$

the symmetries of degree 1 *by*

$$\begin{aligned} X_1 &= u_1\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v} + \varphi_1\frac{\partial}{\partial \varphi} + \psi_1\frac{\partial}{\partial \psi} \\ \bar{X}_1 &= \Big((c+4k)(\varphi_1 q_{\frac{1}{2}} + \psi q_{\frac{3}{2}}) + 2k(c-4k)\bar{p}_1 v\Big)\frac{\partial}{\partial u} \\ &\quad + \Big((c+4k)(\psi_1 q_{\frac{1}{2}} - \varphi q_{\frac{3}{2}}) - 2k(c-4k)\bar{p}_1 u\Big)\frac{\partial}{\partial v} \\ &\quad + \Big(-4kuq_{\frac{1}{2}} + 2\psi_1 + 2k(c-4k)\bar{p}_1\psi\Big)\frac{\partial}{\partial \varphi} \\ &\quad + \Big(-4kvq_{\frac{1}{2}} - 2\varphi_1 - 2k(c-4k)\bar{p}_1\varphi\Big)\frac{\partial}{\partial \psi}; \end{aligned} \tag{6.204}$$

the symmetries of degree $3/2$ *by*

$$\begin{aligned} Y_{\frac{3}{2}} &= \Big(vq_{\frac{3}{2}} + u_1 q_{\frac{1}{2}} - \frac{1}{2}k^{-1}\psi_2 + u^2\psi - uv\varphi - \frac{1}{2}k^{-1}c\varphi\psi\varphi_1\Big)\frac{\partial}{\partial u} \\ &\quad + \Big(-uq_{\frac{3}{2}} + v_1 q_{\frac{1}{2}} + \frac{1}{2}k^{-1}\varphi_2 - v^2\varphi + uv\psi - \frac{1}{2}k^{-1}c\varphi\psi\psi_1\Big)\frac{\partial}{\partial v} \\ &\quad + \Big(q_{\frac{3}{2}}\psi + q_{\frac{1}{2}}\varphi_1\Big)\frac{\partial}{\partial \psi} + \Big(-q_{\frac{3}{2}}\varphi + q_{\frac{1}{2}}\psi_1\Big)\frac{\partial}{\partial \psi}, \\ \bar{Y}_{\frac{3}{2}} &= \Big(4kcu_1 q_{\frac{1}{2}} - (c-4k)(\psi_2 + c\varphi\psi\varphi_1) + k(c-12k)(u^2+v^2)\psi\Big)\frac{\partial}{\partial u} \end{aligned}$$

$$
\begin{aligned}
&+ \Big(4kcv_1 q_{\frac{1}{2}} + (c-4k)(\varphi_2 - c\varphi\psi\psi_1) - k(c-12k)(u^2+v^2)\varphi\Big)\frac{\partial}{\partial u} \\
&+ \Big(4kcq_{\frac{1}{2}}\varphi_1 + 2k(c-12k)u\varphi\psi + 4kv_1\Big)\frac{\partial}{\partial \varphi} \\
&+ \Big(4kcq_{\frac{1}{2}}\psi_1 + 2k(c-12k)v\varphi\psi - 4ku_1\Big)\frac{\partial}{\partial \psi},
\end{aligned}
\tag{6.205}
$$

and finally the symmetries of degree 2 by

$$
\begin{aligned}
X_2 = {} & \Big(v_2 - kv(u^2+v^2) + (c-4k)u_1\varphi\psi + 4k(v\psi\psi_1 - u\varphi\psi_1) \\
&+ (c+8k)u\psi\varphi_1 - cv\varphi\varphi_1\Big)\frac{\partial}{\partial u} \\
&+ \Big(-u_2 + kv(u^2+v^2) + (c-4k)v_1\varphi\psi - 4k(u\varphi\varphi_1 - v\psi\varphi_1) \\
&- (c+8k)v\varphi\psi_1 + cu\varphi\varphi_1\Big)\frac{\partial}{\partial v} \\
&+ \Big(\psi_2 - k(3u^2+v^2)\psi + (c-4k)\varphi\psi\varphi_1 + 2kuv\varphi\Big)\frac{\partial}{\partial \varphi} \\
&+ \Big(-\varphi_2 - k(u^2+3v^2)\varphi + (c-4k)\varphi\psi\psi_1 - 2kuv\psi\Big)\frac{\partial}{\partial \psi}, \\
\bar{X}_2 = {} & \Big((c+4k)(-k\psi q_{\frac{5}{2}} + \psi_2 q_{\frac{1}{2}} - 3ku^2\psi q_{\frac{1}{2}} + c\varphi\psi\varphi_1 q_{\frac{1}{2}}) \\
&+ (c-4k)(4k\bar{p}_2 v + v\psi\varphi_1 - v\varphi\psi_1) \\
&+ 16k^2 v q_{\frac{1}{2}} q_{\frac{3}{2}} + (c-12k)kv^2\psi q_{\frac{1}{2}} + 4ckuv\varphi q_{\frac{1}{2}}\Big)\frac{\partial}{\partial u} \\
&+ \Big((c+4k)(k\varphi q_{\frac{5}{2}} - \varphi_2 q_{\frac{1}{2}} + 3kv^2\varphi q_{\frac{1}{2}} + c\varphi\psi\psi_1 q_{\frac{1}{2}}) \\
&+ (c-4k)(-4k\bar{p}_2 u - u\psi\varphi_1 + u\varphi\psi_1) \\
&- 16k^2 u q_{\frac{1}{2}} q_{\frac{3}{2}} - (c-12k)ku^2\varphi q_{\frac{1}{2}} - 4ckuv\psi q_{12}\Big)\frac{\partial}{\partial v} \\
&+ \Big(-2\varphi_2 + (c-4k)(4k\bar{p}_2\psi + 2\varphi\psi\psi_1) - 2(c-12k)ku\varphi\psi q_{\frac{1}{2}} \\
&+ (4ku + 16k^2\psi q_{\frac{1}{2}})q_{\frac{3}{2}} - 4kv_1 q_{\frac{1}{2}} - 4kuv\psi + 4kv^2\varphi\Big)\frac{\partial}{\partial \varphi} \\
&+ \Big(-2\psi_2 - (c-4k)(4k\bar{p}_2\varphi + 2\varphi\psi\varphi_1) - 2(c-12k)kv\varphi\psi q_{\frac{1}{2}} \\
&+ (4kv - 16k^2\varphi q_{\frac{1}{2}})q_{\frac{3}{2}} + 4ku_1 q_{\frac{1}{2}} - 4kuv\varphi + 4kv^2\psi\Big)\frac{\partial}{\partial \psi}.
\end{aligned}
\tag{6.206}
$$

Analogously to case A, we have the following result.

THEOREM 6.44. *The nonlocal even symmetry $\bar{X}_1$ given by* (6.204) *acts as a recursion symmetry on the hierarchies of odd symmetries.*

In order to compute the graded commutators $[\bar{X}_1, Y_{\frac{1}{2}}]$ and $[\bar{X}_1, \bar{Y}_{\frac{1}{2}}]$, we have to compute the components of $Y_{\frac{1}{2}}$ and $\bar{Y}_{\frac{1}{2}}$ with respect to the nonlocal variables $q_{\frac{1}{2}}$, $q_{\frac{3}{2}}$ and $\bar{p}_1$. Analogously to the computations in Subsection 8, we find

$$\begin{aligned} Y_{\frac{1}{2}}^{q_{\frac{1}{2}}} &= \varphi\psi, \\ Y_{\frac{1}{2}}^{q_{\frac{3}{2}}} &= \frac{1}{4}(u^2+v^2), \\ Y_{\frac{1}{2}}^{\bar{p}_1} &= \varphi\psi q_{\frac{1}{2}} + \frac{1}{4}k^{-1}(u\varphi+v\psi) - \frac{1}{2}k^{-1}q_{\frac{3}{2}} \end{aligned} \tag{6.207}$$

and

$$\begin{aligned} \bar{Y}_{\frac{1}{2}}^{q_{\frac{1}{2}}} &= 0, \\ \bar{Y}_{\frac{1}{2}}^{q_{\frac{3}{2}}} &= -\frac{1}{8}k^{-1}(u^2+v^2), \\ \bar{Y}_{\frac{1}{2}}^{\bar{p}_1} &= -\frac{1}{8}k^{-2}(u\varphi+v\psi) + \frac{1}{4}k^{-2}q_{\frac{3}{2}}. \end{aligned} \tag{6.208}$$

Moreover, the computation of $[\bar{X}_1, \bar{Y}_{\frac{1}{2}}]$ requires the $\partial/\partial\bar{q}_{\frac{1}{2}}$-component of $\bar{X}_1$ which is given to be

$$\bar{X}_1^{q_{\frac{1}{2}}} = -(c+4k)\varphi\psi q_{\frac{1}{2}} - 2q_{\frac{3}{2}}. \tag{6.209}$$

Now the computation of the commutators leads to

$$\begin{aligned} [\bar{X}_1, Y_{\frac{1}{2}}] &= \frac{1}{2}(c-12k)Y_{\frac{3}{2}} - \frac{1}{4}k^{-1}\bar{Y}_{\frac{3}{2}}, \\ [\bar{X}_1, \bar{Y}_{\frac{1}{2}}] &= (-\frac{1}{4}k^{-1}c+1)Y_{\frac{3}{2}} + \frac{1}{8}k^{-2}\bar{Y}_{\frac{3}{2}}, \end{aligned} \tag{6.210}$$

indicating that $\bar{X}_1$ acts as a recursion operator on the Y, $\bar{Y}$ hierarchies.

It is our conjecture that $\bar{X}_1$ is a Hamiltonian symmetry for equation (6.172). We refer to the concluding remarks for more comments on this issue.

9. Concluding remarks

In the previous sections we proposed a construction for supersymmetric generalizations of the cubic nonlinear Schrödinger equation (6.160) and discussed symmetries, conserved quantities for the resulting interesting cases A and B. In both cases we found an infinite set of (higher order) local and nonlocal symmetries. These facts indicate the complete integrability of both systems.

It is possible to transform the results obtained thus far in the superfield formulation. Namely, if we introduce the odd quantity Φ by

$$\Phi = \omega + \theta q, \tag{6.211}$$

where θ is an additional odd variable, and put

$$D_\theta = \frac{1}{2}\frac{\partial}{\partial\theta} + \theta D_x, \tag{6.212}$$

then

$$[D_\theta, D_\theta] = D_x$$

and it is clear that D_θ corresponds to the supersymmetry $Y_{\frac{1}{2}}$ given by (6.203). Notice that our definition of D_θ differs a factor $\frac{1}{2}$ in the $\partial/\partial\theta$ term. This is caused by our requirement that $[D_\theta, D_\theta] = D_x$, whereas the operator D_θ introduced by Mathieu satisfies $[D_\theta, D_\theta] = 2D_x$. In this setting the general complex equation (6.174) takes the form

$$\begin{aligned} i\Phi_t = -4D_\theta^4\Phi + 2(c_1 - c_2)\Phi\Phi^* D_\theta^2\Phi \\ + 2(c_2 + 2k)\Phi D_\theta \Phi D_\theta \Phi^* - 2c_2\Phi^*(D_\theta\Phi)^2 \end{aligned} \tag{6.213}$$

Our hypothesis is that there exist Hamiltonian structures of the systems of Cases A and B in this setting. Due to the conjecture that the nonlocal recursion symmetry $\bar{X}_1$ given by (6.204) is a Hamiltonian symmetry associated to a linear combination of P_2 and $\bar{P}_2$ we hope to prove the formal construction and the Lie superalgebra structure of the local and nonlocal symmetries and the Poisson structure of the associated hierarchies of conserved quantities.

REMARK 6.16. The contents of this section clearly indicates how to construct supersymmetric extensions of classical integrable systems, which can be termed completely integrable by the existence of infinite hierarchies of local and/or nonlocal symmetries and conservation Laws.

CHAPTER 7

Deformations of supersymmetric equations

We shall illustrate the developed theory of deformations of supersymmetric equations and systems through a number of examples.

First of all we shall continue the theory for the supersymmetric extension of the KdV equation [**35, 72, 74, 87**] started in Section 6 of the previous chapter. We shall construct the recursion operator for symmetries, which is just realized by the contraction of a symmetry and the deformation. Moreover we construct a new hierarchy of conserved quantities and a hierarchy of (x,t)-dependent symmetries.

As a second application, we consider the two supersymmetric extensions of the nonlinear Schrödinger equation (Section 2) leading to the recursion operators for symmetries and new hierarchies of odd and even symmetries.

We shall also construct a supersymmetric extension of the Boussinesq equation, construct deformations for this system and eventually arrive at the recursion operator for symmetries and at hierarchies of odd and even symmetries and conservation laws.

Finally, we construct two-dimensional supersymmetric extensions (i.e., extensions including two odd dependent variables) of the KdV and study their symmetries, conservation laws, and deformations, obtaining recursion operators and hierarchies of symmetries.

1. Supersymmetric KdV equation

We start at the supersymmetric extension of the KdV equation [**72, 74**] and restrict our considerations to the case $a = 3$ in the system

$$\begin{aligned} u_t &= -u_3 + 6uu_1 - a\varphi\varphi_2, \\ \varphi_t &= -\varphi_3 + (6-a)\varphi_1 u + a\varphi u_1 \end{aligned} \tag{7.1}$$

(see Section 6 of Chapter 6).

Features and properties of the equation were discussed in several papers, cf. [**35, 87**].

1.1. Nonlocal variables. In order to construct a deformation of (7.1), we have to construct an appropriate covering by the introduction of a number of nonlocal variables. These nonlocal variables, which arise classically from conserved densities related to conservation laws, have been computed to be

$$q_{\frac{1}{2}} = D^{-1}(\varphi),$$

$$
\begin{aligned}
q_{\frac{3}{2}} =& D^{-1}(p_1\varphi),\\
q_{\frac{5}{2}} =& D^{-1}\left(\frac{1}{2}p_1^2\varphi - u\varphi\right)
\end{aligned}
\tag{7.2}
$$

and

$$
\begin{aligned}
p_1 =& D^{-1}(u),\\
p_0 =& D^{-1}(p_1),\\
\overline{p}_1 =& D^{-1}(\varphi q_{\frac{1}{2}}),\\
p_3 =& D^{-1}(u^2 - \varphi\varphi_1),\\
\overline{p}_3 =& D^{-1}(u^2 - 2u\varphi q_{\frac{1}{2}} + uq_{\frac{1}{2}}q_{\frac{3}{2}}),
\end{aligned}
\tag{7.3}
$$

where $D = D_x$.

Odd nonlocal variables will be denoted by q, while even nonlocal variables will be denoted by p and $\overline{p}$. We mention that, in effect, the total derivative operator D_x should be lifted to an appropriate covering, where it is denoted by the same symbol D_x, i.e.,

$$
\begin{aligned}
D_x = & \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + \dots\\
& + (q_{\frac{1}{2}})_x\frac{\partial}{\partial q_{\frac{1}{2}}} + (q_{\frac{3}{2}})_x\frac{\partial}{\partial q_{\frac{3}{2}}} + (q_{\frac{5}{2}})_x\frac{\partial}{\partial q_{\frac{5}{2}}}\\
& + (p_0)_x\frac{\partial}{\partial p_0} + (p_1)_x\frac{\partial}{\partial p_1} + (p_3)_x\frac{\partial}{\partial p_3}\\
& + (\overline{p}_1)_x\frac{\partial}{\partial \overline{p}_1} + (\overline{p}_3)_x\frac{\partial}{\partial \overline{p}_3}.
\end{aligned}
\tag{7.4}
$$

Other odd nonlocal variables, $q_{\frac{7}{2}}$ and $q_{\frac{9}{2}}$, are given by

$$
\begin{aligned}
q_{\frac{7}{2}} &= D^{-1}\Big(p_3\varphi + q_{\frac{1}{2}}\big(\frac{1}{2}p_1^2u - p_1u_1 + u_2 - u^2\big)\Big),\\
q_{\frac{9}{2}} &= D^{-1}\Big(6p_3p_1\varphi + q_{\frac{1}{2}}\big(p_1^3u - 3p_1^2u_1 + 6p_1u_2 - 6p_1u^2\\
&\quad + 36uu_1 - 6u_3\big)\Big).
\end{aligned}
\tag{7.5}
$$

Note that the variables $q_{\frac{3}{2}}, q_{\frac{5}{2}}, q_{\frac{7}{2}}, q_{\frac{9}{2}}, p_0, \overline{p}_1, \overline{p}_3$ contain higher nonlocalities.

1.2. Symmetries. For hierarchies $\{Y_{\frac{2n+1}{2}}\}$, $\{X_{2n+1}\}$, $n \in \mathbb{N}$, of symmetries of equation (7.1) we refer to 4 of Chapter 6. Recall that

$$
\begin{aligned}
Y_{\frac{1}{2}} =& \varphi_1\frac{\partial}{\partial u} + u\frac{\partial}{\partial\varphi},\\
Y_{\frac{3}{2}} =& (2q_{\frac{1}{2}}u_1 - p_1\varphi_1 + u\varphi - \varphi_2)\frac{\partial}{\partial u} + (2q_{\frac{1}{2}}\varphi_1 - p_1u + u_1)\frac{\partial}{\partial\varphi},\\
X_1 =& u_1\frac{\partial}{\partial u} + \varphi_1\frac{\partial}{\partial\varphi},
\end{aligned}
$$

$$
\begin{aligned}
X_3 =& - u_t \frac{\partial}{\partial u} - \varphi_t \frac{\partial}{\partial \varphi}, \\
X_5 =& - (u_5 - 10u_3 u - 20u_2 u_1 + 30u_1 u^2 + 5\varphi\varphi_4 + 5\varphi_1\varphi_3 \\
& - 20u\varphi\varphi_2 - 20u_1\varphi\varphi_1) \frac{\partial}{\partial u} \\
& - (\varphi_5 - 5u\varphi_3 - 10u_1\varphi_2 - 10u_2\varphi_1 + 10u^2\varphi_1 + 20u_1 u\varphi - 5u_3\varphi) \frac{\partial}{\partial \varphi}.
\end{aligned} \tag{7.6}
$$

Moreover we found the supersymmetric analogue of the (x, t)-dependent symmetry which acts as recursion on the even hierarchy $\{X_{2n+1}\}$, $n \in \mathbb{N}$, i.e.,

$$
V_2 = -6tX_5 - 2xX_3 + H_2, \tag{7.7}
$$

where

$$
\begin{aligned}
H_2 =& \big(- q_{\frac{1}{2}}(\varphi_2 + p_1\varphi_1 - \varphi u) + 3q_{\frac{3}{2}}\varphi_1 - 13\varphi\varphi_1 \\
& + 4p_1 u_1 - 2\overline{p}_1 u_1 - 8u_2 + 16u^2 \big) \frac{\partial}{\partial u} \\
& + \big(- q_{\frac{1}{2}}(p_1 u - u_1) + 3q_{\frac{3}{2}} u \\
& + 2p_1\varphi_1 - 2\overline{p}_1\varphi_1 - 7\varphi_2 + 14\varphi u \big) \frac{\partial}{\partial \varphi}.
\end{aligned} \tag{7.8}
$$

It should be noted that the vector fields

$$
\begin{aligned}
Y_{-\frac{1}{2}} =& \frac{\partial}{\partial q_{\frac{1}{2}}} - q_{\frac{1}{2}} \frac{\partial}{\partial \overline{p}_1} + (p_1 q_{\frac{3}{2}} - 2q_{\frac{5}{2}}) \frac{\partial}{\partial \overline{p}_3}, \\
\overline{X}_{-1} =& \frac{\partial}{\partial p_1} + q_{\frac{1}{2}} \frac{\partial}{\partial q_{\frac{3}{2}}} + q_{\frac{3}{2}} \frac{\partial}{\partial q_{\frac{5}{2}}} + x \frac{\partial}{\partial p_0}, \\
X_{-1} =& = \frac{\partial}{\partial \overline{p}_1}
\end{aligned} \tag{7.9}
$$

are symmetries of equation (7.1) in the covering defined by (7.2), (7.3). These symmetries are vertical in the covering under consideration.

Computation of graded Lie brackets leads to the identities

$$
\begin{aligned}
[Y_{-\frac{1}{2}}, V_2] =& Y_{\frac{3}{2}}, \\
[X_{-1}, V_2] =& 2Z_1 + 4X_1, \\
[\overline{X}_{-1}, V_2] =& - 2X_1,
\end{aligned} \tag{7.10}
$$

where Z_1 is the nonlocal symmetry of degree 1 (cf. 4 of Chapter 6), which acts, by its Lie bracket, as a recursion operator on the odd hierarchy $\{Y_{n+\frac{1}{2}}\}$, $n \in \mathbb{N}$. Recall that

$$
Z_1 = (q_{\frac{1}{2}}\varphi_1) \frac{\partial}{\partial u} + (q_{\frac{1}{2}} u - \varphi_1) \frac{\partial}{\partial \varphi} + \dots \tag{7.11}
$$

1.3. Deformations. In order to construct a deformation of (7.1), we formally construct the infinite-dimensional Cartan covering (see Subsection 3.5 of Chapter 6) over the infinite covering of (7.1) by (7.2), (7.3).

In the setting under consideration, the Cartan covering is described by the Cartan forms $\omega_0, \dots, \omega_k, \dots$ on the infinite prolongation of the supersymmetric KdV equation together with the forms corresponding to the nonlocal variables (7.2), (7.3):

$$\omega_{q_{\frac{1}{2}}},\ \omega_{q_{\frac{3}{2}}},\ \omega_{q_{\frac{5}{2}}},\ \omega_{p_0},\ \omega_{p_1},\ \omega_{\overline{p}_1},\ \omega_{p_3},\ \omega_{\overline{p}_3}, \tag{7.12}$$

where $\omega_f = \mathrm{L}_{U_\varphi}(f)$ denotes the Cartan form corresponding to the potential f (see (2.13) on p. 66). According to (7.12), we search for a generalized vector field which is linear with respect to the Cartan forms. Applying the deformation condition on this vector field and taking into account the grading of (7.1), (7.2), (7.3), and (7.12), we arrive at the following deformation

$$\begin{aligned}
U_1 = \big(&\omega_{u_2} + \omega_u(-4u) + \omega_{\varphi_1}(-2\varphi) + \omega_\varphi(\varphi_1)\\
&+ \omega_{q_{\frac{1}{2}}}(q_{\frac{1}{2}}u_1 + p_1\varphi_1 + \varphi_2 - u\varphi)\\
&+ \omega_{p_1}(-2u_1) + \omega_{\overline{p}_1}(u_1) + \omega_{q_{\frac{3}{2}}}(-\varphi_1)\big)\frac{\partial}{\partial u}\\
&+ \big(\omega_{\varphi_2} + \omega_\varphi(-2u) + \omega_u(-2\varphi)\\
&+ \omega_{q_{\frac{1}{2}}}(-q_{\frac{1}{2}}\varphi_1 + p_1 u - u_1)\\
&+ \omega_{p_1}(-\varphi_1) + \omega_{\overline{p}_1}(\varphi_1) + \omega_{q_{\frac{3}{2}}}(-u)\big)\frac{\partial}{\partial \varphi}.
\end{aligned} \tag{7.13}$$

Similar to the results of Subsection 2.8 of Chapter 6, the element U_1 satisfies the identity

$$[\![U_1, U_1]\!]^{\mathrm{fn}} = 0, \tag{7.14}$$

which means that U_1 is a graded Nijenhuis operator in the sense [**49**].

We now redefine our hierarchies in the following way. First we put

$$\begin{aligned}
Y_{\frac{1}{2}} =& \varphi_1\frac{\partial}{\partial u} + u\frac{\partial}{\partial \varphi},\\
Y_{\frac{3}{2}} =& (2q_{\frac{1}{2}}u_1 - p_1\varphi_1 + u\varphi - \varphi_2)\frac{\partial}{\partial u} + (2q_{\frac{1}{2}}\varphi_1 - p_1 u + u_1)\frac{\partial}{\partial \varphi},\\
X_1 =& u_1\frac{\partial}{\partial u} + \varphi_1\frac{\partial}{\partial \varphi},\\
\overline{X}_1 =& (q_{\frac{1}{2}}\varphi_1)\frac{\partial}{\partial u} + (q_{\frac{1}{2}}u - \varphi_1)\frac{\partial}{\partial \varphi} = Z_1,\\
V_0 =& (2u + xu_1 + 3tu_t)\frac{\partial}{\partial u} + (\frac{3}{2}\varphi + x\varphi_1 + 3t\varphi_t)\frac{\partial}{\partial \varphi}
\end{aligned} \tag{7.15}$$

and define the odd and even hierarchies of symmetries by

$$\begin{aligned} Y_{2n+\frac{1}{2}} &= \underbrace{((\dots(Y_{\frac{1}{2}} \lrcorner U_1) \lrcorner U_1)\dots) \lrcorner U_1)}_{n \text{ times}} = Y_{\frac{1}{2}} \lrcorner U_1^n, \\ Y_{2n+\frac{3}{2}} &= Y_{\frac{3}{2}} \lrcorner U_1^n, \\ X_{2n+1} &= X_1 \lrcorner U_1^n, \\ \overline{X}_{2n+1} &= \overline{X}_1 \lrcorner U_1^n, \\ V_{2n} &= V_0 \lrcorner U_1^n. \end{aligned} \tag{7.16}$$

1.4. Passing from deformations to "classical" recursion operators. Here we rewrite the main result of the previous subsection in more conventional terms, i.e., as formal matrix integro-differential operators. We shall see that this representation is far less "economical" than representation (7.13). Moreover, if one uses conventional left action of differential operators, additional parasitic signs arise, which makes this representation even more cumbersome.

Let $X = \mathcal{D}_{(F,G)}$ be a nonlocal symmetry of (7.1) in the covering defined by (7.2), (7.3) with 2-component generating function (F, G) and let $|X|$ be the degree of X; then one has $|F| = |X|$ and $|G| = |X| + 1$.

It means that X is of the form

$$X = \sum_{i=0}^{\infty} \Big(D^i(F) \frac{\partial}{\partial u_i} + D^i(G) \frac{\partial}{\partial \varphi_i} \Big), \tag{7.17}$$

where F and G satisfy the shadow equation for the covering in question and D denotes the extension of the total derivative D_x onto the covering. Then one has

$$\begin{aligned} \mathrm{i}_X(\omega_{u_i}) &= D^i(F), \\ \mathrm{i}_X(\omega_{\varphi_i}) &= D^i(G) \end{aligned} \tag{7.18}$$

for all $i = 0, 1, \dots$ From the definition of nonlocal variables (see (7.2) and (7.3)) one also has

$$\begin{aligned} \mathrm{i}_X(\omega_{p_1}) &= D^{-1}(F), \\ \mathrm{i}_X(\omega_{q_{\frac{1}{2}}}) &= D^{-1}(G), \\ \mathrm{i}_X(\omega_{p_0}) &= D^{-1}(D^{-1}(F)), \\ \mathrm{i}_X(\omega_{q_{\frac{3}{2}}}) &= D^{-1}(D^{-1}(F)\varphi + Gp_1), \\ \mathrm{i}_X(\omega_{\overline{p}_1}) &= D^{-1}(Gq_{\frac{1}{2}} - D^{-1}(G)\varphi), \\ \mathrm{i}_X(\omega_{q_{\frac{5}{2}}}) &= D^{-1}(D^{-1}(F)p_1\varphi + \frac{1}{2}Gp_1^2 - F\varphi - Gu), \\ i_X(\omega_{p_3}) &= D^{-1}(2uF - G\varphi_1 + D(G)\varphi), \end{aligned} \tag{7.19}$$

while

$$\begin{aligned}\mathrm{i}_X(\omega_{\overline{p}_3}) = D^{-1}\Big(&2Fu - 2F\varphi q_{\frac{1}{2}} - 2Guq_{\frac{1}{2}} + 2D^{-1}(G)u\varphi \\ &+ Fq_{\frac{1}{2}}q_{\frac{3}{2}} + D^{-1}(G)uq_{\frac{3}{2}} - D^{-1}\big(D^{-1}(F)\varphi + Gp_1\big)uq_{\frac{1}{2}}\Big) \end{aligned}\tag{7.20}$$

(the last equality is given for reasons of completeness only and will not be used below).

Then the recursion operator $\mathcal{R}$ corresponding to the deformation U_1, (7.13) acts as

$$\mathcal{R}(X) = \mathrm{i}_X(U_1) \tag{7.21}$$

and is of the form

$$\mathcal{R}(F, G) = (F_1, G_1), \tag{7.22}$$

where

$$\begin{aligned}F_1 &= D^2(F) + F(-4u) + D(G)(-2\varphi) + G(\varphi_1) \\ &\quad + D^{-1}(G)(-q_{\frac{1}{2}}u_1 + p_1\varphi_1 + \varphi_2 - u\varphi) + D^{-1}(F)(-2u_1) \\ &\quad + D^{-1}(Gq_{\frac{1}{2}} - D^{-1}(G)\varphi)u_1 + D^{-1}(D^{-1}(F)\varphi + Gp_1)(-\varphi_1), \\ G_1 &= D^2(G) + G(-2u) + F(-2\varphi) \\ &\quad + D^{-1}(G)(-q_{\frac{1}{2}}\varphi_1 + p_1u - u_1) + D^{-1}(F)(-\varphi_1) \\ &\quad + D^{-1}(Gq_{\frac{1}{2}} - D^{-1}(G)\varphi)(\varphi_1) + D^{-1}(D^{-1}(F)\varphi + Gp_1)(-u). \end{aligned}\tag{7.23}$$

Due to the relations

$$\begin{aligned}D^{-1}(Gq_{\frac{1}{2}}) &= D^{-1}(G)q_{\frac{1}{2}} - D^{-1}(D^{-1}(G)\varphi) \\ &= -(-1)^{|X|}q_{\frac{1}{2}}D^{-1}(G) + (-1)^{|X|}D^{-1}(\varphi D^{-1}(G)), \\ D^{-1}(Gp_1) &= p_1D^{-1}(G) - D^{-1}(uD^{-1}(G)), \end{aligned}\tag{7.24}$$

we rewrite F_1, G_1 in a left action notation as

$$\begin{aligned}F_1 &= D^2(F) - 4uF + (-1)^{|X|}2\varphi D(G) - (-1)^{|X|}\varphi_1 G \\ &\quad - (-1)^{|X|}(-q_{\frac{1}{2}}u_1 + p_1\varphi_1 + \varphi_2 - u\varphi)D^{-1}(G) - 2u_1D^{-1}(F) \\ &\quad - (-1)^{|X|}u_1q_{\frac{1}{2}}D^{-1}(G) + (-1)^{|X|}u_1D^{-1}(\varphi D^{-1}(G)) \\ &\quad + (-1)^{|X|}u_1D^{-1}(\varphi D^{-1}(G)) + \varphi_1D^{-1}(\varphi D^{-1}(F) \\ &\quad + (-1)^{|X|}\varphi_1p_1D^{-1}(G) - (-1)^{|X|}\varphi_1D^{-1}(uD^{-1}(G))), \\ G_1 &= D^2(G) - 2uG - (-1)^{|X|}2\varphi F \\ &\quad + (-q_{\frac{1}{2}}\varphi_1 + p_1u - u_1)D^{-1}(G) - (-1)^{|X|}\varphi_1D^{-1}(F) \\ &\quad + (-1)^{|X|}\varphi_1((-1)^{|X|+1}q_{\frac{1}{2}}D^{-1}(G) - (-1)^{|X|+1}D^{-1}(\varphi D^{-1}(G))) \\ &\quad + (-1)^{2|X|}\varphi_1D^{-1}(\varphi D^{-1}(G)) - (-1)^{|X|}uD^{-1}(\varphi D^{-1}(F)) \\ &\quad - up_1D^{-1}(G) + uD^{-1}(uD^{-1}(G)). \end{aligned}\tag{7.25}$$

From this we finally arrive at

$$\begin{aligned}
F_1 &= D^2(F) - 4uF - 2u_1D^{-1}(F) + \varphi_1 D^{-1}(\varphi D^{-1}(F)) \\
&\quad + (-1)^{|X|} 2\varphi D(G) - (-1)^{|X|}\varphi_1 G - (-1)^{|X|}(\varphi_2 - u\varphi)D^{-1}(G) \\
&\quad + (-1)^{|X|} 2u_1 D^{-1}(\varphi D^{-1}(G)) \\
&\quad - (-1)^{|X|}\varphi_1 D^{-1}(uD^{-1}(G)), \\
G_1 &= -(-1)^{|X|} 2\varphi F - (-1)^{|X|}\varphi_1 D^{-1}(F) \\
&\quad - (-1)^{|X|} u D^{-1}(\varphi D^{-1}(F)) \\
&\quad + D^2(G) - 2uG - u_1 D^{-1}(G) + 2\varphi_1 D^{-1}(\varphi D^{-1}(G)) \\
&\quad + uD^{-1}(uD^{-1}(G)),
\end{aligned} \tag{7.26}$$

or

$$\begin{aligned}
F_1 &= D^2(F) - 4uF - 2u_1D^{-1}(F) + \varphi_1 D^{-1}(\varphi D^{-1}(F)) \\
&\quad + (-1)^{|X|}\big(2\varphi D(G) - \varphi_1 G + (-\varphi_2 + u\varphi)D^{-1}(G) \\
&\quad + 2u_1 D^{-1}(\varphi D^{-1}(G)) - \varphi_1 D^{-1}(uD^{-1}(G))\big), \\
G_1 &= (-1)^{|X|}\big(- 2\varphi F - \varphi_1 D^{-1}(F) - uD^{-1}(\varphi D^{-1}(F))\big) \\
&\quad D^2(G) - 2uG - u_1 D^{-1}(G) + 2\varphi_1 D^{-1}(\varphi D^{-1}(G)) + uD^{-1}(uD^{-1}(G)),
\end{aligned} \tag{7.27}$$

leading to the recursion operator $\mathcal{R} = \mathcal{R}_{ij}$, where

$$\begin{aligned}
\mathcal{R}_{11} &= D^2 - 4u - 2u_1D^{-1} + \varphi_1 D^{-1}\varphi D^{-1}, \\
\mathcal{R}_{12} &= (-1)^{|X|}(2\varphi D - \varphi_1 - \varphi_2 D^{-1} + u\varphi D^{-1} + 2u_1 D^{-1}\varphi D^{-1} \\
&\quad - \varphi_1 D^{-1} u D^{-1}), \\
\mathcal{R}_{21} &= (-1)^{|X|}(-2\varphi - \varphi_1 D^{-1} - uD^{-1}\varphi D^{-1}), \\
\mathcal{R}_{22} &= D^2 - 2u - u_1 D^{-1} + 2\varphi_1 D^{-1}\varphi D^{-1} + uD^{-1}uD^{-1}.
\end{aligned} \tag{7.28}$$

Note that the classical recursion operator for the KdV equation is just the φ-independent part of $\mathcal{R}_{11}$:

$$\mathcal{R}_0 = D^2 - 4u - 2u_1 D^{-1}. \tag{7.29}$$

From the above representation it becomes clear that the action of the recursion operator considered as action from the left, requires introduction of the sign $(-1)^{|X|}$, which makes the operation not natural. Therefore we shall restrict ourselves to representations similar to (7.13).

2. Supersymmetric extensions of the NLS equation

In this section, we shall discuss deformations and recursion operators for the two supersymmetric extensions of the nonlinear Schrödinger equation [**88**]

$$u_t = -v_2 + kv(u^2 + v^2) - u_1(c_1 - c_2)\varphi\psi - 4kv\psi\psi_1$$

$$
\begin{aligned}
&- u(c_1 + c_2 + 4k)\psi\varphi_1 + c_2 u\varphi\psi_1 + c_1 v\varphi\varphi_1,\\
v_t &= u_2 - ku(u^2 + v^2) - v_1(c_1 - c_2)\varphi\psi + 4ku\varphi\varphi_1\\
&+ v(c_1 + c_2 + 4k)\varphi\psi_1 - c_1 u\psi\psi_1 - c_2 v\psi\varphi_1,\\
\varphi_t &= -\psi_2 + (\frac{1}{2}c_2u^2 + ku^2 + kv^2)\psi - \frac{1}{2}c_2uv\varphi - (c_1 - c_2)\varphi\psi\varphi_1,\\
\psi_t &= \varphi_2 - (\frac{1}{2}c_2v^2 + ku^2 + kv^2)\varphi + \frac{1}{2}c_2uv\psi - (c_1 - c_2)\varphi\psi\psi_1,
\end{aligned}
$$

where in

Case A:	$c_1 = -4k,$	$c_2 = 0,$
Case B:	$c_1 = c,$	$c_2 = 4k.$

The construction of deformations will follow exactly the same lines as for the supersymmetric KdV equation presented in Section 1, so for the nonlinear Schrödinger equation we shall only present the results.

2.1. Case A. In order to work in the appropriate covering for the supersymmetric extension of the Nonlinear Schrödinger Equation we did construct the following set of nonlocal variables, associated to conserved quantities

$$
\begin{gathered}
p_0,\ p_1,\ p_2\ ,\overline{p}_0,\ \overline{p}_1,\ \overline{p}_2,\\
q_{\frac{1}{2}},\ \overline{q}_{\frac{1}{2}},\ q_{\frac{3}{2}},\ \overline{q}_{\frac{3}{2}},\ q_{\frac{5}{2}},\ \overline{q}_{\frac{5}{2}},
\end{gathered}
$$

which are defined by

$$
\begin{aligned}
p_0 &= D^{-1}(\varphi\psi),\\
\overline{p}_0 &= D^{-1}(p_1),\\
p_1 &= D^{-1}(u^2 + v^2 - 2\varphi\varphi_1 - 2\psi\psi_1),\\
\overline{p}_1 &= D^{-1}\big(k(\psi v + \varphi u)\overline{q}_{\frac{1}{2}} + k(\psi u - \varphi v)q_{\frac{1}{2}} - 2\psi\psi_1 - 2\varphi\varphi_1\big),\\
p_2 &= D^{-1}(uv_1 + 2\varphi_1\psi_1),\\
\overline{p}_2 &= D^{-1}\big(k(2\psi_1 v + 2\varphi_1 u + k\psi v p_1 + k\varphi u p_1)q_{\frac{1}{2}}\\
&\quad + k(-2\psi_1 u + 2\varphi_1 v - k\psi u p_1 + k\varphi v p_1)\overline{q}_{\frac{1}{2}} + 2uv_1\big),\\
q_{\frac{1}{2}} &= D^{-1}(\psi u - \varphi v),\\
\overline{q}_{\frac{1}{2}} &= D^{-1}(\psi v + \varphi u),\\
q_{\frac{3}{2}} &= D^{-1}(k\psi u p_1 - k\varphi v p_1 + 2\psi_1 u - 2\varphi_1 v),\\
\overline{q}_{\frac{3}{2}} &= D^{-1}(k\psi v p_1 + k\varphi u p_1 + 2\psi_1 v + 2\varphi_1 u).
\end{aligned}
$$

After introduction of the associated Cartan forms, we found the deformation, or Nijenhuis operator, for this case to be

$$
U_1 = \Big(\omega_{v_1} + \omega_{p_1}(-kv) - 2\omega_{p_0}ku_1 + \omega_u(-2k\varphi\psi)
$$

$$
\begin{aligned}
&+\omega_{\varphi}(-ku\psi-kv\varphi)+\omega_{\psi}(-kv\psi+ku\varphi)\\
&-\omega_{q_{\frac12}}(k\varphi_1)+\omega_{\overline{q}_{\frac12}}(k\psi_1)\Big)\frac{\partial}{\partial u}\\
&+\Big(-\omega_{u_1}+\omega_{p_1}(ku)-2\omega_{p_0}kv_1+\omega_v(-2k\varphi\psi)\\
&+\omega_{\varphi}(-kv\psi+ku\varphi)+\omega_{\psi}(ku\psi+kv\varphi)\\
&+\omega_{q_{\frac12}}(-k\psi)+\omega_{\overline{q}_{\frac12}}(-k\varphi_1)\Big)\frac{\partial}{\partial v}\\
&+\Big(\omega_{\psi_1}+\omega_{\varphi}(k\varphi\psi)+\omega_{p_1}(-\frac{k}{2}\psi)+\omega_{p_0}(-2k\varphi_1)\\
&+\omega_{q_{\frac12}}(-\frac{k}{2}u)+\omega_{\overline{q}_{\frac12}}(-\frac{k}{2}v)\Big)\frac{\partial}{\partial\varphi}\\
&+\Big(-\omega_{\varphi_1}+\omega_{\psi}(-k\varphi\psi)+\omega_{p_1}(+\frac{k}{2}\varphi)+\omega_{p_0}(-2k\psi_1)\\
&+\omega_{q_{\frac12}}(-\frac{k}{2}v)+\omega_{\overline{q}_{\frac12}}(\frac{k}{2}u)\Big)\frac{\partial}{\partial\psi}.
\end{aligned}
$$

By starting at the symmetries (see [**88**])

$$
\begin{aligned}
X_0&=v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v}+\dots,\\
\overline{X}_0&=\psi\frac{\partial}{\partial\varphi}-\varphi\frac{\partial}{\partial\psi}+\dots,\\
Y_{\frac12}&=-\psi_1\frac{\partial}{\partial u}+\varphi_1\frac{\partial}{\partial v}+\frac12 v\frac{\partial}{\partial\varphi}-\frac12 u\frac{\partial}{\partial\psi}+\dots,\\
\overline{Y}_{\frac12}&=\varphi_1\frac{\partial}{\partial u}+\psi_1\frac{\partial}{\partial v}+\frac12 u\frac{\partial}{\partial\varphi}+\frac12 v\frac{\partial}{\partial\psi}+\dots
\end{aligned}
$$

and

$$
\begin{aligned}
S_0=(u+xu_1+2tu_t)\frac{\partial}{\partial u}&+(v+xv_1+2tv_t)\frac{\partial}{\partial v}\\
&+(\frac12\varphi+x\varphi_1+2t\varphi_t)\frac{\partial}{\partial\varphi}+(\frac12\psi+x\psi_1+2t\psi_t)\frac{\partial}{\partial\psi}+\dots,
\end{aligned}
$$

the recursion operator $U_1=\mathcal{R}$ generates five hierarchies of symmetries

$$
\begin{aligned}
X_n&=X_0\mathcal{R}^n,\\
Y_{n+\frac12}&=Y_{\frac12}\mathcal{R}^n,\\
\overline{X}_n&=X_0\mathcal{R}^n,\\
\overline{Y}_{n+\frac12}&=\overline{Y}_{\frac12}\mathcal{R}^n,\\
S_n&=S_0\mathcal{R}^n,
\end{aligned}
$$

where $X_0\mathcal{R}^n,\dots$ should be understood as

$$X_n=X_0\mathcal{R}^n=(\dots((X_0\,\lrcorner\,\underbrace{U_1)\,\lrcorner\,U_1)\,\lrcorner\,\dots)\,\lrcorner\,U_1}_{n\text{ times}}.$$

2.2. Case B. In this case the supersymmetric nonlinear Schrödinger equation is

$$\begin{aligned}
u_t &= -v_2 + kv(u^2+v^2) - (c_1-4k)u_1\varphi\psi - 4kv\psi\psi_1,\\
&\quad - (c_1+8k)u\psi\varphi_1 + 4ku\varphi\psi_1 + c_1 v\varphi\varphi_1,\\
v_t &= u_2 - ku(u^2+v^2) - (c_1-4k)v_1\varphi\psi + 4ku\varphi\varphi_1,\\
&\quad (c_1+8k)v\varphi\psi_1 - c_1 u\psi\psi_1 - 4kv\psi\varphi_1,\\
\varphi_1 &= -\psi_2 + (3ku^2+kv^2)\psi - 2kuv\varphi - (c_1-4k)\varphi\psi\varphi_1,\\
\psi_1 &= \varphi_2 - (ku^2+3kv^2)\varphi + 2kuv\psi - (c_1-4k)\varphi\psi\psi_1.
\end{aligned}$$

We introduce the following nonlocal variables, resulting from computed conservation laws,

$$\begin{aligned}
p_0 &= D^{-1}(\varphi\psi),\\
\overline{p}_0 &= D^{-1}\big(p_1 + (c_1+4k)\overline{p}_1\big),\\
p_1 &= D^{-1}\Big(u^2+v^2+\frac{1}{2k}(c_1+4k)(\varphi\varphi_1+\psi\psi_1)\Big),\\
\overline{p}_1 &= D^{-1}\Big((u\psi-v\varphi)q_{\frac12} - \frac{1}{2k}(\varphi\varphi_1+\psi\psi_1)\Big),\\
q_{\frac12} &= D^{-1}(u\psi-v\varphi),\\
\overline{q}_{\frac12} &= D^{-1}\Big(q_{\frac32}+\frac12(c_1+4k)\varphi\psi q_{\frac12}\Big),\\
q_{\frac32} &= D^{-1}(v\psi_1+u\varphi_1)
\end{aligned}$$

and additionally

$$\begin{aligned}
q_{-\frac12} &= D^{-1}(q_{\frac12}),\\
p_2 &= D^{-1}\Big(-uv_1+\frac{1}{4k}(c_1+4k)\varphi_1\psi_1+\frac14(c_1+12k)(u^2+v^2)\varphi\psi\Big),\\
\overline{p}_2 &= D^{-1}\Big(-(v\psi_1+u\varphi_1)q_{\frac12}+\frac{1}{2k}\varphi_1\psi_1\Big).
\end{aligned}$$

Within this covering, we constructed a deformation of the form

$$\begin{aligned}
U_1 = \Big(&\omega_{v_1} + \omega_u\Big(\frac12(c_1-4k)\varphi\psi\Big)\\
&+\omega_\varphi\Big(-4ku\psi+\frac14(c_1-4k)v\varphi\Big)+\omega_\psi\Big(\frac14(c_1-4k)v\psi+4ku\varphi\Big)\\
&+\omega_{p_0}\Big(\frac12(c_1-4k)u_1\Big)+\omega_{p_1}\big(-kv\big)+\omega_{\overline{p}_1}\Big(-\frac12 k(c_1+12k)v\Big)\\
&+\omega_{q_{\frac12}}\Big(\frac12 k(c_1+12k)vq_{\frac12}+\frac12(c_1+4k)\varphi_1\Big)\\
&+\omega_{q_{\frac32}}\Big(-\frac12(c_1+4k)\psi\Big)\Big)\frac{\partial}{\partial u}
\end{aligned}$$

$$
\begin{aligned}
&+\Big(-\omega_{u_1}+\omega_v\frac{1}{2}(c_1-4k)\varphi\psi+\omega_\varphi(-4kv\psi-\frac{1}{4}(c_1-4k)u\varphi)\\
&+\omega_\psi(-\frac{1}{4}(c_1-4k)u\psi+4kv\varphi)\\
&+\omega_{p_0}\frac{1}{2}(c_1-4k)v_1+\omega_{p_1}(ku)+\omega_{\overline{p}_1}\frac{1}{2}k(c_1+12k)u\\
&+\omega_{q_{\frac{1}{2}}}(-\frac{1}{2}k(c_1+12k)uq_{\frac{1}{2}}+\frac{1}{2}(c_1+4k)\psi_1)\\
&+\omega_{q_{\frac{3}{2}}}\frac{1}{2}(c_1+4k)\varphi\Big)\frac{\partial}{\partial v}\\
&+\Big(\omega_{\psi_1}+\frac{1}{4}\omega_\varphi(c_1-4k)\varphi\psi+\omega_{p_0}\frac{1}{2}(c_1-4k)\varphi_1-\omega_{p_1}k\psi\\
&-\omega_{\overline{p}_1}\frac{1}{2}k(c_1+12k)\psi+\omega_{q_{\frac{1}{2}}}(-2ku-\frac{1}{2}k(c_1+12k)\psi q_{\frac{1}{2}})\Big)\frac{\partial}{\partial\varphi}\\
&+\Big(-\omega_{\varphi_1}+\omega_\psi\frac{1}{4}(c_1-4k)\varphi\psi+\omega_{p_0}\frac{1}{2}(c_1-4k)\psi_1+\omega_{p_1}k\varphi\\
&+\omega_{\overline{p}_1}\frac{1}{2}k(c_1+12k)\varphi+\omega_{q_{\frac{1}{2}}}(-2kv+\frac{1}{2}k(c_1+12k)\varphi q_{\frac{1}{2}})\Big)\frac{\partial}{\partial\psi}.
\end{aligned}
$$

The action of U_1 on the symmetries

$$
\begin{aligned}
X_1&=u_1\frac{\partial}{\partial u}+v_1\frac{\partial}{\partial v}+\varphi_1\frac{\partial}{\partial\varphi}+\psi_1\frac{\partial}{\partial\psi}+\dots,\\
\overline{X}_1&=\big((c+4k)(\varphi_1q_{\frac{1}{2}}+\psi q_{\frac{3}{2}})+2k(c-4k)\bar{p}_1v\big)\frac{\partial}{\partial u}\\
&\quad+\big((c+4k)(\psi_1q_{\frac{1}{2}}-\varphi q_{\frac{3}{2}})-2k(c-4k)\bar{p}_1u\big)\frac{\partial}{\partial v}\\
&\quad+\big(-4kuq_{\frac{1}{2}}+2\psi_1+2k(c-4k)\bar{p}_1\psi\big)\frac{\partial}{\partial\varphi}\\
&\quad+\big(-4kvq_{\frac{1}{2}}-2\varphi_1-2k(c-4k)\bar{p}_1\varphi\big)\frac{\partial}{\partial\psi}+\dots,\\
Y_{\frac{1}{2}}&=\varphi_1\frac{\partial}{\partial u}+\psi_1\frac{\partial}{\partial v}+\frac{1}{2}u\frac{\partial}{\partial\varphi}+\frac{1}{2}v\frac{\partial}{\partial\psi}+\dots,\\
\overline{Y_{\frac{1}{2}}}&=q_{\frac{1}{2}}v\frac{\partial}{\partial u}-q_{\frac{1}{2}}u\frac{\partial}{\partial v}+(q_{\frac{1}{2}}\psi-\frac{1}{4k}u)\frac{\partial}{\partial\varphi}+(-q_{\frac{1}{2}}\varphi-\frac{1}{4k}v)\frac{\partial}{\partial\psi}+\dots,\\
X_0&=v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v}+\psi\frac{\partial}{\partial\varphi}-\varphi\frac{\partial}{\partial\psi}+\dots,\\
\overline{X}_0&=-q_{\frac{1}{2}}\psi\frac{\partial}{\partial u}+q_{\frac{1}{2}}\varphi\frac{\partial}{\partial v}-\frac{1}{8k}\varphi\frac{\partial}{\partial\varphi}-\frac{1}{8k}\psi\frac{\partial}{\partial\psi}+\dots,\\
Y_{-\frac{1}{2}}&=\psi\frac{\partial}{\partial u}-\varphi\frac{\partial}{\partial v}+\dots
\end{aligned}
$$

creates hierarchies of symmetries in a similar way as in the preceding subsection. Note that $\overline{X}_1$ is the nonlocal recursion symmetry constructed in Section 8.2 of Chapter 6.

3. Supersymmetric Boussinesq equation

We discuss the construction of a supersymmetric extension of the Boussinesq equation. Conservation laws, nonlocal variables, symmetries and recursion operators for this supersymmetric system will be discussed too.

3.1. Construction of supersymmetric extensions. We start our discussion from the classical system [**14, 80**]

$$\begin{aligned} u_t &= -\frac{1}{2}u_{xx} + uu_x + v_x, \\ v_t &= \frac{1}{2}v_{xx} + uv_x + u_x v. \end{aligned} \tag{7.30}$$

We construct a so-called fermionic extension [**35**] by setting

$$\begin{aligned} \Phi &= \varphi + \theta u, \\ \Psi &= \psi + \theta v, \end{aligned} \tag{7.31}$$

where φ ψ, θ are odd variables.

Due to the classical grading of equation (7.30), i.e.,

$$\deg(u) = 1,\ \deg(v) = 2,\ \deg(x) = -1,\ \deg(t) = -2,$$

and the grading of the odd variables

$$\deg(\theta) = -\frac{1}{2},\quad \deg(\varphi) = \frac{1}{2},\quad \deg(\psi) = \frac{3}{2},$$

the variables Φ, Ψ are graded by

$$\deg(\Phi) = \frac{1}{2},\quad \deg(\Psi) = \frac{3}{2}.$$

Now we construct a formal extension of (7.30) by setting

$$\begin{aligned} u_t &= f_1[u, v, \varphi, \psi] \\ v_t &= f_2[u, v, \varphi, \psi] \\ \varphi_t &= f_3[u, v, \varphi, \psi] \\ \psi_t &= f_4[u, v, \varphi, \psi] \end{aligned} \tag{7.32}$$

where f_1, f_2, f_3, f_4 are functions of degrees 3, 4, 5/2, 7/2 respectively defined on the jet bundle $J^\infty(\pi)$, $\pi\colon (x, t, u, v) \mapsto (x, t)$, extended by the odd variables φ and ψ. The construction of f_1 and f_2 should be done in such a way that in the absence of odd variables f_1, f_2 reduce to the right-hand sides of (7.30). We now put on the following requirements on system (7.32), see [**88**]:

1. The existence of an odd symmetry of (7.32), i.e.,
$$Y_{\frac{1}{2}} = \varphi_1\frac{\partial}{\partial u} + \psi_1\frac{\partial}{\partial v} + u\frac{\partial}{\partial \varphi} + v\frac{\partial}{\partial \psi} + \dots,$$
$$[Y_{\frac{1}{2}}, Y_{\frac{1}{2}}] = 2(u_1\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v} + \varphi_1\frac{\partial}{\partial \varphi} + \psi_1\frac{\partial}{\partial \psi}) + \dots \doteq -2\frac{\partial}{\partial x}.$$
2. The existence of an even symmetry of (7.32) of appropriate degree which reduces to the classical first higher order symmetry of (7.30) in the absence of odd variables, i.e.,
$$\begin{aligned} X_3^{\text{clas}} = &\left(\frac{1}{3}u_3 - u_1^2 + 2uv_1 + 2vu_1 - uu_2 + u^2u_1\right)\frac{\partial}{\partial u} \\ &+ \left(\frac{1}{3}v_3 + u_1v_1 + 2vv_1 + uv_2 + 2uu_1v + u^2v_1\right)\frac{\partial}{\partial v}. \end{aligned} \tag{7.33}$$

From the above requirements we obtained the following supersymmetric extension of (7.30):
$$\begin{aligned} u_t &= -\frac{1}{2}u_2 + uu_1 + v_1, \\ v_t &= \frac{1}{2}v_2 + u_1v + uv_1 + \varphi_1\psi_1 + \varphi_2\psi, \\ \varphi_t &= -\frac{1}{2}\varphi_2 + \psi_1 + u\varphi_1, \\ \psi_t &= \frac{1}{2}\psi_2 + u\psi_1 + u_1\psi, \end{aligned} \tag{7.34}$$
while the symmetry X_3 is given by
$$\begin{aligned} X_3 = &\Big(\frac{1}{3}u_3 - u_1^2 + 2vu_1 - uu_2 + 2uv_1 + u^2u_1 + \varphi_1\psi_1 + \varphi_2\psi\Big)\frac{\partial}{\partial u} \\ &+ \Big(\frac{1}{3}v_3 + u_1v_1 + 2vv_1 + uv_2 + 2uvu_1 + u^2v_1 + \varphi_2\psi_1 + \varphi_1\psi_2 \\ &+ 2u\varphi_1\psi_1 - \psi\psi_2 + 2\varphi_2\psi u + 2u_1\varphi_1\psi\Big)\frac{\partial}{\partial v} \\ &+ \Big(\frac{1}{3}\varphi_3 - u\varphi_2 + 2u\psi_1 + u^2\varphi_1 + v\varphi_1 - u_1\varphi_1 + u_1\psi\Big)\partial\varphi \\ &+ \Big(\frac{1}{3}\psi_3 + u\psi_2 + u^2\psi_1 + v\psi_1 + u_1\psi_1 + 2uu_1\psi + v_1\psi\Big)\frac{\partial}{\partial \psi}. \end{aligned} \tag{7.35}$$
The resulting supersymmetric extension of the Boussinesq equation is just the same as mentioned in [**67**].

3.2. Construction of conserved quantities and nonlocal variables. For the supersymmetric extension (7.34) of the Boussinesq equation we constructed the following set of conserved densities (X), associated conserved quantities $(\int_{-\infty}^{\infty} X\,dx)$ and nonlocal variables $D^{-1}(X)$, i.e, the variables p_i of degree i, q_j of degree j:
$$p_0 = D^{-1}(u),$$

$$\begin{aligned}
p_1 &= D^{-1}(v),\\
p_2 &= D^{-1}(uv + \varphi_1\psi),\\
p_3 &= D^{-1}(v^2 + uv_1 + u^2v + 2u\varphi_1\psi + \varphi_1\psi_1 - \psi\psi_1),\\
\overline{p}_0 &= D^{-1}(p_1),\\
\overline{p}_1 &= D^{-1}(\psi q_{\frac{1}{2}} + \varphi\psi),\\
\overline{p}_2 &= D^{-1}(p_1\varphi\psi - u\varphi\psi + \varphi_1\psi - u\psi q_{\frac{1}{2}} + p_1\varphi_1 q_{\frac{1}{2}}),\\
\overline{p}_3 &= 2D^{-1}\Big((p_2\varphi_1 - u^2\psi - 2v\psi + u_1\psi - p_1v\varphi - \varphi v_1)q_{\frac{1}{2}}\\
&\quad + (u\psi - p_1\psi)\overline{q}_{\frac{3}{2}} + (-u^2 + p_1u - 2v + u_1 - p_1^2 + p_2)\varphi\psi\\
&\quad - u^2v - uv_1 - v^2\Big)
\end{aligned}$$

and

$$\begin{aligned}
q_{\frac{1}{2}} &= D^{-1}(\psi),\\
q_{\frac{3}{2}} &= D^{-1}(u\psi + v\varphi),\\
\overline{q}_{1\frac{1}{2}} &= D^{-1}(q_{\frac{1}{2}}v + p_1\varphi_1),\\
q_{\frac{5}{2}} &= D^{-1}(-\varphi_1p_1^2 + p_2(2\psi - 2\varphi_1) - 2(p_1v + v_1)q_{\frac{1}{2}} - 2v\varphi_1),\\
\overline{q}_{\frac{5}{2}} &= D^{-1}\Big(\frac{1}{2}\varphi_1p_1^2 + p_2(-2\psi + \varphi_1) + (uv - 2uu_1\\
&\quad + u_1p_1 + u_2)q_{\frac{1}{2}} + v\varphi_1\Big).
\end{aligned}$$

Note that the variables $\overline{p}_0$, $\overline{p}_1, \dots$ contain higher order nonlocalities. In fact, introduction of the nonlocal variables p_0, $\overline{p}_0, \dots, q_{\frac{1}{2}}$, $q_{\frac{3}{2}}$, $\overline{q}_{\frac{3}{2}}, \dots$ is essential for the construction of nonlocal symmetries, while the associated Cartan forms $\omega_{p_0}, \omega_{\overline{p}_0}, \dots, \omega_{q_{\frac{1}{2}}}, \dots$ play a significant role in the construction of deformations or recursion operators.

3.3. Symmetries. We obtained the following symmetries for the supersymmetric extension of Boussinesq equation (7.34):

$$\begin{aligned}
Y_{\frac{1}{2}} &= \varphi_1\frac{\partial}{\partial u} + \psi_1\frac{\partial}{\partial v} + u\frac{\partial}{\partial \varphi} + v\frac{\partial}{\partial \psi} + \dots,\\
\overline{Y}_{\frac{1}{2}} &= \psi\frac{\partial}{\partial u} + \psi_1\frac{\partial}{\partial v} + (u - p_1)\frac{\partial}{\partial \varphi} + \dots,\\
X_1 &= u_1\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v} + \varphi_1\frac{\partial}{\partial \varphi} + \psi_1\frac{\partial}{\partial \psi} + \dots,\\
\overline{X}_1 &= (\varphi\psi + \varphi_1 q_{\frac{1}{2}})\frac{\partial}{\partial u} + (\varphi\psi_1 + \varphi_1\psi + \psi_1 q_{\frac{1}{2}})\frac{\partial}{\partial v}\\
&\quad + (-uq_{\frac{1}{2}} - q_{\frac{3}{2}} - \varphi_1 + u\varphi)\frac{\partial}{\partial \varphi} + (-vq_{\frac{1}{2}} - \psi_1 - u\psi)\frac{\partial}{\partial \psi} + \dots,
\end{aligned}$$

$$
\begin{aligned}
Y_{\frac{3}{2}} &= (-2q_{\frac{1}{2}}u_1 - \varphi_2 + u\varphi_1 + p_1\varphi_1 - 3u\psi + u_1\varphi)\frac{\partial}{\partial u} \\
&+ (-2q_{\frac{1}{2}}v_1 - \psi_2 + 2u\psi_1 + p_1\psi_1 - v\varphi_1 - v\psi - 2u_1\psi + v_1\varphi)\frac{\partial}{\partial v} \\
&+ (-2q_{\frac{1}{2}}\varphi_1 + \varphi\varphi_1 - u^2 + p_1u + u_1 + 2p_2)\frac{\partial}{\partial \varphi} \\
&+ (-2q_{\frac{1}{2}}\psi_1 + 2\varphi_1\psi + \varphi\psi_1 + uv + p_1v + v_1)\frac{\partial}{\partial \psi} + \dots, \\
\overline{Y}_{\frac{3}{2}} &= (-q_{\frac{1}{2}}u_1 - \psi_1 - 2u\psi + p_1\psi)\frac{\partial}{\partial u} \\
&+ (-q_{\frac{1}{2}}v_1 - \psi_2 - 2u\psi_1 + p_1\psi_1 - 2u_1\psi)\frac{\partial}{\partial v} \\
&+ (-q_{\frac{1}{2}}\varphi_1 - u^2 + p_1u - v + u_1 - \frac{1}{2}p_1^2 + p_2)\frac{\partial}{\partial \varphi} - q_{\frac{1}{2}}\psi_1\frac{\partial}{\partial \psi} + \dots,
\end{aligned}
$$

3.4. Deformation and recursion operator. In a way, analogously to previous applications, we construct a deformation of the equation structure U related to the supersymmetric Boussinesq equation, i.e.,

$$
\begin{aligned}
U_1 = &\Big(\omega_{u_1} - 2\omega_v - \omega_u u - \omega_{p_0}u_1 - \omega_\varphi\psi + \omega_{q_{\frac{1}{2}}}(2\psi - \varphi_1)\Big)\frac{\partial}{\partial u} \\
&+ \Big(- \omega_{v_1} - \omega_v u - 2\omega_u v - 2\omega_{\varphi_1}\psi - \omega_\varphi\psi_1 + \omega_\psi(\varphi_1 + \psi) \\
&- \omega_{p_0}v_1 + \omega_{q_{\frac{1}{2}}}\psi_1\Big)\frac{\partial}{\partial v} \\
&+ \Big(\omega_{\varphi_1} - 2\omega_\psi + \omega_\varphi(2p_1 - u) - \omega_{p_0}\varphi_1 + \omega_{p_1}(2q_{\frac{1}{2}} + \varphi) \\
&- \omega_{q_{\frac{3}{2}}} - 2\omega_{\overline{q}_{\frac{3}{2}}} + \omega_{q_{\frac{1}{2}}}u\Big)\frac{\partial}{\partial \varphi} \\
&+ \Big(- \omega_{\psi_1} - \omega_\psi u - 2\omega_u\psi - \omega_{p_0}\psi_1 + \omega_{p_1}\psi - \omega_{q_{\frac{1}{2}}}v\Big)\frac{\partial}{\partial \psi}.
\end{aligned}
$$

From the deformation U, we obtain four hierarchies of (x, t)-independent symmetries $\{Y_{n+\frac{1}{2}}\}$, $\{\overline{Y}_{n+\frac{1}{2}}\}$, $\{X_{n+1}\}$, $\{\overline{X}_{n+1}\}$, $n \in \mathbb{N}$, by

$$
\begin{aligned}
Y_{n+\frac{1}{2}} &= (\dots(Y_{\frac{1}{2}} \lrcorner U_1)\dots \lrcorner U_1), \\
\overline{Y}_{n+\frac{1}{2}} &= (\dots(\overline{Y}_{\frac{1}{2}} \lrcorner U_1)\dots \lrcorner U_1), \\
X_{n+1} &= (\dots(X_1 \lrcorner U_1)\dots \lrcorner U_1), \\
\overline{X}_{n+1} &= (\dots(\overline{X}_1 \lrcorner U_1)\dots \lrcorner U_1),
\end{aligned}
$$

and an (x, t)-dependent hierarchy defined by

$$S_n = (\dots(S_0 \lrcorner U_1)\dots \lrcorner U_1),$$

where S_0 is defined by

$$S_0 = (u + xu_1 + 2tu_t)\frac{\partial}{\partial u} + (2v + xv_1 + 2tv_t)\frac{\partial}{\partial v}$$

$$+\Big(\frac{1}{2}\varphi + x\varphi_1 + 2t\varphi_t\Big)\frac{\partial}{\partial\varphi} + \Big(\frac{3}{2}\psi + x\psi_1 + 2t\psi_t\Big)\frac{\partial}{\partial\psi} + \cdots$$

In effect, the hierarchies $\{\overline{Y}_{n+\frac{1}{2}}\}$ and $\{\overline{X}_{n+1}\}$ start at symmetries

$$\overline{Y}_{-\frac{1}{2}} = \frac{\partial}{\partial\varphi}$$

and

$$\overline{X}_0 = (2q_{\frac{1}{2}} - \varphi)\frac{\partial}{\partial\varphi} + \psi\frac{\partial}{\partial\psi}$$

respectively.

4. Supersymmetric extensions of the KdV equation, $N = 2$

In this chapter we shall discuss the supersymmetric extensions of the classical KdV equation

$$u_t = -u_{xxx} + 6uu_x \tag{7.36}$$

with two odd variables, the situation $N = 2$. The construction of such supersymmetric systems runs along similar lines as has been explained for the supersymmetric extension of the classical nonlinear Schrödinger equation, cf. Section 8 of Chapter 6. For additional references see also [**68, 87, 64, 65, 63, 82, 79**].

The extension is obtained by considering two odd (pseudo) total derivative operators D_1 and D_2 given by

$$D_1 = \partial_{\theta_1} + \theta_1 D_x, \quad D_2 = \partial_{\theta_2} + \theta_2 D_x, \tag{7.37}$$

where θ_1, θ_2 are two odd parameters. Obviously, these operators satisfy the relations $D_1^2 = D_2^2 = D_x$ and $[D_1, D_2] = 0$.

The $N = 2$ supersymmetric extension of the KdV equation is obtained by taking an even homogeneous field Φ

$$\Phi = w + \theta_1\psi + \theta_2\varphi + \theta_2\theta_1 u \tag{7.38}$$

with degrees $\deg(\Phi) = 1$, $\deg(u) = 2$, $\deg(w) = 1$, $\deg(\varphi) = \deg(\psi) = 3/2$, $\deg(\theta_1) = \deg(\theta_2) = -1/2$, and considering the most general evolution equation for Φ, which reduces to the KdV equation in the absence of the odd variables φ, ψ.

Proceeding in this way, we arrive at the system

$$\Phi_t = D_x\left(-D_x^2\Phi + 3\Phi D_1D_2\Phi + \frac{1}{2}(a-1)D_1D_2\Phi^2 + a\Phi^3\right). \tag{7.39}$$

Rewriting this system in components, we arrive at a system of partial differential equations for the two even variables u, w and the two odd variables φ, ψ, i.e.,

$$\begin{aligned} u_t = D_x\big(&-u_2 + 3u^2 - 3\varphi\varphi_1 - 3\psi\psi_1 - (a-1)w_1^2 \\ &- (a+2)ww_2 + 3auw^2 + 6aw\psi\varphi\big), \end{aligned}$$

$$
\begin{aligned}
\varphi_t &= D_x\big(- \varphi_2 + 3u\varphi + 3aw^2\varphi - (a+2)w\psi_1 - (a-1)w_1\psi\big),\\
\psi_t &= D_x\big(- \psi_2 + 3u\psi + 3aw^2\psi + (a+2)w\varphi_1 + (a-1)w_1\varphi\big),\\
w_t &= D_x\big(- w_2 + aw^3 + (a+2)uw + (a-1)\psi\varphi\big),
\end{aligned} \tag{7.40}
$$

or equivalently,

$$
\begin{aligned}
u_t &= -u_3 + 6uu_1 - 3\varphi\varphi_2 - 3\psi\psi_2 - 3aw_1w_2 - (a+2)ww_3 + 3au_1w^2\\
&\quad + 6auww_1 + 6aw_1\psi\varphi + 6aw\psi_1\varphi + 6aw\psi\varphi_1,\\
\varphi_t &= -\varphi_3 + 3u_1\varphi + 3u\varphi_1 + 6aww_1\varphi + 3aw^2\varphi_1 - (a+2)w_1\psi_1\\
&\quad - (a+2)w\psi_2 - (a-1)w_2\psi - (a-1)w_1\psi_1,\\
\psi_t &= -\psi_3 + 3u_1\psi + 3u\psi_1 + 6aww_1\psi + 3aw^2\psi_1 + (a+2)w_1\varphi_1\\
&\quad + (a+2)w\varphi_2 + (a-1)w_2\varphi + (a-1)w_1\varphi_1,\\
w_t &= -w_3 + 3aw^2w_1 + (a+2)u_1w + (a+2)uw_1 + (a-1)\psi_1\varphi\\
&\quad + (a-1)\psi\varphi_1.
\end{aligned} \tag{7.41}
$$

It has been demonstrated by several authors [**87, 74**] that the interesting equations from the point of view of complete integrability are the special cases $a = -2, 1, 4$.

In Subsection 4.1 we discuss the case $a = -2$. We shall present in the respective subsections results for the construction of local and nonlocal conservation laws, nonlocal symmetries and finally present the recursion operator for symmetries. A similar presentation is chosen for Subsections 4.2, where we deal with the case $a = 4$, and finally in Subsections 4.3 we present the results for the most intriguing case $a = 1$.

The structure is extremely complicated in this case, which can be illustrated from the fact that in order to find a good setting for the recursion operator for symmetries, we had to introduce a total of 16 nonlocal variables associated to the respective conservation laws, while the complete computation for the recursion operation required the introduction and fixing of more than 20,000 constants.

4.1. Case $a = -2$. In this subsection we discuss the case $a = -2$, which leads to the following system of partial differential equations

$$
\begin{aligned}
u_t &= -u_3 + 6uu_1 - 3\varphi\varphi_2 - 3\psi\psi_2 + 6w_1w_2 - 6u_1w^2 - 12uww_1\\
&\quad - 12w_1\psi\varphi - 12w\psi_1\varphi - 12w\psi\varphi_1,\\
\varphi_t &= -\varphi_3 + 3u_1\varphi + 3u\varphi_1 - 12ww_1\varphi - 6w^2\varphi_1 + 3w_2\psi + 3w_1\psi_1,\\
\psi_t &= -\psi_3 + 3u_1\psi + 3u\psi_1 - 12ww_1\psi - 6w^2\psi_1 - 3w_2\varphi - 3w_1\varphi_1,\\
w_t &= -w_3 - 6w^2w_1 - 3\psi_1\varphi - 3\psi\varphi_1.
\end{aligned} \tag{7.42}
$$

The results obtained in this case for conservation laws, higher symmetries and deformations or recursion operator will be presented in subsequent subsections.

4.1.1. *Conservation laws.* For the even conservation laws and the associated even nonlocal variables we obtained the following results.

1. Nonlocal variables $p_{0,1}$ and $p_{0,2}$ of degree 0 defined by

$$\begin{aligned}(p_{0,1})_x &= w,\\ (p_{0,1})_t &= 3\varphi\psi - 2w^3 - w_2;\\ (p_{0,2})_x &= p_{1,1},\\ (p_{0,2})_t &= 12p_{3,1} - u_1 + 3ww_1\end{aligned} \tag{7.43}$$

(see the definition of $p_{1,1}$ and $p_{3,1}$ below).

2. Nonlocal variables $p_{1,1}$, $p_{1,2}$, $p_{1,3}$, $p_{1,4}$ of degree 1 defined by the relations

$$\begin{aligned}(p_{1,1})_x &= u,\\ (p_{1,1})_t &= -3\psi\psi_1 - 3\varphi\varphi_1 + 12\varphi\psi w + 3u^2 - 6uw^2 - u_2 + 3w_1^2;\\ (p_{1,2})_x &= \psi\overline{q}_{\frac{1}{2}} - \varphi q_{\frac{1}{2}},\\ (p_{1,2})_t &= -\psi_2\overline{q}_{\frac{1}{2}} + \varphi_2 q_{\frac{1}{2}} + 3\psi\overline{q}_{\frac{1}{2}}u\\ &\quad - 6\psi\overline{q}_{\frac{1}{2}}w^2 - 3\psi q_{\frac{1}{2}}w_1 - 2\psi\psi_1 - 3\varphi\overline{q}_{\frac{1}{2}}w_1 - 3\varphi q_{\frac{1}{2}}u + 6\varphi q_{\frac{1}{2}}w^2 + 2\varphi\varphi_1;\\ (p_{1,3})_x &= \psi q_{\frac{1}{2}},\\ (p_{1,3})_t &= -\psi_2 q_{\frac{1}{2}} + 3\psi q_{\frac{1}{2}}u - 6\psi q_{\frac{1}{2}}w^2 + \varphi_1\psi - 3\varphi q_{\frac{1}{2}}w_1 - \varphi\psi_1;\\ (p_{1,4})_x &= \varphi q_{\frac{1}{2}} + w^2,\\ (p_{1,4})_t &= -\varphi_2 q_{\frac{1}{2}} + 3\psi q_{\frac{1}{2}}w_1 + 3\varphi q_{\frac{1}{2}}u - 6\varphi q_{\frac{1}{2}}w^2\\ &\quad - 2\varphi\varphi_1 + 6\varphi\psi w - 3w^4 - 2ww_2 + w_1^2\end{aligned} \tag{7.44}$$

(the variables $q_{\frac{1}{2}}$ and $\overline{q}_{\frac{1}{2}}$ are defined below).

3. Nonlocal variable $p_{2,1}$ of degree 2 defined by

$$\begin{aligned}(p_{2,1})_x &= q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}}u + \psi_1 q_{\frac{1}{2}} + \psi\overline{q}_{\frac{1}{2}}w + \varphi q_{\frac{1}{2}}w,\\ (p_{2,1})_t &= 3q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}}u^2 - 6q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}}uw^2 - q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}}u_2 + 3q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}}w_1^2 - \psi_3 q_{\frac{1}{2}} - \psi_2\overline{q}_{\frac{1}{2}}w\\ &\quad - \varphi_2 q_{\frac{1}{2}}w + \psi_1\overline{q}_{\frac{1}{2}}w_1 + 4\psi_1 q_{\frac{1}{2}}u - 6\psi_1 q_{\frac{1}{2}}w^2 - \varphi_1\overline{q}_{\frac{1}{2}}u - 2\varphi_1 q_{\frac{1}{2}}w_1\\ &\quad + \varphi_1\psi_1 + 3\psi\overline{q}_{\frac{1}{2}}uw - 6\psi\overline{q}_{\frac{1}{2}}w^3 - \psi\overline{q}_{\frac{1}{2}}w_2 + 2\psi q_{\frac{1}{2}}u_1 - 9\psi q_{\frac{1}{2}}ww_1\\ &\quad - 3\psi\psi_1 q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}} - 2\psi\psi_1 w + \varphi\overline{q}_{\frac{1}{2}}u_1 - 3\varphi\overline{q}_{\frac{1}{2}}ww_1 + 3\varphi q_{\frac{1}{2}}uw - 6\varphi q_{\frac{1}{2}}w^3\\ &\quad - 4\varphi q_{\frac{1}{2}}w_2 - \varphi\psi_2 - 3\varphi\varphi_1 q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}} - 2\varphi\varphi_1 w + 12\varphi\psi q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}}w + \varphi\psi u.\end{aligned} \tag{7.45}$$

4. Finally, the variable $p_{3,1}$ of degree 3 defined by

$$(p_{3,1})_x = \frac{1}{4}(-\psi\psi_1 - \varphi\varphi_1 + 4\varphi\psi w + u^2 - 2uw^2 - ww_2),$$

$$\begin{aligned}(p_{3,1})_t = \frac{1}{4}(&-2\psi_1\psi_2 - 2\varphi_1\varphi_2 - 2\varphi_1\psi_1 w + \psi\psi_3 + 7\psi\varphi_2 w - 9\psi\psi_1 u \\ &+ 12\psi\psi_1 w^2 + 4\varphi_1\psi w_1 + \varphi\varphi_3 - 7\varphi\psi_2 w + 4\varphi\psi_1 w_1 - 9\varphi\varphi_1 u \\ &+ 12\varphi\varphi_1 w^2 + 24\varphi\psi u w - 48\varphi\psi w^3 - 10\varphi\psi w_2 + 4u^3 - 12u^2 w^2 \\ &- 2uu_2 + 12uw^4 + 4uww_2 + 4uw_1^2 + u_1^2 - 4u_1 w w_1 + 2u_2 w^2 \\ &+ 6w^3 w_2 + 6w^2 w_1^2 + ww_4 - w_1 w_3 + w_2^2).\end{aligned} \tag{7.46}$$

REMARK 7.1. It should be noted that the first lower index refers to the degree of the object (in this case the nonlocal variable), while the second lower index is referring to the numbering of the objects of that specific degree. The number of nonlocal variables of degree 3 is 4, since this number is the same as for nonlocal variables of degree 1, cf. (7.44). This total number will arise after introduction of these nonlocal variables and computation of the conservation laws and the associated nonlocal variables in this augmented setting. These conservation laws and their associated nonlocal variables are of a higher nonlocality. We shall not pursue this further here, because the number of nonlocal variables found will turn out to be sufficient to compute the deformation of the system of equations (7.42), or equivalently the construction of the recursion operator for symmetries. We refer for a more comprehensive computation to Subsection 4.3, where *all* nonlocal variables at the levels turn out to be essential in the computation of the recursion operator for that case.

For the odd conservation laws and the associated odd nonlocal variables we derived the following results.

1. At degree 1/2 we computed the variables $q_{\frac{1}{2}}$ and $\overline{q}_{\frac{1}{2}}$ defined by

$$\begin{aligned}(q_{\frac{1}{2}})_x &= \varphi, \\ (q_{\frac{1}{2}})_t &= -\varphi_2 + 3\psi w_1 + 3\varphi u - 6\varphi w^2; \\ (\overline{q}_{\frac{1}{2}})_x &= \psi, \\ (\overline{q}_{\frac{1}{2}})_t &= -\psi_2 + 3\psi u - 6\psi w^2 - 3\varphi w_1.\end{aligned} \tag{7.47}$$

2. At degree 3/2 we have the variables $q_{\frac{3}{2}}$ and $\overline{q}_{\frac{3}{2}}$ defined by

$$\begin{aligned}(q_{\frac{3}{2}})_x &= \overline{q}_{\frac{1}{2}} u - \varphi w, \\ (q_{\frac{3}{2}})_t &= 3\overline{q}_{\frac{1}{2}} u^2 - 6\overline{q}_{\frac{1}{2}} u w^2 - \overline{q}_{\frac{1}{2}} u_2 + 3\overline{q}_{\frac{1}{2}} w_1^2 + \varphi_2 w - \psi_1 u - \varphi_1 w_1 - 3\psi\psi_1\overline{q}_{\frac{1}{2}} \\ &\quad + \psi u_1 - 3\psi w w_1 - 3\varphi\varphi_1\overline{q}_{\frac{1}{2}} + 12\varphi\psi\overline{q}_{\frac{1}{2}} w - 3\varphi u w + 6\varphi w^3 + \varphi w_2; \\ (\overline{q}_{\frac{3}{2}})_x &= -(q_{\frac{1}{2}} u + \psi w), \\ (\overline{q}_{\frac{3}{2}})_t &= -3q_{\frac{1}{2}} u^2 + 6q_{\frac{1}{2}} u w^2 + q_{\frac{1}{2}} u_2 - 3q_{\frac{1}{2}} w_1^2 + \psi_2 w - \psi_1 w_1 + \varphi_1 u + 3\psi\psi_1 q_{\frac{1}{2}} \\ &\quad - 3\psi u w + 6\psi w^3 + \psi w_2 + 3\varphi\varphi_1 q_{\frac{1}{2}} - 12\varphi\psi q_{\frac{1}{2}} w - \varphi u_1 + 3\varphi w w_1.\end{aligned} \tag{7.48}$$

3. Finally, at degree 5/2 we obtained $q_{\frac{5}{2}}$ and $\overline{q}_{\frac{5}{2}}$ defined by the relations

$$
\begin{aligned}
(q_{\frac{5}{2}})_x &= \overline{q}_{\frac{1}{2}}p_{1,1}u + 3\overline{q}_{\frac{1}{2}}ww_1 + \varphi_1 w + \psi u - \varphi p_{1,1}w,\\
(q_{\frac{5}{2}})_t &= 3\overline{q}_{\frac{1}{2}}p_{1,1}u^2 - 6\overline{q}_{\frac{1}{2}}p_{1,1}uw^2 - \overline{q}_{\frac{1}{2}}p_{1,1}u_2 + 3\overline{q}_{\frac{1}{2}}p_{1,1}w_1^2 - 18\overline{q}_{\frac{1}{2}}w^3w_1\\
&\quad - 3\overline{q}_{\frac{1}{2}}ww_3 - \varphi_3 w - \psi_2 u + \varphi_2 p_{1,1}w + \varphi_2 w_1 - \psi_1 p_{1,1}u + \psi_1 u_1\\
&\quad - \varphi_1 p_{1,1}w_1 + 2\varphi_1 uw - 6\varphi_1 w^3 - \varphi_1 w_2 - 3\psi\psi_1\overline{q}_{\frac{1}{2}}p_{1,1} - 9\psi\varphi_1\overline{q}_{\frac{1}{2}}w\\
&\quad + \psi p_{1,1}u_1 - 3\psi p_{1,1}ww_1 + 4\psi u^2 - 6\psi uw^2 - \psi u_2 + 6\psi ww_2\\
&\quad + 9\varphi\psi_1\overline{q}_{\frac{1}{2}}w - 3\varphi\varphi_1\overline{q}_{\frac{1}{2}}p_{1,1} + 12\varphi\psi\overline{q}_{\frac{1}{2}}p_{1,1}w + 3\varphi\psi\varphi_1\\
&\quad - 3\varphi p_{1,1}uw + 6\varphi p_{1,1}w^3 + \varphi p_{1,1}w_2 - 4\varphi uw_1 + 4\varphi u_1 w - 12\varphi w^2 w_1;\\
(\overline{q}_{\frac{5}{2}})_x &= -q_{\frac{1}{2}}p_{1,1}u + q_{\frac{1}{2}}u_1 - 3q_{\frac{1}{2}}ww_1 + \psi_1 w - \psi p_{1,1}w,\\
(\overline{q}_{\frac{5}{2}})_t &= -3q_{\frac{1}{2}}p_{1,1}u^2 + 6q_{\frac{1}{2}}p_{1,1}uw^2 + q_{\frac{1}{2}}p_{1,1}u_2 - 3q_{\frac{1}{2}}p_{1,1}w_1^2 + 6q_{\frac{1}{2}}uu_1\\
&\quad - 12q_{\frac{1}{2}}uww_1 - 6q_{\frac{1}{2}}u_1w^2 - q_{\frac{1}{2}}u_3 + 18q_{\frac{1}{2}}w^3w_1 + 3q_{\frac{1}{2}}ww_3 + 6q_{\frac{1}{2}}w_1w_2\\
&\quad - \psi_3 w + \psi_2 p_{1,1}w + \psi_2 w_1 - \psi_1 p_{1,1}w_1 + 2\psi_1 uw - 6\psi_1 w^3 - \psi_1 w_2\\
&\quad + \varphi_1 p_{1,1}u - \varphi_1 u_1 - 3\psi\psi_2 q_{\frac{1}{2}} + 3\psi\psi_1 q_{\frac{1}{2}}p_{1,1} - 3\psi\varphi_1 q_{\frac{1}{2}}w - 3\psi p_{1,1}uw\\
&\quad + 6\psi p_{1,1}w^3 + \psi p_{1,1}w_2 - \psi uw_1 + 4\psi u_1 w - 12\psi w^2 w_1 - 3\varphi\varphi_2 q_{\frac{1}{2}}\\
&\quad + 3\varphi\psi_1 q_{\frac{1}{2}}w + 3\varphi\varphi_1 q_{\frac{1}{2}}p_{1,1} - 12\varphi\psi q_{\frac{1}{2}}p_{1,1}w + 12\varphi\psi q_{\frac{1}{2}}w_1 + 3\varphi\psi\psi_1\\
&\quad - \varphi p_{1,1}u_1 + 3\varphi p_{1,1}ww_1 - \varphi u^2 + \varphi u_2 - 6\varphi ww_2.
\end{aligned}
\tag{7.49}
$$

Thus the entire nonlocal setting comprises the following 14 nonlocal variables:

$$
\begin{array}{ll}
p_{0,1},\ p_{0,2} & \text{of degree } 0,\\
p_{1,1},\ p_{1,2},\ p_{1,3},\ p_{1,4} & \text{of degree } 1,\\
p_{2,1} & \text{of degree } 2,\\
p_{3,1} & \text{of degree } 3,\\
q_{\frac{1}{2}},\ \overline{q}_{\frac{1}{2}} & \text{of degree } \dfrac{1}{2},\\
q_{\frac{3}{2}},\ \overline{q}_{\frac{3}{2}} & \text{of degree } \dfrac{3}{2},\\
q_{\frac{5}{2}},\ \overline{q}_{\frac{5}{2}} & \text{of degree } \dfrac{5}{2}.
\end{array}
\tag{7.50}
$$

In the next subsections the augmented system of equations associated to the local and the nonlocal variables denoted above will be considered in computing higher and nonlocal symmetries and the recursion operator.

4.1.2. *Higher and nonlocal symmetries.* In this subsection, we present results for higher and nonlocal symmetries for the $N = 2$ supersymmetric extension of KdV equation (7.42),

$$Y = Y^u \frac{\partial}{\partial u} + Y^w \frac{\partial}{\partial w} + Y^\varphi \frac{\partial}{\partial \varphi} + Y^\psi \frac{\partial}{\partial \psi} + \dots$$

We obtained the following odd symmetries, just giving here the components of their generating functions,

$$\begin{aligned} Y^u_{\frac{1}{2},1} &= \psi_1, & Y^u_{\frac{1}{2},2} &= \varphi_1, \\ Y^w_{\frac{1}{2},1} &= -\varphi, & Y^u_{\frac{1}{2},2} &= \varphi_1, \\ Y^\varphi_{\frac{1}{2},1} &= -w_1, & Y^\varphi_{\frac{1}{2},2} &= u, \\ Y^\psi_{\frac{1}{2},1} &= u; & Y^\psi_{\frac{1}{2},2} &= w_1 \end{aligned} \tag{7.51}$$

and

$$\begin{aligned} Y^u_{\frac{3}{2},1} &= 2q_{\frac{1}{2}}u_1 - \varphi_2 + 3\psi_1 w - \varphi_1 p_{1,1} + 3\psi w_1 + \varphi u, \\ Y^w_{\frac{3}{2},1} &= 2q_{\frac{1}{2}}w_1 + \psi_1 - \psi p_{1,1} + \varphi w, \\ Y^\varphi_{\frac{3}{2},1} &= -2\varphi_1 q_{\frac{1}{2}} - p_{1,1}u + u_1 - 3ww_1, \\ Y^\psi_{\frac{3}{2},1} &= -2\psi_1 q_{\frac{1}{2}} + 2\varphi\psi - p_{1,1}w_1 - uw - w_2; \\ Y^u_{\frac{3}{2},2} &= 2\overline{q}_{\frac{1}{2}}u_1 - \psi_2 - \psi_1 p_{1,1} - 3\varphi_1 w + \psi u - 3\varphi w_1, \\ Y^w_{\frac{3}{2},2} &= 2\overline{q}_{\frac{1}{2}}w_1 - \varphi_1 + \psi w + \varphi p_{1,1}, \\ Y^\varphi_{\frac{3}{2},2} &= -2\varphi_1 \overline{q}_{\frac{1}{2}} - 2\varphi\psi + p_{1,1}w_1 + uw + w_2, \\ Y^\psi_{\frac{3}{2},2} &= -2\psi_1 \overline{q}_{\frac{1}{2}} - p_{1,1}u + u_1 - 3ww_1. \end{aligned} \tag{7.52}$$

We also obtained the following even symmetries:

$$\begin{aligned} Y^u_{0,1} &= 0, \\ Y^w_{0,1} &= 0, \\ Y^\varphi_{0,1} &= \psi, \\ Y^\psi_{0,1} &= -\varphi; \\ Y^u_{1,1} &= u_1, \\ Y^w_{1,1} &= w_1, \\ Y^\varphi_{1,1} &= \varphi_1, \\ Y^\psi_{1,1} &= \psi_1; \\ Y^u_{1,2} &= \varphi_1 q_{\frac{1}{2}} + 2ww_1, \\ Y^w_{1,2} &= \psi q_{\frac{1}{2}} + w_1, \\ Y^\varphi_{1,2} &= -q_{\frac{1}{2}}u + \varphi_1 - \psi w, \end{aligned}$$

$$
\begin{aligned}
Y^{\psi}_{1,2} &= -q_{\frac{1}{2}} w_1 - \varphi w;\\[1ex]
Y^{u}_{1,3} &= \psi_1 \overline{q}_{\frac{1}{2}} - \varphi_1 q_{\frac{1}{2}},\\
Y^{w}_{1,3} &= -\psi q_{\frac{1}{2}} - \varphi \overline{q}_{\frac{1}{2}},\\
Y^{\varphi}_{1,3} &= \overline{q}_{\frac{1}{2}} w_1 + q_{\frac{1}{2}} u - \varphi_1 + 2\psi w,\\
Y^{\psi}_{1,3} &= -\overline{q}_{\frac{1}{2}} u + q_{\frac{1}{2}} w_1 + \psi_1 + 2\varphi w;\\[1ex]
Y^{u}_{1,4} &= \psi_1 q_{\frac{1}{2}} + \varphi_1 \overline{q}_{\frac{1}{2}},\\
Y^{w}_{1,4} &= \psi \overline{q}_{\frac{1}{2}} - \varphi q_{\frac{1}{2}},\\
Y^{\varphi}_{1,4} &= -\overline{q}_{\frac{1}{2}} u + q_{\frac{1}{2}} w_1 + \psi_1 + 2\varphi w,\\
Y^{\psi}_{1,4} &= -\overline{q}_{\frac{1}{2}} w_1 - q_{\frac{1}{2}} u + \varphi_1 - 2\psi w.
\end{aligned}
\tag{7.53}
$$

Moreover there is a symmetry of degree 2 with the generating function

$$
\begin{aligned}
Y^{u}_{2,1} &= 2q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}} u_1 + \psi_2 q_{\frac{1}{2}} - \varphi_2 \overline{q}_{\frac{1}{2}} - \psi_1 \overline{q}_{\frac{3}{2}} + 3\psi_1 \overline{q}_{\frac{1}{2}} w - \varphi_1 q_{\frac{3}{2}} + 3\varphi_1 q_{\frac{1}{2}} w\\
&\quad + 3\psi \overline{q}_{\frac{1}{2}} w_1 - \psi q_{\frac{1}{2}} u + \varphi \overline{q}_{\frac{1}{2}} u + 3\varphi q_{\frac{1}{2}} w_1 + \varphi_1 \psi + \varphi \psi_1,\\
Y^{w}_{2,1} &= 2q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}} w_1 + \psi_1 \overline{q}_{\frac{1}{2}} + \varphi_1 q_{\frac{1}{2}} - \psi q_{\frac{3}{2}} - \psi q_{\frac{1}{2}} w + \varphi \overline{q}_{\frac{3}{2}} + \varphi \overline{q}_{\frac{1}{2}} w,\\
Y^{\varphi}_{2,1} &= -\overline{q}_{\frac{3}{2}} w_1 + q_{\frac{3}{2}} u - \overline{q}_{\frac{1}{2}} u_1 + 3\overline{q}_{\frac{1}{2}} w w_1 + q_{\frac{1}{2}} u w + q_{\frac{1}{2}} w_2 + \psi_2 + 2\varphi_1 q_{\frac{1}{2}} \overline{q}_{\frac{1}{2}}\\
&\quad - 2\psi u + 4\psi w^2 - 2\varphi\psi q_{\frac{1}{2}} + 2\varphi w_1,\\
Y^{\psi}_{2,1} &= \overline{q}_{\frac{3}{2}} u + q_{\frac{3}{2}} w_1 + \overline{q}_{\frac{1}{2}} u w + \overline{q}_{\frac{1}{2}} w_2 + q_{\frac{1}{2}} u_1 - 3q_{\frac{1}{2}} w w_1 - \varphi_2 + 2\psi_1 q_{\frac{1}{2}} \overline{q}_{\frac{1}{2}}\\
&\quad + 2\psi w_1 - 2\varphi\psi \overline{q}_{\frac{1}{2}} + 2\varphi u - 4\varphi w^2.
\end{aligned}
\tag{7.54}
$$

4.1.3. *Recursion operator.* Here we present the recursion operator $\mathcal{R}$ for symmetries for this case obtained as a higher symmetry in the Cartan covering of the augmented system of equations (7.50). The result is

$$
\mathcal{R} = R^u \frac{\partial}{\partial u} + R^w \frac{\partial}{\partial w} + R^{\varphi} \frac{\partial}{\partial \varphi} + R^{\psi} \frac{\partial}{\partial \psi} + \dots, \tag{7.55}
$$

where the components R^u, R^w, R^{φ}, R^{ψ} are given by

$$
\begin{aligned}
R_u &= \omega_{u_2} + \omega_u(-4u + 4w^2)\\
&\quad + \omega_{w_1}(-4w_1) + \omega_w(8uw - 2w_2 - 6\varphi\psi)\\
&\quad + \omega_{\varphi_1}(-2\varphi) + \omega_{\varphi}(\varphi_1 - 8\psi w) + \omega_{\psi_1}(-2\psi) + \omega_{\psi}(\psi_1 + 8\varphi w)\\
&\quad + \omega_{q_{\frac{1}{2}}}(\varphi_2 - 3\psi_1 w - 3\psi w_1 - \varphi u - q_{\frac{1}{2}} u_1)\\
&\quad + \omega_{\overline{q}_{\frac{1}{2}}}(\psi_2 + 3\varphi_1 w + 3\varphi w_1 - \psi u - \overline{q}_{\frac{1}{2}} u_1)\\
&\quad + \omega_{q_{\frac{3}{2}}}(\psi_1) + \omega_{\overline{q}_{\frac{3}{2}}}(-\varphi_1) + \omega_{p_{1,4}}(2u_1) + \omega_{p_{1,2}}(u_1)\\
&\quad + \omega_{p_{1,1}}(-2u_1 + 4ww_1 + \varphi_1 q_{\frac{1}{2}} + \psi_1 \overline{q}_{\frac{1}{2}}),\\
R_w &= \omega_{w_2} + \omega_w(4w^2) + \omega_{\varphi}(-2\psi) + \omega_{\psi}(2\varphi)
\end{aligned}
$$

$$
\begin{aligned}
&+ \omega_{q_{\frac12}}(-\psi_1 - \varphi w - q_{\frac12} w_1) + \omega_{\overline{q}_{\frac12}}(\varphi_1 - \psi w - \overline{q}_{\frac12} w_1)\\
&+ \omega_{q_{\frac32}}(-\varphi) + \omega_{\overline{q}_{\frac32}}(-\psi) + \omega_{p_{1,4}}(2w_1) + \omega_{p_{1,2}}(w_1)\\
&+ \omega_{p_{1,1}}(\psi q_{\frac12} - \varphi \overline{q}_{\frac12}),\\
R_\varphi &= \omega_u(-2\varphi) + \omega_{w_1}(-2\psi) + \omega_w(-\psi_1 + 8\varphi w)\\
&+ \omega_{\varphi_2} + \omega_\varphi(-2u + 4w^2) + \omega_\psi(-2w_1)\\
&+ \omega_{q_{\frac12}}(-u_1 + 3ww_1 + \varphi_1 q_{\frac12})\\
&+ \omega_{\overline{q}_{\frac12}}(-uw - w_2 + 2\varphi\psi + \varphi_1 \overline{q}_{\frac12})\\
&+ \omega_{q_{\frac32}}(-w_1) + \omega_{\overline{q}_{\frac32}}(-u) + \omega_{p_{1,4}}(2\varphi_1) + \omega_{p_{1,2}}(\varphi_1)\\
&+ \omega_{p_{1,1}}(-\varphi_1 - q_{\frac12} u + \overline{q}_{\frac12} w_1),\\
R_\psi &= \omega_u(-2\psi) + \omega_{w_1}(2\varphi) + \omega_w(\varphi_1 + 8\psi w)\\
&+ \omega_\varphi(2w_1) + \omega_{\psi_2} + \omega_\psi(-2u + 4w^2)\\
&+ \omega_{q_{\frac12}}(uw + w_2 - 2\varphi\psi + \psi_1 q_{\frac12})\\
&+ \omega_{\overline{q}_{\frac12}}(-u_1 + 3ww_1 + \psi_1 \overline{q}_{\frac12})\\
&+ \omega_{q_{\frac32}}(u) + \omega_{\overline{q}_{\frac32}}(-w_1) + \omega_{p_{1,4}}(2\psi_1) + \omega_{p_{1,2}}(\psi_1)\\
&+ \omega_{p_{1,1}}(-\psi_1 - q_{\frac12} w_1 - \overline{q}_{\frac12} u).
\end{aligned}
\tag{7.56}
$$

It should be noted that the components are given in the right-module structure (see Chapter 6).

4.2. Case $a = 4$**.** In this subsection we discuss the case $a = 4$, which does lead to the following system of partial differential equations:

$$
\begin{aligned}
u_t &= -u_3 + 6uu_1 - 3\varphi\varphi_2 - 3\psi\psi_2 - 6ww_3 - 12w_1w_2 + 24uww_1 + 12u_1w^2\\
&\quad + 24\psi\varphi_1 w - 24\varphi\psi_1 w - 24\varphi\psi w_1,\\
\varphi_t &= -\varphi_3 + 3\varphi u_1 + 3\varphi_1 u - 6\psi_2 w - 9\psi_1 w_1 - 3\psi w_2 + 12\varphi_1 w^2 + 24\varphi w w_1,\\
\psi_t &= -\psi_3 + 3\psi u_1 + 3\psi_1 u + 6\varphi_2 w + 9\varphi_1 w_1 + 3\varphi w_2 + 12\psi_1 w^2 + 24\psi w w_1,\\
w_t &= -w_3 + 12w^2 w_1 + 6u_1 w + 6uw_1 + 3\psi\varphi_1 - 3\varphi\psi_1.
\end{aligned}
\tag{7.57}
$$

The results obtained in this case for conservation laws, higher symmetries and deformations or recursion operator will be presented in subsequent subsections.

4.2.1. *Conservation laws.* For the even conservation laws and the associated even nonlocal variables we obtained the following results.

1. Nonlocal variables $p_{0,1}$ and $p_{0,2}$ of degree 0 are

$$
\begin{aligned}
(p_{0,1})_x &= w,\\
(p_{0,1})_t &= -3\varphi\psi + 6uw + 4w^3 - w_2;
\end{aligned}
$$

$$
(p_{0,2})_x = p_{1,1},
$$

$$(p_{0,2})_t = -24p_{3,1} - u_1 - 3ww_1. \tag{7.58}$$

2. Nonlocal variables $p_{1,1}$ and $p_{1,2}$ of degree 1 are defined by

$$\begin{aligned}
(p_{1,1})_x &= u,\\
(p_{1,1})_t &= -3\psi\psi_1 - 3\varphi\varphi_1 - 24\varphi\psi w + 3u^2 + 12uw^2 - u_2 - 6ww_2 - 3w_1^2;\\
(p_{1,2})_x &= \psi q_{\frac12} + \varphi\overline{q}_{\frac12},\\
(p_{1,2})_t &= -\psi_2 q_{\frac12} - \varphi_2\overline{q}_{\frac12} - 6\psi_1\overline{q}_{\frac12}w + 6\varphi_1 q_{\frac12}w - 3\psi\overline{q}_{\frac12}w_1 + 3\psi q_{\frac12}u\\
&\quad + 12\psi q_{\frac12}w^2 - 2\psi\psi_1 + 3\varphi\overline{q}_{\frac12}u + 12\varphi\overline{q}_{\frac12}w^2 + 3\varphi q_{\frac12}w_1\\
&\quad - 2\varphi\varphi_1 - 12\varphi\psi w.
\end{aligned} \tag{7.59}$$

3. Nonlocal variables $p_{2,1}$ and $p_{2,2}$ of degree 2 are

$$\begin{aligned}
(p_{2,1})_x &= \varphi\psi - uw,\\
(p_{2,1})_t &= \varphi_1\psi_1 + \psi\varphi_2 + 9\psi\psi_1 w - \varphi\psi_2 + 9\varphi\varphi_1 w + 6\varphi\psi u + 36\varphi\psi w^2\\
&\quad - 6u^2w - 12uw^3 + uw_2 - u_1w_1 + u_2w + 6w^2w_2;\\
(p_{2,2})_x &= \frac13(-q_{\frac12}\overline{q}_{\frac12}u - \psi q_{\frac12}w - \varphi\overline{q}_{\frac12}w + uw),\\
(p_{2,2})_t &= \frac13(-3q_{\frac12}\overline{q}_{\frac12}u^2 - 12q_{\frac12}\overline{q}_{\frac12}uw^2 + q_{\frac12}\overline{q}_{\frac12}u_2 + 6q_{\frac12}\overline{q}_{\frac12}ww_2 + 3q_{\frac12}\overline{q}_{\frac12}w_1^2\\
&\quad + \psi_2 q_{\frac12}w + \varphi_2\overline{q}_{\frac12}w + \psi_1\overline{q}_{\frac12}u + 6\psi_1\overline{q}_{\frac12}w^2 - \psi_1 q_{\frac12}w_1 - \varphi_1\overline{q}_{\frac12}w_1\\
&\quad - \varphi_1 q_{\frac12}u - 6\varphi_1 q_{\frac12}w^2 - \psi\overline{q}_{\frac12}u_1 - 3\psi\overline{q}_{\frac12}ww_1 - 9\psi q_{\frac12}uw\\
&\quad - 12\psi q_{\frac12}w^3 + \psi q_{\frac12}w_2 + 3\psi\psi_1 q_{\frac12}\overline{q}_{\frac12} - \psi\psi_1 w - 9\varphi\overline{q}_{\frac12}uw - 12\varphi\overline{q}_{\frac12}w^3\\
&\quad + \varphi\overline{q}_{\frac12}w_2 + \varphi q_{\frac12}u_1 + 3\varphi q_{\frac12}ww_1 + 3\varphi\varphi_1 q_{\frac12}\overline{q}_{\frac12} - \varphi\varphi_1 w + 24\varphi\psi q_{\frac12}\overline{q}_{\frac12}w\\
&\quad - 2\varphi\psi u - 12\varphi\psi w^2 + 6u^2w + 12uw^3 - uw_2 + u_1w_1\\
&\quad - u_2w - 6w^2w_2).
\end{aligned} \tag{7.60}$$

4. Finally, the variables $p_{3,1}$ and $p_{3,2}$ of degree 3 are defined by

$$\begin{aligned}
(p_{3,1})_x &= \frac18(\psi\psi_1 + \varphi\varphi_1 + 8\varphi\psi w - u^2 - 4uw^2 + ww_2),\\
(p_{3,1})_t &= \frac18(2\psi_1\psi_2 + 2\varphi_1\varphi_2 + 14\varphi_1\psi_1 w - \psi\psi_3 + 17\psi\varphi_2 w + 9\psi\psi_1 u\\
&\quad + 72\psi\psi_1 w^2 - 2\psi\varphi_1 w_1 - \varphi\varphi_3 - 17\varphi\psi_2 w + 2\varphi\psi_1 w_1 + 9\varphi\varphi_1 u\\
&\quad + 72\varphi\varphi_1 w^2 + 96\varphi\psi uw + 192\varphi\psi w^3 - 14\varphi\psi w_2 - 4u^3 - 48u^2w^2\\
&\quad + 2uu_2 - 48uw^4 + 26uww_2 + 2uw_1^2 - u_1^2 - 2u_1ww_1 + 10u_2w^2\\
&\quad + 36w^3w_2 + 12w^2w_1^2 - ww_4 + w_1w_3 - w_2^2);\\
(p_{3,2})_x &= \frac1{27}(27\overline{q}_{\frac12}\overline{q}_{\frac32}u - 27\overline{q}_{\frac12}q_{\frac32}w_1 - 45q_{\frac12}\overline{q}_{\frac32}w_1 + 27q_{\frac12}q_{\frac32}u - 8q_{\frac12}\overline{q}_{\frac12}p_{1,1}w_1\\
&\quad + 6q_{\frac12}\overline{q}_{\frac12}uw - 10q_{\frac12}\overline{q}_{\frac12}w_2 - 9\psi_2 q_{\frac12} - 9\varphi_2\overline{q}_{\frac12} - 186\psi_1 q_{\frac32} + 16\psi_1\overline{q}_{\frac12}w
\end{aligned}$$

$$+ 52\psi_1 q_{\frac{1}{2}} p_{1,1} + 36\psi_1 q_{\frac{1}{2}} p_{1,2} + 18\varphi_1 \overline{q}_{\frac{3}{2}} - 24\psi\overline{q}_{\frac{5}{2}} - 72\psi\overline{q}_{\frac{3}{2}} w$$
$$- 48\psi\overline{q}_{\frac{1}{2}} p_{2,1} + 288\varphi q_{\frac{5}{2}}). \tag{7.61}$$

For the odd conservation laws and the associated odd nonlocal variables we derived the following results.

1. At degree 1/2, we have the variables $q_{\frac{1}{2}}$ and $\overline{q}_{\frac{1}{2}}$ defined by the relations

$$\begin{aligned}
(q_{\frac{1}{2}})_x &= \psi,\\
(q_{\frac{1}{2}})_t &= -\psi_2 + 6\varphi_1 w + 3\psi u + 12\psi w^2 + 3\varphi w_1;\\
(\overline{q}_{\frac{1}{2}})_x &= \varphi,\\
(\overline{q}_{\frac{1}{2}})_t &= -\varphi_2 - 6\psi_1 w - 3\psi w_1 + 3\varphi u + 12\varphi w^2.
\end{aligned} \tag{7.62}$$

2. At degree 3/2, the variables are $q_{\frac{3}{2}}$ and $\overline{q}_{\frac{3}{2}}$:

$$\begin{aligned}
(q_{\frac{3}{2}})_x &= \frac{1}{3}(q_{\frac{1}{2}} u + \varphi w),\\
(q_{\frac{3}{2}})_t &= \frac{1}{3}(3q_{\frac{1}{2}} u^2 + 12 q_{\frac{1}{2}} u w^2 - q_{\frac{1}{2}} u_2 - 6q_{\frac{1}{2}} w w_2 - 3q_{\frac{1}{2}} w_1^2 - \varphi_2 w - \psi_1 u\\
&\quad - 6\psi_1 w^2 + \varphi_1 w_1 - 3\psi\psi_1 q_{\frac{1}{2}} + \psi u_1 + 3\psi w w_1 - 3\varphi\varphi_1 q_{\frac{1}{2}} - 24\varphi\psi q_{\frac{1}{2}} w\\
&\quad + 9\varphi u w + 12\varphi w^3 - \varphi w_2);\\
(\overline{q}_{\frac{3}{2}})_x &= \frac{1}{3}(\overline{q}_{\frac{1}{2}} u - \psi w),\\
(\overline{q}_{\frac{3}{2}})_t &= \frac{1}{3}(3\overline{q}_{\frac{1}{2}} u^2 + 12\overline{q}_{\frac{1}{2}} u w^2 - \overline{q}_{\frac{1}{2}} u_2 - 6\overline{q}_{\frac{1}{2}} w w_2 - 3\overline{q}_{\frac{1}{2}} w_1^2 + \psi_2 w - \psi_1 w_1\\
&\quad - \varphi_1 u - 6\varphi_1 w^2 - 3\psi\psi_1 \overline{q}_{\frac{1}{2}} - 9\psi u w - 12\psi w^3 + \psi w_2 - 3\varphi\varphi_1 \overline{q}_{\frac{1}{2}}\\
&\quad - 24\varphi\psi\overline{q}_{\frac{1}{2}} w + \varphi u_1 + 3\varphi w w_1).
\end{aligned} \tag{7.63}$$

3. Finally, at degree 5/2 we have $q_{\frac{5}{2}}$ and $\overline{q}_{\frac{5}{2}}$ which are defined by the relations, i.e.,

$$\begin{aligned}
(q_{\frac{5}{2}})_x &= \frac{1}{24}(2\overline{q}_{\frac{1}{2}} p_{1,1} u - 2\psi_1 w - 2\psi p_{1,1} w + 4\psi p_{2,1} + 2\varphi u + 3\varphi w^2),\\
(q_{\frac{5}{2}})_t &= \frac{1}{24}(6\overline{q}_{\frac{1}{2}} p_{1,1} u^2 + 24\overline{q}_{\frac{1}{2}} p_{1,1} u w^2 - 2\overline{q}_{\frac{1}{2}} p_{1,1} u_2 - 12\overline{q}_{\frac{1}{2}} p_{1,1} w w_2\\
&\quad - 6\overline{q}_{\frac{1}{2}} p_{1,1} w_1^2 + 2\psi_3 w + 2\psi_2 p_{1,1} w - 4\psi_2 p_{2,1} - 2\psi_2 w_1 - 2\varphi_2 u\\
&\quad - 15\varphi_2 w^2 - 2\psi_1 p_{1,1} w_1 - 24\psi_1 u w - 42\psi_1 w^3 + 2\psi_1 w_2 - 2\varphi_1 p_{1,1} u\\
&\quad - 12\varphi_1 p_{1,1} w^2 + 24\varphi_1 p_{2,1} w + 2\varphi_1 u_1 - 6\psi\psi_1 \overline{q}_{\frac{1}{2}} p_{1,1} - 18\psi p_{1,1} u w\\
&\quad - 24\psi p_{1,1} w^3 + 2\psi p_{1,1} w_2 + 12\psi p_{2,1} u + 48\psi p_{2,1} w^2 - 4\psi u w_1\\
&\quad - 21\psi w^2 w_1 - 6\varphi\varphi_1 \overline{q}_{\frac{1}{2}} p_{1,1} - 48\varphi\psi\overline{q}_{\frac{1}{2}} p_{1,1} w + 2\varphi\psi\psi_1 + 2\varphi p_{1,1} u_1\\
&\quad + 6\varphi p_{1,1} w w_1 + 12\varphi p_{2,1} w_1 + 8\varphi u^2 + 69\varphi u w^2
\end{aligned}$$

$$
\begin{aligned}
&- 2\varphi u_2 + 36\varphi w^4 - 24\varphi w w_2);\\
(\overline{q}_{\frac{5}{2}})_x &= \frac{1}{6}(-4\overline{q}_{\frac{1}{2}}p_{1,1}w_1 - 2q_{\frac{1}{2}}p_{1,1}u + 2\psi_1 p_{1,1} + 4\varphi_1 q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}} - 2\varphi_1 w\\
&- 3\psi w^2 - 6\varphi p_{1,1}w). \qquad (7.64)
\end{aligned}
$$

We omitted explicit expressions for $(p_{3,2})_t$ and $(q_{\frac{5}{2}})_t$ in (7.61) and (7.64) because they are too massive.

Thus, we obtained the following 14 nonlocal variables:

$$
\begin{array}{ll}
p_{0,1},\ p_{0,2} & \text{of degree } 0,\\
p_{1,1},\ p_{1,2} & \text{of degree } 1,\\
p_{2,1},\ p_{2,2} & \text{of degree } 2,\\
p_{3,1},\ p_{3,2} & \text{of degree } 3,\\
q_{\frac{1}{2}},\ \overline{q}_{\frac{1}{2}} & \text{of degree } \frac{1}{2},\\
q_{\frac{3}{2}},\ \overline{q}_{\frac{3}{2}} & \text{of degree } \frac{3}{2},\\
q_{\frac{5}{2}},\ \overline{q}_{\frac{5}{2}} & \text{of degree } \frac{5}{2}.
\end{array} \qquad (7.65)
$$

In the next subsections the augmented system of equations associated to the local and the nonlocal variables denoted above will be considered in computing higher and nonlocal symmetries and the recursion operator.

4.2.2. *Higher and nonlocal symmetries.* In this subsection we present results for higher and nonlocal symmetries for the $N = 2$ supersymmetric extension of the KdV equation (7.57) in the case $a = 4$,

$$
Y = Y^u\frac{\partial}{\partial u} + Y^w\frac{\partial}{\partial w} + Y^\varphi\frac{\partial}{\partial \varphi} + Y^\psi\frac{\partial}{\partial \psi} + \dots
$$

We obtained the following odd symmetries. The components of their generating functions are given below:

$$
\begin{array}{ll}
Y^u_{\frac{1}{2},1} = \psi_1, & Y^u_{\frac{1}{2},2} = \varphi_1,\\
Y^w_{\frac{1}{2},1} = -\varphi, & Y^w_{\frac{1}{2},2} = \psi,\\
Y^\varphi_{\frac{1}{2},1} = -w_1, & Y^\varphi_{\frac{1}{2},2} = u,\\
Y^\psi_{\frac{1}{2},1} = u; & Y^\psi_{\frac{1}{2},2} = w_1
\end{array} \qquad (7.66)
$$

and

$$
\begin{aligned}
Y^u_{\frac{3}{2},1} &= -2\overline{q}_{\frac{1}{2}}u_1 + \varphi_2 + 3\psi_1 w + \varphi_1 p_{1,1} + 3\psi w_1 - \varphi u,\\
Y^w_{\frac{3}{2},1} &= -2\overline{q}_{\frac{1}{2}}w_1 + \psi_1 + \psi p_{1,1} - 3\varphi w,\\
Y^\varphi_{\frac{3}{2},1} &= 2\varphi_1\overline{q}_{\frac{1}{2}} + p_{1,1}u - u_1 - 3ww_1,\\
Y^\psi_{\frac{3}{2},1} &= 2\psi_1\overline{q}_{\frac{1}{2}} - 4\varphi\psi + p_{1,1}w_1 + 3uw - w_2;
\end{aligned}
$$

$$
\begin{aligned}
Y^u_{\frac{3}{2},2} &= 2q_{\frac{1}{2}}u_1 - \psi_2 - \psi_1 p_{1,1} + 3\varphi_1 w + \psi u + 3\varphi w_1,\\
Y^w_{\frac{3}{2},2} &= 2q_{\frac{1}{2}}w_1 + \varphi_1 + 3\psi w + \varphi p_{1,1},\\
Y^\varphi_{\frac{3}{2},2} &= -2\varphi_1 q_{\frac{1}{2}} - 4\varphi\psi + p_{1,1}w_1 + 3uw - w_2,\\
Y^\psi_{\frac{3}{2},2} &= -2\psi_1 q_{\frac{1}{2}} - p_{1,1}u + u_1 + 3ww_1.
\end{aligned} \tag{7.67}
$$

We also obtained the following even symmetries:

$$
\begin{aligned}
Y^u_{0,1} &= 0,\\
Y^w_{0,1} &= 0,\\
Y^\varphi_{0,1} &= \psi,\\
Y^\psi_{0,1} &= -\varphi;\\
Y^u_{1,1} &= \psi_1 q_{\frac{1}{2}} + \varphi_1 \overline{q}_{\frac{1}{2}},\\
Y^w_{1,1} &= \psi \overline{q}_{\frac{1}{2}} - \varphi q_{\frac{1}{2}},\\
Y^\varphi_{1,1} &= -\overline{q}_{\frac{1}{2}}u + q_{\frac{1}{2}}w_1 + \varphi_1 + 2\psi w,\\
Y^\psi_{1,1} &= -\overline{q}_{\frac{1}{2}}w_1 - q_{\frac{1}{2}}u + \psi_1 - 2\varphi w;\\
Y^u_{1,2} &= u_1,\\
Y^w_{1,2} &= w_1,\\
Y^\varphi_{1,2} &= \varphi_1,\\
Y^\psi_{1,2} &= \psi_1.
\end{aligned} \tag{7.68}
$$

4.2.3. *Recursion operator.* Here we present the recursion operator $\mathcal{R}$ for symmetries for the case $a = 4$ obtained as a higher symmetry in the Cartan covering of the augmented system of equations (7.65). This operator is of the form

$$
\mathcal{R} = R^u \frac{\partial}{\partial u} + R^w \frac{\partial}{\partial w} + R^\varphi \frac{\partial}{\partial \varphi} + R^\psi \frac{\partial}{\partial \psi} + \dots, \tag{7.69}
$$

where the components R^u, R^w, R^φ, R^ψ are given by

$$
\begin{aligned}
R_u &= \omega_{u_2} + \omega_u(-4u - 4w^2) + \omega_{w_2}(4w)\\
&\quad + \omega_{w_1}(6w_1) + \omega_w(-16uw + 6w_2 + 18\varphi\psi)\\
&\quad + \omega_{\varphi_1}(-2\varphi) + \omega_\varphi(\varphi_1 + 12\psi w) + \omega_{\psi_1}(-2\psi) + \omega_\psi(\psi_1 - 12\varphi w)\\
&\quad + \omega_{q_{\frac{1}{2}}}(\psi_2 - 3\varphi_1 w - 3\varphi w_1 - \psi u - q_{\frac{1}{2}}u_1)\\
&\quad + \omega_{\overline{q}_{\frac{1}{2}}}(\varphi_2 + 3\psi_1 w + 3\psi w_1 - \varphi u - \overline{q}_{\frac{1}{2}}u_1)\\
&\quad + \omega_{q_{\frac{3}{2}}}(3\psi_1) + \omega_{\overline{q}_{\frac{3}{2}}}(3\varphi_1) + \omega_{p_{1,2}}(u_1)\\
&\quad + \omega_{p_{1,1}}(-2u_1 + \psi_1 q_{\frac{1}{2}} + \varphi_1 \overline{q}_{\frac{1}{2}})\\
&\quad + \omega_{p_{0,1}}(2w_3 - 8uw_1 - 8u_1 w + 8\varphi_1\psi + 8\varphi\psi_1),
\end{aligned}
$$

$$
\begin{aligned}
R_w &= \omega_u(-4w) + \omega_{w_2} + \omega_w(-4u - 4w^2) + \omega_\varphi(2\psi) + \omega_\psi(-2\varphi)\\
&+ \omega_{q_{\frac{1}{2}}}(-\varphi_1 - 3\psi w - q_{\frac{1}{2}} w_1) + \omega_{\overline{q}_{\frac{1}{2}}}(\psi_1 - 3\varphi w - \overline{q}_{\frac{1}{2}} w_1)\\
&+ \omega_{q_{\frac{3}{2}}}(-3\varphi) + \omega_{\overline{q}_{\frac{3}{2}}}(3\psi) + \omega_{p_{1,2}}(w_1)\\
&+ \omega_{p_{1,1}}(-2w_1 + \psi\overline{q}_{\frac{1}{2}} - \varphi q_{\frac{1}{2}}) + \omega_{p_{0,1}}(-2u_1 - 8ww_1),\\
R_\varphi &= \omega_u(-2\varphi) + \omega_{w_1}(2\psi) + \omega_w(5\psi_1 - 12\varphi w)\\
&+ \omega_{\varphi_2} + \omega_\varphi(-2u - 4w^2) + \omega_{\psi_1}(4w) + \omega_\psi(4w_1)\\
&+ \omega_{q_{\frac{1}{2}}}(w_2 - 3uw + 4\varphi\psi + \varphi_1 q_{\frac{1}{2}}) + \omega_{\overline{q}_{\frac{1}{2}}}(-u_1 - 3ww_1 + \varphi_1\overline{q}_{\frac{1}{2}})\\
&+ \omega_{q_{\frac{3}{2}}}(-3w_1) + \omega_{\overline{q}_{\frac{3}{2}}}(3u) + \omega_{p_{1,2}}(\varphi_1)\\
&+ \omega_{p_{1,1}}(-\varphi_1 + 2\psi w + q_{\frac{1}{2}} w_1 - \overline{q}_{\frac{1}{2}} u)\\
&+ \omega_{p_{0,1}}(2\psi_2 - 8\varphi_1 w - 8\varphi w_1) + \omega_{p_{2,1}}(-2\psi),\\
R_\psi &= \omega_u(-2\psi) + \omega_{w_1}(-2\varphi) + \omega_w(-5\varphi_1 - 12\psi w)\\
&+ \omega_{\varphi_1}(-4w) + \omega_\varphi(-4w_1) + \omega_{\psi_2} + \omega_\psi(-2u - 4w^2)\\
&+ \omega_{q_{\frac{1}{2}}}(-u_1 - 3ww_1 + \psi_1 q_{\frac{1}{2}})\\
&+ \omega_{\overline{q}_{\frac{1}{2}}}(3uw - w_2 - 4\varphi\psi + \psi_1\overline{q}_{\frac{1}{2}})\\
&+ \omega_{q_{\frac{3}{2}}}(3u) + \omega_{\overline{q}_{\frac{3}{2}}}(3w_1) + \omega_{p_{1,2}}(\psi_1)\\
&+ \omega_{p_{1,1}}(-\psi_1 - 2\varphi w - q_{\frac{1}{2}} u - \overline{q}_{\frac{1}{2}} w_1) + \omega_{p_{0,1}}(-2\varphi_2 - 8\psi_1 w - 8\psi w_1)\\
&+ \omega_{p_{2,1}}(2\varphi).
\end{aligned} \tag{7.70}
$$

It should be noted that the components are again given here in the right-module structure (see Chapter 6).

REMARK 7.2. Personal communication with Prof. A. Sorin informed us about existence of a deformation, or recursion operator of order 1 in this specific case, a fact which might be indicated by the structure of the existing nonlocal variables. The result is given by

$$
\begin{aligned}
\mathcal{R}_1 = &\Big(\omega_u(2w) - \omega_{w_2} + \omega_w(4u) + \omega_\varphi(-3\psi) + \omega_\psi(3\varphi)\\
&+ \omega_{q_{\frac{1}{2}}}(\varphi_1) + \omega_{\overline{q}_{\frac{1}{2}}}(-\psi_1) + \omega_{p_{0,1}}(2u_1)\Big)\frac{\partial}{\partial u}\\
&+ \Big(\omega_u + \omega_w(2w) + \omega_{q_{\frac{1}{2}}}(\psi)\\
&+ \omega_{\overline{q}_{\frac{1}{2}}}(\varphi) + \omega_{p_{0,1}}(2w_1)\Big)\frac{\partial}{\partial w}\\
&+ \Big(\omega_w(3\varphi) - \omega_{\psi_1} + \omega_\varphi(2w) + \omega_{q_{\frac{1}{2}}}(u)\\
&+ \omega_{\overline{q}_{\frac{1}{2}}}(w_1) + \omega_{p_{0,1}}(2\varphi_1) + \omega_{p_{1,1}}(-\psi)\Big)\frac{\partial}{\partial\varphi}
\end{aligned}
$$

$$+\Big(\omega_w(3\psi)+\omega_{\varphi_1}+\omega_\psi(2w)+\omega_{q_{\frac12}}(w_1)$$
$$+\omega_{\overline{q}_{\frac12}}(-u)+\omega_{p_{0,1}}(2\psi_1)+\omega_{p_{1,1}}(\varphi)\Big)\frac{\partial}{\partial\psi}.$$

4.3. Case $a=1$**.** In this section we discuss the case $a=1$, which does lead to the following system of partial differential equations:

$$\begin{aligned}
u_t &= -u_3+6uu_1-3\varphi\varphi_2-3\psi\psi_2-3ww_3-3w_1w_2+3u_1w^2+6uww_1\\
&\quad +6\psi\varphi_1 w-6\varphi\psi_1 w-6\varphi\psi w_1,\\
\varphi_t &= -\varphi_3+3\varphi u_1+3\varphi_1 u-3\psi_2 w-3\psi_1 w_1+3\varphi_1 w^2+6\varphi ww_1,\\
\psi_t &= -\psi_3+3\psi u_1+3\psi_1 u+3\varphi_2 w+3\varphi_1 w_1+3\psi_1 w^2+6\psi ww_1,\\
w_t &= -w_3+3w^2w_1+3uw_1+3u_1w.
\end{aligned}\tag{7.71}$$

The results obtained in this case for conservation laws, higher symmetries and recursion symmetries will be presented in subsequent subsections.

4.3.1. *Conservation laws.* For the even conservation laws and the associated even nonlocal variables we obtained the following results.

1. Nonlocal variables $p_{0,1}$ and $p_{0,2}$ of degree 0 are

$$\begin{aligned}
(p_{0,1})_x &= w,\\
(p_{0,1})_t &= 3uw+w^3-w_2;\\[4pt]
(p_{0,2})_x &= p_1,\\
(p_{0,2})_t &= -6p_3-u_1.
\end{aligned}\tag{7.72}$$

2. Nonlocal variables $p_{1,1}$, $p_{1,2}$, $p_{1,3}$, and $p_{1,4}$ of degree 1 are defined by

$$\begin{aligned}
(p_1)_x &= u,\\
(p_1)_t &= -3\psi\psi_1-3\varphi\varphi_1-6\varphi\psi w+3u^2+3uw^2-u_2-3ww_2;
\end{aligned}$$

$$\begin{aligned}
(p_{1,1})_x &= \cos(2p_{0,1})(\varphi q_{\frac12,2}+p_1w)+\sin(2p_{0,1})(\psi q_{\frac12,2}+w^2),\\
(p_{1,1})_t &= \cos(2p_{0,1})(-\varphi_2 q_{\frac12,2}-\psi_1 q_{\frac12,2}w-2\psi q_{\frac12,2}w_1-\psi\varphi_1+3\varphi q_{\frac12,2}u\\
&\quad +\varphi q_{\frac12,2}w^2-\varphi\psi_1+3p_1uw+p_1w^3-p_1w_2+uw_1-u_1w-w^2w_1)\\
&\quad +\sin(2p_{0,1})(-\psi_2 q_{\frac12,2}+\varphi_1 q_{\frac12,2}w+3\psi q_{\frac12,2}u+\psi q_{\frac12,2}w^2-2\psi\psi_1\\
&\quad +2\varphi q_{\frac12,2}w_1-2\varphi\psi w+4uw^2+w^4-2ww_2+w_1^2);
\end{aligned}$$

$$\begin{aligned}
(p_{1,2})_x &= \cos(2p_{0,1})(\psi q_{\frac12,2}+w^2)-\sin(2p_{0,1})(\varphi q_{\frac12,2}+p_1w),\\
(p_{1,2})_t &= \cos(2p_{0,1})(-\psi_2 q_{\frac12,2}+\varphi_1 q_{\frac12,2}w+3\psi q_{\frac12,2}u\\
&\quad +\psi q_{\frac12,2}w^2-2\psi\psi_1+2\varphi q_{\frac12,2}w_1-2\varphi\psi w+4uw^2+w^4-2ww_2+w_1^2)\\
&\quad +\sin(2p_{0,1})(\varphi_2 q_{\frac12,2}+\psi_1 q_{\frac12,2}w+2\psi q_{\frac12,2}w_1+\psi\varphi_1-3\varphi q_{\frac12,2}u\\
&\quad -\varphi q_{\frac12,2}w^2+\varphi\psi_1-3p_1uw-p_1w^3+p_1w_2-uw_1+u_1w+w^2w_1);
\end{aligned}$$

$$
\begin{aligned}
(p_{1,3})_x &= -2\cos(2p_{0,1})\varphi q_{\frac12,2} + \sin(2p_{0,1})(2q_{\frac12,1}q_{\frac12,2}w - \psi q_{\frac12,2} + \varphi q_{\frac12,1}),\\
(p_{1,3})_t &= 2\cos(2p_{0,1})(\varphi_2 q_{\frac12,2} + 2\psi_1 q_{\frac12,2}w + \varphi_1 q_{\frac12,1}w + \psi q_{\frac12,2}w_1\\
&\quad + \psi q_{\frac12,1}w^2 + \psi\varphi_1 - 3\varphi q_{\frac12,2}u - 2\varphi q_{\frac12,2}w^2 - \varphi q_{\frac12,1}w_1 + \varphi\psi_1)\\
&\quad + \sin(2p_{0,1})(6q_{\frac12,1}q_{\frac12,2}uw + 2q_{\frac12,1}q_{\frac12,2}w^3 - 2q_{\frac12,1}q_{\frac12,2}w_2 + \psi_2 q_{\frac12,2}\\
&\quad - \varphi_2 q_{\frac12,1} - \psi_1 q_{\frac12,1}w - \varphi_1 q_{\frac12,2}w - 3\psi q_{\frac12,2}u - \psi q_{\frac12,1}w^2\\
&\quad - 2\psi q_{\frac12,1}w_1 + 2\psi\psi_1 - 2\varphi q_{\frac12,2}w_1 + 3\varphi q_{\frac12,1}u + \varphi q_{\frac12,1}w^2 - 2\varphi\varphi_1);
\end{aligned}
$$

$$
\begin{aligned}
(p_{1,4})_x &= \cos(2p_{0,1})(2q_{\frac12,1}q_{\frac12,2}w - \psi q_{\frac12,2} + \varphi q_{\frac12,1}) + 2\sin(2p_{0,1})\varphi q_{\frac12,2},\\
(p_{1,4})_t &= \cos(2p_{0,1})(6q_{\frac12,1}q_{\frac12,2}uw + 2q_{\frac12,1}q_{\frac12,2}w^3 - 2q_{\frac12,1}q_{\frac12,2}w_2 + \psi_2 q_{\frac12,2}\\
&\quad - \varphi_2 q_{\frac12,1} - \psi_1 q_{\frac12}w - \varphi_1 q_{\frac12,2}w - 3\psi q_{\frac12,2}u - \psi q_{\frac12,2}w^2 - 2\psi q_{\frac12,1}w_1\\
&\quad + 2\psi\psi_1 - 2\varphi q_{\frac12,2}w_1 + 3\varphi q_{\frac12,1}u + \varphi q_{\frac12,1}w^2 - 2\varphi\varphi_1)\\
&\quad + 2\sin(2p_{0,1})(-\varphi_2 q_{\frac12,2} - 2\psi_1 q_{\frac12,2}w - \varphi_1 q_{\frac12,1}w - \psi q_{\frac12,2}w_1 - \psi q_{\frac12,1}w^2\\
&\quad - \psi\varphi_1 + 3\varphi q_{\frac12,2}u + 2\varphi q_{\frac12,2}w^2 + \varphi q_{\frac12,1}w_1 - \varphi\psi_1).
\end{aligned}
\tag{7.73}
$$

3. The variable $p_{3,1}$ of degree 3 is

$$
\begin{aligned}
(p_{3,1})_x &= \frac12(\psi\psi_1 + \varphi\varphi_1 + 2\varphi\psi w - u^2 - uw^2 + ww_2),\\
(p_{3,1})_t &= \frac12(2\psi_1\psi_2 + 2\varphi_1\varphi_2 + 8\varphi_1\psi_1 w - \psi\psi_3 + 5\psi\varphi_2 w + 9\psi\psi_1 u\\
&\quad + 12\psi\psi_1 w^2 + \psi\varphi_1 w_1 - \varphi\varphi_3 - 5\varphi\psi_2 w - \varphi\psi_1 w_1 + 9\varphi\varphi_1 u\\
&\quad + 12\varphi\varphi_1 w^2 + 18\varphi\psi uw + 12\varphi\psi w^3 - 2\varphi\psi w_2 - 4u^3 - 9u^2w^2 + 2uu_2\\
&\quad - 3uw^4 + 11uww_2 - uw_1^2 - u_1^2 + u_1ww_1 + 4u_2w^2 + 6w^3w_2\\
&\quad + 3w^2w_1^2 - ww_4 + w_1w_3 - w_2^2).
\end{aligned}
\tag{7.74}
$$

For the odd conservation laws and the associated odd nonlocal variables we derived the following results.

1. At degree 1/2 we have the variables $q_{\frac12,1}$, $q_{\frac12,2}$, $q_{\frac12,3}$, and $q_{\frac12,4}$, defined by the relations

$$
\begin{aligned}
(q_{\frac12,1})_x &= \varphi,\\
(q_{\frac12,1})_t &= -\varphi_2 - 3\psi_1 w + 3\varphi u + 3\varphi w^2;
\end{aligned}
$$

$$
\begin{aligned}
(q_{\frac12,2})_x &= \psi,\\
(q_{\frac12,2})_t &= -\psi_2 + 3\varphi_1 w + 3\psi u + 3\psi w^2;
\end{aligned}
$$

$$
\begin{aligned}
(q_{\frac12,3})_x &= \cos(2p_{0,1})q_{\frac12,1}w + \sin(2p_{0,1})q_{\frac12,2}w,\\
(q_{\frac12,3})_t &= \cos(2p_{0,1})(3q_{\frac12,1}uw + q_{\frac12,1}w^3 - q_{\frac12,1}w_2 - \varphi_1 w - \psi w^2 + \varphi w_1)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sin(2p_{0,1})(3q_{\frac{1}{2},2}uw + q_{\frac{1}{2},2}w^3 - q_{\frac{1}{2},2}w_2 - \psi_1 w + \psi w_1 + \varphi w^2);\\
(q_{\frac{1}{2},4})_x &= \cos(2p_{0,1})q_{\frac{1}{2},2}w - \sin(2p_{0,1})q_{\frac{1}{2},1}w,\\
(q_{\frac{1}{2},4})_t &= \cos(2p_{0,1})(3q_{\frac{1}{2},2}uw + q_{\frac{1}{2},2}w^3 - q_{\frac{1}{2},2}w_2 - \psi_1 w + \psi w_1 + \varphi w^2)\\
&+ \sin(2p_{0,1})(-3q_{\frac{1}{2},1}uw - q_{\frac{1}{2},1}w^3 + q_{\frac{1}{2},1}w_2 + \varphi_1 w + \psi w^2 - \varphi w_1)
\end{aligned}
\tag{7.75}
$$

2. At degree $3/2$, we have $q_{\frac{3}{2},1}$ and $q_{\frac{3}{2},2}$:

$$
\begin{aligned}
(q_{\frac{3}{2},1})_x &= \cos(2p_{0,1})(q_{\frac{1}{2},2}p_1 w + q_{\frac{1}{2},1}u - q_{\frac{1}{2},1}w^2 + \psi q_{\frac{1}{2},1}q_{\frac{1}{2},2} + \psi w)\\
&+ \sin(2p_{0,1})(q_{\frac{1}{2},2}u - q_{\frac{1}{2},2}w^2 - q_{\frac{1}{2},1}p_1 w - \varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2} - \varphi w),\\
(q_{\frac{3}{2},1})_t &= \cos(2p_{0,1})(3q_{\frac{1}{2},2}p_1 uw + q_{\frac{1}{2},2}p_1 w^3 - q_{\frac{1}{2},2}p_1 w_2 - q_{\frac{1}{2},2}uw_1\\
&+ q_{\frac{1}{2},2}u_1 w + q_{\frac{1}{2},2}w^2 w_1 + 3q_{\frac{1}{2},1}u^2 - q_{\frac{1}{2},1}uw^2 - q_{\frac{1}{2},1}u_2 - q_{\frac{1}{2},1}w^4\\
&- q_{\frac{1}{2},1}ww_2 - q_{\frac{1}{2},1}w_1^2 - \psi_2 q_{\frac{1}{2},1}q_{\frac{1}{2},2} - \psi_2 w - \psi_1 p_1 w + \psi_1 w_1\\
&+ \varphi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},2}w - \varphi_1 u + 2\varphi_1 w^2 + 3\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}u + \psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w^2\\
&- \psi\psi_1 q_{\frac{1}{2},1} - \psi\varphi_1 q_{\frac{1}{2},2} + \psi p_1 w_1 + 3\psi uw + 2\psi w^3 - \psi w_2\\
&+ 2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w_1 - \varphi\psi_1 q_{\frac{1}{2},2} - 3\varphi\varphi_1 q_{\frac{1}{2},1} - 4\varphi\psi q_{\frac{1}{2},1}w + \varphi p_1 w^2\\
&+ \varphi u_1 + \varphi ww_1)\\
&+ \sin(2p_{0,1})(3q_{\frac{1}{2},2}u^2 - q_{\frac{1}{2},2}uw^2 - q_{\frac{1}{2},2}u_2 - q_{\frac{1}{2},2}w^4 - q_{\frac{1}{2},2}ww_2\\
&- q_{\frac{1}{2},2}w_1^2 - 3q_{\frac{1}{2},1}p_1 uw - q_{\frac{1}{2},1}p_1 w^3 + q_{\frac{1}{2},1}p_1 w_2 + q_{\frac{1}{2},1}uw_1\\
&- q_{\frac{1}{2},1}u_1 w - q_{\frac{1}{2},1}w^2 w_1 + \varphi_2 q_{\frac{1}{2},1}q_{\frac{1}{2},2} + \varphi_2 w + \psi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},2}w - \psi_1 u\\
&+ 2\psi_1 w^2 + \varphi_1 p_1 w - \varphi_1 w_1 + 2\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w_1 - 3\psi\psi_1 q_{\frac{1}{2},2}\\
&- \psi\varphi_1 q_{\frac{1}{2},1} + \psi p_1 w^2 + \psi u_1 + \psi ww_1 - 3\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}u - \varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w^2\\
&- \varphi\psi_1 q_{\frac{1}{2},1} - \varphi\varphi_1 q_{\frac{1}{2},2} - 4\varphi\psi q_{\frac{1}{2},2}w - \varphi p_1 w_1 - 3\varphi uw\\
&- 2\varphi w^3 + \varphi w_2);
\end{aligned}
$$

$$
\begin{aligned}
(q_{\frac{3}{2},2})_x &= \cos(2p_{0,1})(-q_{\frac{1}{2},2}u + q_{\frac{1}{2},2}w^2 + q_{\frac{1}{2},1}p_1 w + \varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2} + \varphi w)\\
&+ \sin(2p_{0,1})(q_{\frac{1}{2},2}p_1 w + q_{\frac{1}{2},1}u - q_{\frac{1}{2},1}w^2 + \psi q_{\frac{1}{2},1}q_{\frac{1}{2},2} + \psi w),\\
(q_{\frac{3}{2},2})_t &= \cos(2p_{0,1})(-3q_{\frac{1}{2},2}u^2 + q_{\frac{1}{2},2}uw^2 + q_{\frac{1}{2},2}u_2 + q_{\frac{1}{2},2}w^4 + q_{\frac{1}{2},2}ww_2\\
&+ q_{\frac{1}{2},2}w_1^2 + 3q_{\frac{1}{2},1}p_1 uw + q_{\frac{1}{2},1}p_1 w^3 - q_{\frac{1}{2},1}p_1 w_2 - q_{\frac{1}{2},1}uw_1 + q_{\frac{1}{2},1}u_1 w\\
&+ q_{\frac{1}{2},1}w^2 w_1 - \varphi_2 q_{\frac{1}{2},1}q_{\frac{1}{2},2} - \varphi_2 w - \psi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},2}w + \psi_1 u - 2\psi_1 w^2\\
&- \varphi_1 p_1 w + \varphi_1 w_1 - 2\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w_1 + 3\psi\psi_1 q_{\frac{1}{2},2} + \psi\varphi_1 q_{\frac{1}{2},1}\\
&- \psi p_1 w^2 - \psi u_1 - \psi ww_1 + 3\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}u + \varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w^2 + \varphi\psi_1 q_{\frac{1}{2},1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \varphi\varphi_1 q_{\frac{1}{2},2} + 4\varphi\psi q_{\frac{1}{2},2} w + \varphi p_1 w_1 + 3\varphi u w + 2\varphi w^3 - \varphi w_2) \\
&+ \sin(2p_{0,1})(3q_{\frac{1}{2},2} p_1 u w + q_{\frac{1}{2},2} p_1 w^3 - q_{\frac{1}{2},2} p_1 w_2 - q_{\frac{1}{2},2} u w_1 \\
&+ q_{\frac{1}{2},2} u_1 w + q_{\frac{1}{2},2} w^2 w_1 + 3q_{\frac{1}{2},1} u^2 - q_{\frac{1}{2},1} u w^2 - q_{\frac{1}{2},1} u_2 - q_{\frac{1}{2},1} w^4 \\
&- q_{\frac{1}{2},1} w w_2 - q_{\frac{1}{2},1} w_1^2 - \psi_2 q_{\frac{1}{2},1} q_{\frac{1}{2},2} - \psi_2 w - \psi_1 p_1 w + \psi_1 w_1 \\
&+ \varphi_1 q_{\frac{1}{2},1} q_{\frac{1}{2},2} w - \varphi_1 u + 2\varphi_1 w^2 + 3\psi q_{\frac{1}{2},1} q_{\frac{1}{2},2} u + \psi q_{\frac{1}{2},1} q_{\frac{1}{2},2} w^2 \\
&- \psi\psi_1 q_{\frac{1}{2},1} - \psi\varphi_1 q_{\frac{1}{2},2} + \psi p_1 w_1 + 3\psi u w + 2\psi w^3 - \psi w_2 \\
&+ 2\varphi q_{\frac{1}{2},1} q_{\frac{1}{2},2} w_1 - \varphi\psi_1 q_{\frac{1}{2},2} - 3\varphi\varphi_1 q_{\frac{1}{2},1} - 4\varphi\psi q_{\frac{1}{2},1} w + \varphi p_1 w^2 \\
&+ \varphi u_1 + \varphi w w_1). \qquad (7.76)
\end{aligned}
$$

3. At level 1 and 3/2 there exist three more higher nonlocal conservation laws, of which we only shall present here the x-components:

$$
\begin{aligned}
(p_{1,5})_x &= \cos(2p_{0,1})(w q_{\frac{1}{2},1} q_{\frac{1}{2},3} + w q_{\frac{1}{2},2} q_{\frac{1}{2},4} + p_{1,3} w) \\
&+ \sin(2p_{0,1})(w q_{\frac{1}{2},2} q_{\frac{1}{2},3} - w q_{\frac{1}{2},1} q_{\frac{1}{2},4} - p_{1,4} w) \\
&+ 2w q_{\frac{1}{2},1} q_{\frac{1}{2},2} + \varphi q_{\frac{1}{2},1}; \\
(q_{\frac{3}{2},3})_x &= \cos(2p_{0,1})(q_{\frac{1}{2},4}(-2p_1 w + w_1) + q_{\frac{1}{2},3}(u + 2w^2) + q_{\frac{1}{2},2}(-2p_{1,1} w) \\
&+ q_{\frac{1}{2},1}(2p_{1,2} w + 2p_{1,4} w) + \psi p_{1,4}) \\
&+ \sin(2p_{0,1})(q_{\frac{1}{2},4}(-u - 2w^2) + q_{\frac{1}{2},3}(-2p_1 w + w_1) + q_{\frac{1}{2},2}(2p_{1,2} w) \\
&+ q_{\frac{1}{2},1}(2p_{1,1} w + 2p_{1,3} w + \psi p_{1,3}) \\
&- q_{\frac{1}{2},1} w_1 + q_{\frac{1}{2},2} u; \\
(q_{\frac{3}{2},4})_x &= \cos(2p_{0,1})(q_{\frac{1}{2},4}(-u - 2w^2) + q_{\frac{1}{2},3}(-2p_1 w + w_1) + q_{\frac{1}{2},2}(-2p_{1,2} w) \\
&+ q_{\frac{1}{2},1}(-2p_{1,1} w)) \\
&+ \sin(2p_{0,1})(q_{\frac{1}{2},4}(2p_1 w - w_1) + q_{\frac{1}{2},3}(-u - 2w^2) + q_{\frac{1}{2},2}(-2p_{1,1} w) \\
&+ q_{\frac{1}{2},1}(2p_{1,2} w)) \\
&+ q_{\frac{1}{2},1} u + q_{\frac{1}{2},2} w_1 + \psi q_{\frac{1}{2},1} q_{\frac{1}{2},2}. \qquad (7.77)
\end{aligned}
$$

Thus, we obtained the following 16 nonlocal variables:

$$
\begin{aligned}
&p_{0,1},\ p_{0,2} && \text{of degree } 0, \\
&p_1,\ p_{1,1},\ p_{1,2},\ p_{1,3},\ p_{1,4},\ p_{1,5} && \text{of degree } 1, \\
&p_{3,1} && \text{of degree } 3, \\
&q_{\frac{1}{2},1},\ q_{\frac{1}{2},2},\ q_{\frac{1}{2},3},\ q_{\frac{1}{2},4} && \text{of degree } \frac{1}{2}, \\
&q_{\frac{3}{2},1},\ q_{\frac{3}{2},2},\ q_{\frac{3}{2},3},\ q_{\frac{3}{2},4} && \text{of degree } \frac{1}{2}. \qquad (7.78)
\end{aligned}
$$

In the next subsections the augmented system of equations associated to the local and the nonlocal variables denoted above will be considered in computing higher and nonlocal symmetries and the recursion operator.

4.3.2. *Higher and nonlocal symmetries.* In this subsection, we present results for higher and nonlocal symmetries for the $N = 2$ supersymmetric extension of KdV equation (7.71) in the case $a = -1$,

$$Y = Y^u\frac{\partial}{\partial u} + Y^w\frac{\partial}{\partial w} + Y^\varphi\frac{\partial}{\partial \varphi} + Y^\psi\frac{\partial}{\partial \psi} + \dots$$

We obtained the following odd symmetries whose generating functions are:

$$\begin{aligned}
Y^u_{\frac{1}{2},1} &= -\psi_1,\\
Y^w_{\frac{1}{2},1} &= \varphi,\\
Y^\varphi_{\frac{1}{2},1} &= w_1,\\
Y^\psi_{\frac{1}{2},1} &= -u;
\end{aligned}$$

$$\begin{aligned}
Y^u_{\frac{1}{2},2} &= \cos(2p_{0,1})(\psi_1 - 2\varphi w) + \sin(2p_{0,1})(-\varphi_1 - 2\psi w),\\
Y^w_{\frac{1}{2},2} &= \cos(2p_{0,1})\varphi + \sin(2p_{0,1})\psi,\\
Y^\varphi_{\frac{1}{2},2} &= \cos(2p_{0,1})(2\psi q_{\frac{1}{2},1} + w_1) + sin(2p_{0,1})(2\psi q_{\frac{1}{2},2} - u) - 4\psi q_{\frac{1}{2},4},\\
Y^\psi_{\frac{1}{2},2} &= \cos(2p_{0,1})(-2\varphi q_{\frac{1}{2},1} + u) + \sin(2p_{0,1})(-2\varphi q_{\frac{1}{2},2} + w_1) + 4\varphi q_{\frac{1}{2},4};
\end{aligned}$$

$$\begin{aligned}
Y^u_{\frac{1}{2},3} &= \cos(2p_{0,1})(-\varphi_1 - 2\psi w) + \sin(2p_{0,1})(-\psi_1 + 2\varphi w),\\
Y^w_{\frac{1}{2},3} &= \cos(2p_{0,1})\psi - \sin(2p_{0,1})\varphi),\\
Y^\varphi_{\frac{1}{2},3} &= \cos(2p_{0,1})(2\psi q_{\frac{1}{2},2} - u) + \sin(2p_{0,1})(-2\psi q_{\frac{1}{2},1} - w_1) + 4\psi q_{\frac{1}{2},3},\\
Y^\psi_{\frac{1}{2},3} &= \cos(2p_{0,1})(-2\varphi q_{\frac{1}{2},2} + w_1) + \sin(2p_{0,1})(2\varphi q_{\frac{1}{2},1} - u) - 4\varphi q_{\frac{1}{2},3};
\end{aligned}$$

$$\begin{aligned}
Y^u_{\frac{1}{2},4} &= \varphi_1,\\
Y^w_{\frac{1}{2},4} &= \psi,\\
Y^\varphi_{\frac{1}{2},4} &= u,\\
Y^\psi_{\frac{1}{2},4} &= w_1;
\end{aligned}$$

$$\begin{aligned}
Y^u_{\frac{3}{2},1} &= \cos(2p_{0,1})(-2q_{\frac{1}{2},2}u_1 - 2q_{\frac{1}{2},2}ww_1 + 2q_{\frac{1}{2},1}uw - q_{\frac{1}{2},1}w_2\\
&\quad + \psi_2 + \psi_1 p_1 + \varphi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},2} - 2\varphi_1 w + 2\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w - \psi u\\
&\quad - \psi w^2 - \varphi p_1 w - \varphi w_1)\\
&\quad + \sin(2p_{0,1})(2q_{\frac{1}{2},2}uw - q_{\frac{1}{2},2}w_2 + 2q_{\frac{1}{2},1}u_1 + 2q_{\frac{1}{2},1}ww_1 - \varphi_2\\
&\quad + 2\psi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},2} - 2\psi_1 w - \varphi_1 p_1 - \psi p_1 w - \psi w_1 - 2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w
\end{aligned}$$

$$
\begin{aligned}
&\quad + \varphi u + \varphi w^2) - 2q_{\frac{1}{2},3}u_1 - \psi_1 p_{1,2} - \psi_1 p_{1,4} + \varphi_1 p_{1,1},\\
Y^w_{\frac{3}{2},1} &= \cos(2p_{0,1})(-q_{\frac{1}{2},1}u + \varphi_1 - \psi q_{\frac{1}{2},1}q_{\frac{1}{2},2} + \psi w)\\
&\quad + \sin(2p_{0,1})(-q_{\frac{1}{2},2}u + \psi_1 - \varphi w) - 2q_{\frac{1}{2},3}w_1 + \psi p_{1,1} + \varphi p_{1,2} + \varphi p_{1,4},\\
Y^\varphi_{\frac{3}{2},1} &= \cos(2p_{0,1})(q_{\frac{1}{2},1}q_{\frac{1}{2},2}u + \psi_1 q_{\frac{1}{2},1} + \psi q_{\frac{1}{2},1}p_1 - \varphi q_{\frac{1}{2},1}w\\
&\quad - \varphi\psi + 2uw - w_2)\\
&\quad + \sin(2p_{0,1})(-\psi_1 q_{\frac{1}{2},2} - 2\varphi_1 q_{\frac{1}{2},1} + \psi q_{\frac{1}{2},2}p_1 - 2\psi q_{\frac{1}{2}}w + \varphi q_{\frac{1}{2},2}w\\
&\quad - p_1 u + u_1 + ww_1) + 2\varphi_1 q_{\frac{1}{2},3} - 2\psi q_{\frac{3}{2},1} + p_{1,1}u + p_{1,2}w_1 + p_{1,4}w_1,\\
Y^\psi_{\frac{3}{2},1} &= \cos(2p_{0,1})(-q_{\frac{1}{2},1}q_{\frac{1}{2},2}w_1 + 2\psi_1 q_{\frac{1}{2},2} + \varphi_1 q_{\frac{1}{2},1} + \psi q_{\frac{1}{2},1}w\\
&\quad - 2\varphi q_{\frac{1}{2},2}w - \varphi q_{\frac{1}{2},1}p_1 + p_1 u - u_1 - ww_1)\\
&\quad + \sin(2p_{0,1})(2q_{\frac{1}{2},1}q_{\frac{1}{2},2}u - \varphi_1 q_{\frac{1}{2},2} - \psi q_{\frac{1}{2},2}w - \varphi q_{\frac{1}{2},2}p_1 - \varphi\psi + 2uw - w_2)\\
&\quad + 2\psi_1 q_{\frac{1}{2},3} + 2\varphi q_{\frac{3}{2},1} + p_{1,1}w_1 - p_{1,2}u - p_{1,4}u.
\end{aligned}
\tag{7.79}
$$

We also have

$$
\begin{aligned}
Y^u_{\frac{3}{2},2} &= \cos(2p_{0,1})(-2q_{\frac{1}{2},2}uw + q_{\frac{1}{2},2}w_2 - 2q_{\frac{1}{2},1}u_1 - 2q_{\frac{1}{2},1}ww_1 + \varphi_2\\
&\quad - 2\psi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},2} + 2\psi_1 w + \varphi_1 p_1 + \psi p_1 w + \psi w_1 + 2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w\\
&\quad - \varphi u - \varphi w^2)\\
&\quad + \sin(2p_{0,1})(-2q_{\frac{1}{2},2}u_1 - 2q_{\frac{1}{2},2}ww_1 + 2q_{\frac{1}{2},1}uw - q_{\frac{1}{2},1}w_2 + \psi_2 + \psi_1 p_1\\
&\quad + \varphi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},2} - 2\varphi_1 w + 2\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}w - \psi u - \psi w^2 - \varphi p_1 w - \varphi w_1)\\
&\quad + 2q_{\frac{1}{2},4}u_1 - \psi_1 p_{1,1} - \psi_1 p_{1,3} - \varphi_1 p_{1,2},\\
Y^w_{\frac{3}{2},2} &= \cos(2p_{0,1})(q_{\frac{1}{2},2}u - \psi_1 + \varphi w)\\
&\quad + \sin(2p_{0,1})(-q_{\frac{1}{2},1}u + \varphi_1 - \psi q_{\frac{1}{2},1}q_{\frac{1}{2},2} + \psi w)\\
&\quad + 2q_{\frac{1}{2},4}w_1 - \psi p_{1,2} + \varphi p_{1,1} + \varphi p_{1,3},\\
Y^\varphi_{\frac{3}{2},2} &= \cos(2p_{0,1})(\psi_1 q_{\frac{1}{2},2} + 2\varphi_1 q_{\frac{1}{2},1} - \psi q_{\frac{1}{2},2}p_1 + 2\psi q_{\frac{1}{2},1}w - \varphi q_{\frac{1}{2},2}w\\
&\quad + p_1 u - u_1 - ww_1)\\
&\quad + \sin(2p_{0,1})(q_{\frac{1}{2},1}q_{\frac{1}{2},2}u + \psi_1 q_{\frac{1}{2},1} + \psi q_{\frac{1}{2},1}p_1 - \varphi q_{\frac{1}{2},1}w - \varphi\psi + 2uw - w_2)\\
&\quad - 2\varphi_1 q_{\frac{1}{2},4} - 2\psi q_{\frac{3}{2},2} + p_{1,1}w_1 - p_{1,2}u + p_{1,3}w_1,\\
Y^\psi_{\frac{3}{2},2} &= \cos(2p_{0,1})(-2q_{\frac{1}{2},1}q_{\frac{1}{2},2}u + \varphi_1 q_{\frac{1}{2},2} + \psi q_{\frac{1}{2},2}w + \varphi q_{\frac{1}{2},2}p_1\\
&\quad + \varphi\psi - 2uw + w_2)\\
&\quad + \sin(2p_{0,1})(-q_{\frac{1}{2},1}q_{\frac{1}{2},2}w_1 + 2\psi_1 q_{\frac{1}{2},2} + \varphi_1 q_{\frac{1}{2},1} + \psi q_{\frac{1}{2},1}w - 2\varphi q_{\frac{1}{2},2}w\\
&\quad - \varphi q_{\frac{1}{2},1}p_1 + p_1 u - u_1 - ww_1)\\
&\quad - 2\psi_1 q_{\frac{1}{2},4} + 2\varphi q_{\frac{3}{2},2} - p_{1,1}u - p_{1,2}w_1 - p_{1,3}u
\end{aligned}
\tag{7.80}
$$

and

$$
\begin{aligned}
Y^u_{\frac{3}{2},3} &= \cos(2p_{0,1})(-4q_{\frac{1}{2},4}uw + 2q_{\frac{1}{2},4}w_2 + 2q_{\frac{1}{2},3}u_1 + 4q_{\frac{1}{2},3}ww_1 \\
&\quad - 2\psi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},4} + \psi_1 p_{1,2} + \psi_1 p_{1,4} - 2\varphi_1 q_{\frac{1}{2},2}q_{\frac{1}{2},4} - 4\varphi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},3} + \varphi_1 p_{1,1} \\
&\quad + \varphi_1 p_{1,3} - 4\psi q_{\frac{1}{2},1}q_{\frac{1}{2},3}w + 2\psi p_{1,1}w + 2\psi p_{1,3}w + 4\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},4}w \\
&\quad - 2\varphi p_{1,2}w - 2\varphi p_{1,4}w) \\
&\quad + \sin(2p_{0,1})(-2q_{\frac{1}{2},4}u_1 - 4q_{\frac{1}{2},4}ww_1 - 4q_{\frac{1}{2},3}uw + 2q_{\frac{1}{2},3}w_2 \\
&\quad - 2\psi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},3} + \psi_1 p_{1,1} + \psi_1 p_{1,3} - 2\varphi_1 q_{\frac{1}{2},2}q_{\frac{1}{2},3} + 4\varphi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},4} \\
&\quad - \varphi_1 p_{1,2} - \varphi_1 p_{1,4} + 4\psi q_{\frac{1}{2},1}q_{\frac{1}{2},4}w - 2\psi p_{1,2}w - 2\psi p_{1,4}w \\
&\quad + 4\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},3}w - 2\varphi p_{1,1}w - 2\varphi p_{1,3}w) \\
&\quad + 2q_{\frac{1}{2},2}u_1 + 2q_{\frac{1}{2},2}ww_1 \\
&\quad + 2q_{\frac{1}{2},1}uw - q_{\frac{1}{2},1}w_2 - \psi_2 - \psi_1 p_1 - 4\varphi_1 q_{\frac{1}{2},3}q_{\frac{1}{2},4} \\
&\quad - \varphi_1 q_{\frac{1}{2},1}q_{\frac{1}{2},2} + 2\varphi_1 w + \psi u + \psi w^2 + \varphi p_1 w + \varphi w_1, \\
Y^w_{\frac{3}{2},3} &= \cos(2p_{0,1})(2q_{\frac{1}{2},4}u - 2q_{\frac{1}{2},3}w_1 - 2\psi q_{\frac{1}{2},2}q_{\frac{1}{2},4} - \psi p_{1,1} - \psi p_{1,3} \\
&\quad - 2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},4} + \varphi p_{1,2} + \varphi p_{1,4}) \\
&\quad + \sin(2p_{0,1})(2q_{\frac{1}{2},4}w_1 + 2q_{\frac{1}{2},3}u - 2\psi q_{\frac{1}{2},2}q_{\frac{1}{2},3} + \psi p_{1,2} + \psi p_{1,4} \\
&\quad - 2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},3} + \varphi p_{1,1} + \varphi p_{1,3}) \\
&\quad - q_{\frac{1}{2},1}u + \varphi_1 - 4\psi q_{\frac{1}{2},3}q_{\frac{1}{2},4} - \psi q_{\frac{1}{2},1}q_{\frac{1}{2},2} + \psi w, \\
Y^\varphi_{\frac{3}{2},3} &= \cos(2p_{0,1})(-2q_{\frac{1}{2},2}q_{\frac{1}{2},4}u - 2q_{\frac{1}{2},1}q_{\frac{1}{2},4}w_1 - 4q_{\frac{1}{2},1}q_{\frac{1}{2},3}u + 2\psi_1 q_{\frac{1}{2},4} \\
&\quad + 2\varphi_1 q_{\frac{1}{2},3} + 2\psi q_{\frac{1}{2},4}p_1 - 2\psi q_{\frac{1}{2},2}p_{1,1} - 2\psi q_{\frac{1}{2},2}p_{1,3} - 4\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}q_{\frac{1}{2},3} \\
&\quad + 2\psi q_{\frac{1}{2},1}p_{1,2} + 2\psi q_{\frac{1}{2},1}p_{1,4} - 2\varphi q_{\frac{1}{2},4}w + p_{1,1}u \\
&\quad + p_{1,2}w_1 + p_{1,3}u + p_{1,4}w_1) \\
&\quad + \sin(2p_{0,1})(-2q_{\frac{1}{2},2}q_{\frac{1}{2},3}u + 4q_{\frac{1}{2},1}q_{\frac{1}{2},4}u - 2q_{\frac{1}{2},1}q_{\frac{1}{2},3}w_1 + 2\psi_1 q_{\frac{1}{2},3} \\
&\quad - 2\varphi_1 q_{\frac{1}{2},4} + 2\psi q_{\frac{1}{2},3}p_1 + 2\psi q_{\frac{1}{2},2}p_{1,2} + 2\psi q_{\frac{1}{2},2}p_{1,4} \\
&\quad + 4\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}q_{\frac{1}{2},4} + 2\psi q_{\frac{1}{2},1}p_{1,1} + 2\psi q_{\frac{1}{2},1}p_{1,3} - 2\varphi q_{\frac{1}{2},3}w + p_{1,1}w_1 \\
&\quad - p_{1,2}u + p_{1,3}w_1 - p_{1,4}u) \\
&\quad - 4q_{\frac{1}{2},3}q_{\frac{1}{2},4}u - q_{\frac{1}{2},1}q_{\frac{1}{2},2}u - \psi_1 q_{\frac{1}{2},1} - 4\psi q_{\frac{1}{2},4}p_{1,2} - 4\psi q_{\frac{1}{2},4}p_{1,4} \\
&\quad - 4\psi q_{\frac{1}{2},3}p_{1,1} - 4\psi q_{\frac{1}{2},3}p_{1,3} - \psi q_{\frac{1}{2},1}p_1 + \varphi q_{\frac{1}{2},1}w - \varphi\psi + 2uw - w_2, \\
Y^\psi_{\frac{3}{2},3} &= \cos(2p_{0,1})(-2q_{\frac{1}{2},2}q_{\frac{1}{2},4}w_1 - 2q_{\frac{1}{2},1}q_{\frac{1}{2},4}u - 2\psi_1 q_{\frac{1}{2},3} + 2\varphi_1 q_{\frac{1}{2},4} \\
&\quad + 2\psi q_{\frac{1}{2},4}w - 2\varphi q_{\frac{1}{2},4}p_1 + 4\varphi q_{\frac{1}{2},3}w + 2\varphi q_{\frac{1}{2},2}p_{1,1} + 2\varphi q_{\frac{1}{2},2}p_{1,3} \\
&\quad + 4\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}q_{\frac{1}{2},3} - 2\varphi q_{\frac{1}{2},1}p_{1,2} - 2\varphi q_{\frac{1}{2},1}p_{1,4} - p_{1,1}w_1 + p_{1,2}u \\
&\quad - p_{1,3}w_1 + p_{1,4}u)
\end{aligned}
$$

$$
\begin{aligned}
&+\sin(2p_{0,1})(-2q_{\frac12,2}q_{\frac12,3}w_1-2q_{\frac12,1}q_{\frac12,3}u+2\psi_1q_{\frac12,4}+2\varphi_1q_{\frac12,3}\\
&+2\psi q_{\frac12,3}w-4\varphi q_{\frac12,4}w-2\varphi q_{\frac12,3}p_1-2\varphi q_{\frac12,2}p_{1,2}-2\varphi q_{\frac12,2}p_{1,4}\\
&-4\varphi q_{\frac12,1}q_{\frac12,2}q_{\frac12,4}-2\varphi q_{\frac12,1}p_{1,1}-2\varphi q_{\frac12,1}p_{1,3}+p_{1,1}u\\
&+p_{1,2}w_1+p_{1,3}u+p_{1,4}w_1)\\
&-4q_{\frac12,3}q_{\frac12,4}w_1-q_{\frac12,1}q_{\frac12,2}w_1-2\psi_1q_{\frac12,2}-\varphi_1q_{\frac12,1}-\psi q_{\frac12,1}w\\
&+4\varphi q_{\frac12,4}p_{1,2}+4\varphi q_{\frac12,4}p_{1,4}+4\varphi q_{\frac12,3}p_{1,1}+4\varphi q_{\frac12,3}p_{1,3}+2\varphi q_{\frac12,2}w\\
&+\varphi q_{\frac12,1}p_1-p_1u+u_1+ww_1,
\end{aligned}
\tag{7.81}
$$

together with

$$
\begin{aligned}
Y^u_{\frac32,4}&=\cos(2p_{0,1})(2q_{\frac12,4}u_1+4q_{\frac12,4}ww_1+4q_{\frac12,3}uw-2q_{\frac12,3}w_2-4\psi_1q_{\frac12,2}q_{\frac12,4}\\
&-2\psi_1q_{\frac12,1}q_{\frac12,3}+\psi_1p_{1,1}-2\varphi_1q_{\frac12,2}q_{\frac12,3}-\varphi_1p_{1,2}-4\psi q_{\frac12,2}q_{\frac12,3}w\\
&-2\psi p_{1,2}w+4\varphi q_{\frac12,2}q_{\frac12,4}w-2\varphi p_{1,1}w)\\
&+\sin(2p_{0,1})(-4q_{\frac12,4}uw+2q_{\frac12,4}w_2+2q_{\frac12,3}u_1+4q_{\frac12,3}ww_1-4\psi_1q_{\frac12,2}q_{\frac12,3}\\
&+2\psi_1q_{\frac12,1}q_{\frac12,4}-\psi_1p_{1,2}+2\varphi_1q_{\frac12,2}q_{\frac12,4}-\varphi_1p_{1,1}+4\psi q_{\frac12,2}q_{\frac12,4}w\\
&-2\psi p_{1,1}w+4\varphi q_{\frac12,2}q_{\frac12,3}w+2\varphi p_{1,2}w)\\
&+2q_{\frac12,2}uw-q_{\frac12,2}w_2-2q_{\frac12,1}u_1-2q_{\frac12,1}ww_1+\varphi_2-4\psi_1q_{\frac12,3}q_{\frac12,4}\\
&-2\psi_1q_{\frac12,1}q_{\frac12,2}+2\psi_1w+\varphi_1p_1+\psi p_1w+\psi w_1+2\varphi q_{\frac12,1}q_{\frac12,2}w\\
&-\varphi u-\varphi w^2,\\
Y^w_{\frac32,4}&=\cos(2p_{0,1})(-2q_{\frac12,4}w_1-2q_{\frac12,3}u+2\psi q_{\frac12,2}q_{\frac12,3}+\psi p_{1,2}\\
&+2\varphi q_{\frac12,1}q_{\frac12,3}+\varphi p_{1,1})\\
&+\sin(2p_{0,1})(2q_{\frac12,4}u-2q_{\frac12,3}w_1-2\psi q_{\frac12,2}q_{\frac12,4}+\psi p_{1,1}\\
&-2\varphi q_{\frac12,1}q_{\frac12,4}-\varphi p_{1,2})\\
&-q_{\frac12,2}u+\psi_1+4\varphi q_{\frac12,3}q_{\frac12,4}-\varphi w,\\
Y^\varphi_{\frac32,4}&=\cos(2p_{0,1})(-2q_{\frac12,2}q_{\frac12,3}u+2q_{\frac12,1}q_{\frac12,3}w_1+2\psi_1q_{\frac12,3}-2\varphi_1q_{\frac12,4}\\
&-4\psi q_{\frac12,4}w-2\psi q_{\frac12,3}p_1+2\psi q_{\frac12,2}p_{1,2}+2\psi q_{\frac12,1}p_{1,1}-2\varphi q_{\frac12,3}w\\
&+p_{1,1}w_1-p_{1,2}u)\\
&+\sin(2p_{0,1})(2q_{\frac12,2}q_{\frac12,4}u-2q_{\frac12,1}q_{\frac12,4}w_1-2\psi_1q_{\frac12,4}-2\varphi_1q_{\frac12,3}+2\psi q_{\frac12,4}p_1\\
&-4\psi q_{\frac12,3}w+2\psi q_{\frac12,2}p_{1,1}-2\psi q_{\frac12,1}p_{1,2}+2\varphi q_{\frac12,4}w-p_{1,1}u-p_{1,2}w_1)\\
&+4q_{\frac12,3}q_{\frac12,4}w_1+\psi_1q_{\frac12,2}+2\varphi_1q_{\frac12,1}-4\psi q_{\frac12,4}p_{1,1}+4\psi q_{\frac12,3}p_{1,2}-\psi q_{\frac12,2}p_1\\
&+2\psi q_{\frac12,1}w-\varphi q_{\frac12,2}w+p_1u-u_1-ww_1,\\
Y^\psi_{\frac32,4}&=\cos(2p_{0,1})(-4q_{\frac12,2}q_{\frac12,4}u+2q_{\frac12,2}q_{\frac12,3}w_1-2q_{\frac12,1}q_{\frac12,3}u+2\psi_1q_{\frac12,4}\\
&+2\varphi_1q_{\frac12,3}+2\psi q_{\frac12,3}w+2\varphi q_{\frac12,3}p_1-2\varphi q_{\frac12,2}p_{1,2}-2\varphi q_{\frac12,1}p_{1,1}
\end{aligned}
$$

$$
\begin{aligned}
&+ p_{1,1}u + p_{1,2}w_1) \\
&+ \sin(2p_{0,1})(-2q_{\frac{1}{2},2}q_{\frac{1}{2},4}w_1 - 4q_{\frac{1}{2},2}q_{\frac{1}{2},3}u + 2q_{\frac{1}{2},1}q_{\frac{1}{2},4}u + 2\psi_1 q_{\frac{1}{2},3} \\
&- 2\varphi_1 q_{\frac{1}{2},4} - 2\psi q_{\frac{1}{2},4}w - 2\varphi q_{\frac{1}{2},4}p_1 - 2\varphi q_{\frac{1}{2},2}p_{1,1} + 2\varphi q_{\frac{1}{2},1}p_{1,2} \\
&+ p_{1,1}w_1 - p_{1,2}u) \\
&- 4q_{\frac{1}{2},3}q_{\frac{1}{2},4}u - 2q_{\frac{1}{2},1}q_{\frac{1}{2},2}u + \varphi_1 q_{\frac{1}{2},2} + \psi q_{\frac{1}{2},2}w + 4\varphi q_{\frac{1}{2},4}p_{1,1} \\
&- 4\varphi q_{\frac{1}{2},3}p_{1,2} + \varphi q_{\frac{1}{2},2}p_1 - \varphi\psi + 2uw - w_2.
\end{aligned} \tag{7.82}
$$

Even symmetries are

$$
\begin{aligned}
Y^u_{1,1} &= u_1, \\
Y^w_{1,1} &= w_1, \\
Y^\varphi_{1,1} &= \varphi_1, \\
Y^\psi_{1,1} &= \psi_1; \\
Y^u_{1,2} &= \cos(2p_{0,1})(-\psi_1 q_{\frac{1}{2},1} - \varphi_1 q_{\frac{1}{2},2} + 2\varphi q_{\frac{1}{2},1}w - 2uw + w_2) \\
&\quad + \sin(2p_{0,1})(2\varphi_1 q_{\frac{1}{2},1} + 2\psi q_{\frac{1}{2},1}w - u_1 - 2ww_1) - 2\varphi_1 q_{\frac{1}{2},3}, \\
Y^w_{1,2} &= \cos(2p_{0,1})(-\psi q_{\frac{1}{2},2} - \varphi q_{\frac{1}{2},1} + u) \\
&\quad + \sin(2p_{0,1})w_1 - 2\psi q_{\frac{1}{2},3}, \\
Y^\varphi_{1,2} &= \cos(2p_{0,1})(q_{\frac{1}{2},2}u + q_{\frac{1}{2},1}w_1 - \psi_1 - \psi p_1 + \varphi w) \\
&\quad + \sin(2p_{0,1})(-2q_{\frac{1}{2},1}u + \varphi_1 - 2\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}) + 2(q_{\frac{1}{2},3}u + \psi p_{1,2} + \psi p_{1,4}), \\
Y^\psi_{1,2} &= \cos(2p_{0,1})(q_{\frac{1}{2},2}w_1 + q_{\frac{1}{2},1}u - \varphi_1 - \psi w + \varphi p_1) \\
&\quad + \sin(2p_{0,1})(-\psi_1 + 2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2} + 2\varphi w) + 2(q_{\frac{1}{2},3}w_1 - \varphi p_{1,2} - \varphi p_{1,4}); \\
Y^u_{1,3} &= \cos(2p_{0,1})(2\varphi_1 q_{\frac{1}{2},1} + 2\psi q_{\frac{1}{2}}w - u_1 - 2ww_1) \\
&\quad + \sin(2p_{0,1})(\psi_1 q_{\frac{1}{2},1} + \varphi_1 q_{\frac{1}{2},2} - 2\varphi q_{\frac{1}{2},1}w + 2uw - w_2) - 2\varphi_1 q_{\frac{1}{2},4}, \\
Y^w_{1,3} &= \cos(2p_{0,1})w_1 \\
&\quad + \sin(2p_{0,1})(\psi\overline{q}_{\frac{1}{2}} + \varphi q_{\frac{1}{2},1} - u) - 2\psi q_{\frac{1}{2},4}, \\
Y^\varphi_{1,3} &= \cos(2p_{0,1})(-2q_{\frac{1}{2},1}u + \varphi_1 - 2\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}) \\
&\quad + \sin(2p_{0,1})(-q_{\frac{1}{2},2}u - q_{\frac{1}{2},1}w_1 + \psi_1 + \psi p_1 - \varphi w) \\
&\quad + 2(q_{\frac{1}{2},4}u - \psi p_{1,1} - \psi p_{1,3}), \\
Y^\psi_{1,3} &= \cos(2p_{0,1})(-\psi_1 + 2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2} + 2\varphi w) \\
&\quad + \sin(2p_{0,1})(-q_{\frac{1}{2},2}w_1 - q_{\frac{1}{2},1}u + \varphi_1 + \psi w - \varphi p_1) \\
&\quad + 2(q_{\frac{1}{2},4}w_1 + \varphi p_{1,1} + \varphi p_{1,3});
\end{aligned} \tag{7.83}
$$

$$
Y^u_{1,4} = \cos(2p_{0,1})(-2\psi_1 q_{\frac{1}{2},2} + 2\varphi q_{\frac{1}{2},2}w + u_1 + 2ww_1)
$$

$$
\begin{aligned}
&+\sin(2p_{0,1})(\psi_1 q_{\frac{1}{2},1}+\varphi_1 q_{\frac{1}{2},2}+2\psi q_{\frac{1}{2},2}w-2uw+w_2)-2\psi_1 q_{\frac{1}{2},3},\\
Y^w_{1,4}&=-\cos(2p_{0,1})w_1\\
&+\sin(2p_{0,1})(-\psi q_{\frac{1}{2},2}-\varphi q_{\frac{1}{2},1}+u)+2\varphi q_{\frac{1}{2},3},\\
Y^\varphi_{1,4}&=\cos(2p_{0,1})(\varphi_1+2\psi w)\\
&+\sin(2p_{0,1})(-q_{\frac{1}{2},2}u+q_{\frac{1}{2},1}w_1+\psi_1-\psi p_1-\varphi w)\\
&+2(-q_{\frac{1}{2},3}w_1+\psi p_{1,1}),\\
Y^\psi_{1,4}&=\cos(2p_{0,1})(2q_{\frac{1}{2},2}u-\psi_1)\\
&+\sin(2p_{0,1})(q_{\frac{1}{2},2}w_1-q_{\frac{1}{2},1}u+\varphi_1+\psi w+\varphi p_1)+2(q_{\frac{1}{2},3}u-\varphi p_{1,1}).
\end{aligned}
\tag{7.84}
$$

Finally, we got

$$
\begin{aligned}
Y^u_{1,5}&=\cos(2p_{0,1})(\psi_1 q_{\frac{1}{2},1}+\varphi_1 q_{\frac{1}{2},2}+2\psi q_{\frac{1}{2},2}w-2uw+w_2)\\
&+\sin(2p_{0,1})(2\psi_1 q_{\frac{1}{2},2}-2\varphi q_{\frac{1}{2},2}w-u_1-2ww_1)-2\psi_1 q_{\frac{1}{2},4},\\
Y^w_{1,5}&=\cos(2p_{0,1})(-\psi q_{\frac{1}{2},2}-\varphi q_{\frac{1}{2},1}+u)\\
&+\sin(2p_{0,1})w_1+2\varphi q_{\frac{1}{2},4},\\
Y^\varphi_{1,5}&=\cos(2p_{0,1})(-q_{\frac{1}{2},2}u+q_{\frac{1}{2},1}w_1+\psi_1-\psi p_1-\varphi w)\\
&+\sin(2p_{0,1})(-\varphi_1-2\psi w)+2(-q_{\frac{1}{2},4}w_1+\psi p_{1,2}),\\
Y^\psi_{1,5}&=\cos(2p_{0,1})(q_{\frac{1}{2},2}w_1-q_{\frac{1}{2},1}u+\varphi_1+\psi w+\varphi p_1)\\
&+\sin(2p_{0,1})(-2q_{\frac{1}{2},2}u+\psi_1)+2(q_{\frac{1}{2},4}u-\varphi)p_{1,2});\\
Y^u_{1,6}&=\cos(2p_{0,1})(-\psi_1 q_{\frac{1}{2},3}+\varphi_1 q_{\frac{1}{2},4}+2\psi q_{\frac{1}{2},4}w+2\varphi q_{\frac{1}{2},3}w)\\
&+\sin(2p_{0,1})(\psi_1 q_{\frac{1}{2},4}+\varphi_1 q_{\frac{1}{2},3}+2\psi q_{\frac{1}{2},3}w-2\varphi q_{\frac{1}{2},4}w)\\
&-\psi_1 q_{\frac{1}{2},2}-\varphi_1 q_{\frac{1}{2},1}-\psi q_{\frac{1}{2},1}w+\varphi q_{\frac{1}{2},2}w,\\
Y^w_{1,6}&=-\cos(2p_{0,1})(\psi q_{\frac{1}{2},4}+\varphi q_{\frac{1}{2},3})\\
&+\sin(2p_{0,1})(-\psi q_{\frac{1}{2},3}+\varphi q_{\frac{1}{2},4}),\\
Y^\varphi_{1,6}&=\cos(2p_{0,1})(-q_{\frac{1}{2},4}u+q_{\frac{1}{2},3}w_1+2\psi q_{\frac{1}{2},2}q_{\frac{1}{2},4}+2\psi q_{\frac{1}{2},1}q_{\frac{1}{2},3})\\
&+\sin(2p_{0,1})(-q_{\frac{1}{2},4}w_1-q_{\frac{1}{2},3}u+2\psi q_{\frac{1}{2},2}q_{\frac{1}{2},3}-2\psi q_{\frac{1}{2},1}q_{\frac{1}{2},4})\\
&+q_{\frac{1}{2},1}u-\varphi_1+4\psi q_{\frac{1}{2},3}q_{\frac{1}{2},4}+\psi q_{\frac{1}{2},1}q_{\frac{1}{2},2}-\psi w,\\
Y^\psi_{1,6}&=\cos(2p_{0,1})(q_{\frac{1}{2},4}w_1+q_{\frac{1}{2},3}u-2\varphi q_{\frac{1}{2},2}q_{\frac{1}{2},4}-2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},3})\\
&+\sin(2p_{0,1})(-q_{\frac{1}{2},4}u+q_{\frac{1}{2},3}w_1-2\varphi q_{\frac{1}{2},2}q_{\frac{1}{2},3}+2\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},4})\\
&+q_{\frac{1}{2},2}u-\psi_1-4\varphi q_{\frac{1}{2},3}q_{\frac{1}{2},4}-\varphi q_{\frac{1}{2},1}q_{\frac{1}{2},2}+\varphi w.
\end{aligned}
\tag{7.85}
$$

4.3.3. *Recursion operator.* Here we shall discuss briefly the recursion properties of the nonlocal symmetries $Y_{1,2}$, $Y_{1,3}$, $Y_{1,4}$, $Y_{1,5}$, $Y_{1,6}$ given in (7.83) and (7.85).

We shall discuss their action on the supersymmetry $Y_{\frac{1}{2},1}$ of degree 1/2.

In order to compute the Lie bracket of these symmetries, we have to derive the nonlocal components, just for the vector field $Y_{\frac{1}{2},1}$.

Due to the invariance of the equations, defining the nonlocal variables $p_{0,1}, p_1, q_{\frac{1}{2},1}, q_{\frac{1}{2},2}, q_{\frac{1}{2},3}, q_{\frac{1}{2},4}$ and $p_{1,1}, p_{1,2}, p_{1,3}, p_{1,4}$, the nonlocal components can be obtained.

The prolongation of the vector field $Y_{\frac{1}{2},1}$ is then given as

$$
\begin{aligned}
Y_{\frac{1}{2},1} = {} & -\psi_1\frac{\partial}{\partial u} + \varphi\frac{\partial}{\partial w} + w_1\frac{\partial}{\partial \varphi} - u\frac{\partial}{\partial \psi} \\
& + w\frac{\partial}{\partial q_{\frac{1}{2},1}} - p_1\frac{\partial}{\partial q_{\frac{1}{2},2}} + (p_{1,2}+p_{1,4})\frac{\partial}{\partial q_{\frac{1}{2},3}} - (p_{1,1}+p_{1,3})\frac{\partial}{\partial q_{\frac{1}{2},4}} \\
& + q_{\frac{1}{2},1}\frac{\partial}{\partial p_{0,1}} - \psi\frac{\partial}{\partial p_1} \\
& + (\cos(2p_{0,1})(2q_{\frac{1}{2},1}p_1 + q_{\frac{1}{2},2}w) + \sin(2p_{0,1})(2q_{\frac{1}{2},2}p_1) - 2q_{\frac{3}{2},1})\frac{\partial}{\partial p_{1,1}} \\
& + (\cos(2p_{0,1})(2q_{\frac{1}{2},2}p_1) - \sin(2p_{0,1})(2q_{\frac{1}{2},1}p_1 + q_{\frac{1}{2},2}w) + 2q_{\frac{3}{2},2})\frac{\partial}{\partial p_{1,2}} \\
& + (-\cos(2p_{0,1})(2q_{\frac{1}{2},1}p_1 + 2q_{\frac{1}{2},2}w) + \sin(2p_{0,1}(q_{\frac{1}{2},1}w - q_{\frac{1}{2},2}p_1) \\
& + 2q_{\frac{3}{2},1})\frac{\partial}{\partial p_{1,3}} \\
& + (\cos(2p_{0,1})(q_{\frac{1}{2},1}w - q_{\frac{1}{2},2}p_1) + \sin(2p_{0,1})(2q_{\frac{1}{2},1}p_1 + 2q_{\frac{1}{2},2}w) \\
& - 2q_{\frac{3}{2},1})\frac{\partial}{\partial p_{1,4}}.
\end{aligned}
\tag{7.86}
$$

For the vector fields $Y_{1,i}$, $i = 2, \dots, 6$, prolongation is not required due to the locality of $Y_{\frac{1}{2},1}$.

We obtain the following commutators:

$$
\begin{aligned}
&[Y_{1,2}, Y_{\frac{1}{2},1}] = 0, \\
&[Y_{1,3}, Y_{\frac{1}{2},1}] = 0, \\
&[Y_{1,4}, Y_{\frac{1}{2},1}] = 2Y_{\frac{3}{2},1}, \\
&[Y_{1,5}, Y_{\frac{1}{2},1}] = -2Y_{\frac{3}{2},2}, \\
&[Y_{1,6}, Y_{\frac{1}{2},1}] = -2Y_{\frac{3}{2},3},
\end{aligned}
\tag{7.87}
$$

meaning that $Y_{1,i}$, $i = 2, \dots, 6$, take symmetry $Y_{\frac{1}{2},1}$ higher into the hierarchy.

Similar results are obtained for the local symmetry $Y_{\frac{1}{2},4}$.

In order to compute the Lie brackets for $Y_{\frac{1}{2},2}$ and $Y_{\frac{1}{2},3}$, leading to similar results too, prolongations of these vector fields are required.

The results are related to a similar action of the recursion symmetry for the $n = 1$ supersymmetric KdV equation, discussed in Section 1.

The work on the contruction of the recursion operator as obtained for the cases $a = -2$ by (7.56) and $a = 4$ by (7.70), is still in progress and will be published elsewhere.

CHAPTER 8

Symbolic computations in differential geometry

To introduce this subject, it is nice to tell the story how NN computed the tenth conservation law of the classical KdV equation at the end of the sixties.

From previous results one had obtained nine conservation laws for the KdV equation and the idea was that if one would be able to compute the tenth then people would be convinced that there existed an infinite hierarchy of conservation laws for the KdV equation. At that time, the notion of recursion operators (the first one obtained by Lenard) was not yet known.

Then NN took the decision to retire for two weeks to a nice cabin somewhere high up in the mountains and to try to figure out whether he would be able to find number ten. After two weeks he returned from his exile position having found the next one in the hierarchy, thus "proving" the existence of an infinite hierarchy.

With nowadays modern facilities it is possible to construct the first ten or twenty in few seconds. This is just one of the examples demonstrating the need for computer programs to do in principle simple, but in effect huge algebraic computations to get to final results.

Towards the end of the seventies the first computer programs were constructed. Among them Gragert [**22**], Schrüfer [**90**], Schwarz [**91**], Kersten [**34**], ..., just doing part of the work on computations on differential forms, vector fields, solutions of overdetermined systems of partial differential equations, covering conditions, etc.

Since then, quite a number of programs has been constructed and it seems that nowadays each individual researcher in this field of mathematical physics uses his or her own developed software to do the required computations in more or less the most or almost most efficient way. An overview of existing programs in all distinct related areas was recently given by Hereman in his extensive paper [**30**].

In the following sections, we shall discuss in some detail a number of types of computations which can be carried through on a computer system. The basis of these programs has been constructed by Gragert [**22**], Kersten [**37**], Gragert, Kersten and Martini [**24, 25**], Roelofs [**85, 86**], van Bemmelen [**9, 8**] at the University of Twente, starting in 1979 with exterior differential forms, construction and solution of overdetermined systems of partial differential equations arising from symmetry computations, extension of the software to work in a graded setting, meaning supercalculus, required for the interesting field of super and supersymmetric extensions of classical differential equations and at the end a completely new package, being extremely suitable for classical as well as supersymmetrical systems, together with packages for computation of covering structures of completely integrable systems, and a package to handling the computations with total derivative operators. We should mention here too (super) Lie algebra computations for covering structures by Gragert and Roelofs [**23, 26**].

We prefer to start in Section 1 with setting down the basic notions of the graded or supercalculus, since classical differential geometric computations can be embedded in a very effective way in this more general setting, which will be done in Section 2.

In Section 3 we shall give an idea how the software concerning construction of solutions of overdetermined systems of partial differential equations works, and what the facilities are.

Finally we shall present in Subsection 3.2 a computer session concerning the construction of higher symmetries of third order of the Burgers equation, i.e., defining functions involving derivatives (with respect to x up to order 3), cf. Chapter 2.

1. Super (graded) calculus

We give here a concise exposition of super (or graded) calculus needed for symbolic computations.

At first sight the introduction of graded calculus requires a completely new set of definitions and objects. It has been shown that locally a graded manifold, or equivalently the algebra of functions defined on it, is given as $C^\infty(U)\otimes\Lambda(n)$, where $\Lambda(n)$ is the exterior algebra of n (odd) variables, [**50**]. Below we shall set down the notions involved in the graded calculus and graded differential geometry.

Thus we give a short review of the notions of graded differential geometry as far as they are needed for implementation by means of software procedures, i.e., *graded commutative algebra*, *graded Lie algebra*, *graded manifold*, *graded derivation*, *graded vector field*, *graded differential form*, *exterior differentiation*, *inner differentiation* or *contraction by a vector field*, *Lie derivative along a vector field*, etc.

The notions and notations have been taken from Kostant [**50**] and the reader is referred to this paper for more details, compare with Chapter 6. Throughout this section, the basic field is $\mathbb{R}$ or $\mathbb{C}$ and the grading will be with respect to $\mathbb{Z}_2=\{0,1\}$.

1. A vector space V over $\mathbb{R}$ is a *graded vector space* if one has V_0 and V_1 subspaces of V, such that

$$V=V_0\oplus V_1 \tag{8.1}$$

is a direct sum. Elements of V_0 are called *even*, elements of V_1 are called *odd*. Elements of V_0 or V_1 are called homogeneous elements.

If $v \in V_i$, $i = 0, 1$, then i is called the degree of v, i.e.,

$$|v| = i, \quad i = 0, 1, \text{ or } i \in \mathbb{Z}_2. \tag{8.2}$$

The notation $|v|$ is used for homogeneous elements only.

2. A *graded algebra* B is a graded vector space $B = B_0 \oplus B_1$ with a multiplication such that

$$B_i \cdot B_j \subset B_{i+j}, \quad i, j \in \mathbb{Z}_2. \tag{8.3}$$

3. A graded algebra B is called *graded commutative* if for any two homogeneous elements $x, y \in B$ we have

$$xy = (-1)^{|x||y|} yx. \tag{8.4}$$

4. A graded space V is a *left module* over the graded algebra B, if V is a left module in the usual sense and

$$B_i \cdot V_j \subset V_{i+j}, \quad i, j \in \mathbb{Z}_2; \tag{8.5}$$

right modules are defined similarly.

5. If V is a left module over the graded commutative algebra B, then V inherits a *right module structure*, where we define

$$v \cdot b \overset{\text{def}}{=} (-1)^{|v||b|} b \cdot v, \quad v \in V, \ b \in B. \tag{8.6}$$

Similarly, a left module structure is determined by a right module structure.

6. A graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, together with a bilinear operation $[\cdot, \cdot]$ on $\mathfrak{g}$ such that $[x, y] \in \mathfrak{g}_{|x|+|y|}$ is called a *graded Lie algebra* if

$$\begin{gathered} [x, y] = -(-1)^{|x||y|} [y, x], \\ (-1)^{|x||z|} [x, [y, z]] + (-1)^{|z||y|} [z, [x, y]] + (-1)^{|y||x|} [y, [z, x]] = 0, \end{gathered} \tag{8.7}$$

where the last equality is called the *graded Jacobi identity.*

If V is a graded vector space, then $\mathrm{End}(V)$ has the structure of a graded Lie algebra defined by

$$[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|} \beta\alpha, \quad \alpha, \beta \in \mathrm{End}(V). \tag{8.8}$$

7. If B is a graded algebra, an operator $h \in \mathrm{End}_i(B)$ is called a *graded derivation* of B if

$$h(xy) = h(x)y + (-1)^{|i||x|} x h(y). \tag{8.9}$$

An operator $h \in \mathrm{End}(B)$ is a derivation if its homogeneous components are so.

The graded vector space of derivations of B, denoted by $\mathrm{Der}(B)$, is a graded Lie subalgebra of $\mathrm{End}(B)$. Equality (8.9) is called *graded Leibniz rule.* If B is a graded commutative algebra then $\mathrm{Der}(B)$ is a left B-module: if $\zeta \in \mathrm{Der}(B)$, $f, g \in B$, then $f\zeta \in \mathrm{Der}(B)$, where

$$(f\zeta)g = f(\zeta g). \tag{8.10}$$

8. The local picture of a *graded manifold* is an open neighborhood $U \subset \mathbb{R}^m$ together with the *graded commutative algebra*

$$C^\infty(U) \otimes \Lambda(n), \tag{8.11}$$

where $\Lambda(n)$ is the antisymmetric (exterior) algebra on n elements

$$s_1, \ldots, s_n, \quad |s_i| = 1, \quad s_i s_j = -s_j s_i, \qquad i, j = 1, \ldots, n. \tag{8.12}$$

The pair $(m|n)$ is called the *dimension* of the graded manifold at hand. A particular element $f \in C^\infty(U) \otimes \Lambda(n)$ is represented as

$$f = \sum_\mu f_\mu s_\mu, \tag{8.13}$$

where μ is a multi-index: $\mu \in M_n = \{\mu = (\mu_1, \ldots, \mu_n) \mid \mu_i \in \mathbb{N},\ 1 \le \mu_1 \le \mu_2 \cdots \le \mu_k \le n\}$,

$$s_\mu = s_{\mu_1} \cdot s_{\mu_2} \cdot \ldots \cdot s_{\mu_k}, \quad f_\mu \in C^\infty(U). \tag{8.14}$$

9. *Graded vector fields* on a graded manifold $(U, C^\infty(U) \otimes \Lambda(n))$ are introduced as graded derivations of the algebra $C^\infty(U) \otimes \Lambda(n)$. They constitute a left $C^\infty(U) \otimes \Lambda(n)$-module. Locally, a graded vector field V is represented as

$$V = \sum_{i=1}^m f_i \frac{\partial}{\partial r_i} + \sum_{j=1}^n g_j \frac{\partial}{\partial s_j}, \tag{8.15}$$

where f_i, $g_j \in C^\infty(U) \otimes \Lambda(n)$, and r_i, $i = 1, \ldots, m$, are local coordinates in $U \subset \mathbb{R}^m$.

The derivations $\partial/\partial r_i$, $i = 1, \ldots, m$, are even, while the derivations $\partial/\partial s_j$, $j = 1, \ldots, n$, are odd. They satisfy the relations

$$\frac{\partial r_k}{\partial r_i} = \delta_{ik}, \quad \frac{\partial s_j}{\partial r_i} = 0, \quad \frac{\partial r_k}{\partial s_j} = 0, \quad \frac{\partial s_l}{\partial s_j} = \delta_{jl} \tag{8.16}$$

for all $i, k = 1, \ldots, m$, $j, l = 1, \ldots, n$.

10. A *graded differential k-form* is introduced as k-linear mapping β on $\mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$ which has to satisfy the identities

$$\langle \zeta_1, \ldots, f\zeta_l, \ldots, \zeta_k \mid \beta \rangle = (-1)^{|f| \sum_{i=1}^{l-1} |\zeta_i|} f \langle \zeta_1, \ldots, \zeta_k \mid \beta \rangle \tag{8.17}$$

and

$$\begin{aligned}\langle \zeta_1, \ldots, \zeta_j, \zeta_{j+1} \ldots, \zeta_k \mid \beta \rangle \\ = (-1)^{1+|\zeta_j||\zeta_{j+1}|} \langle \zeta_1, \ldots, \zeta_j, \zeta_{j+1}, \ldots, \zeta_k \mid \beta \rangle,\end{aligned} \tag{8.18}$$

for all $\zeta_i \in \mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$ and $f \in C^\infty(U) \otimes \Lambda(n)$. The set of k-forms is denoted by $\Omega^k(U)$.

REMARK 8.1. Actually we have to write $\Omega^k(U, C^\infty(U) \otimes \Lambda(n))$, but we made our choice for the abbreviated $\Omega^k(U)$.

The set $\Omega^k(U)$ has the structure of a right $C^\infty(U)\otimes\Lambda(n)$-module by

$$\langle\zeta_1,\dots,\zeta_k \mid \beta f\rangle = \langle\zeta_1,\dots,\zeta_k \mid \beta\rangle f. \tag{8.19}$$

We also set $\Omega^0(U) = C^\infty(U)\otimes\Lambda(n)$ and $\Omega(U) = \oplus_{k=0}^\infty \Omega^k(U)$. Moreover $\Omega(U)$ can be given a structure of a *bigraded* $(\mathbb{Z}_+,\mathbb{Z}_2)$-commutative algebra, that is, if $\beta_i \in \Omega^{k_i}(U)_{j_i}$, $i=1,2$, then

$$\beta_1\beta_2 \in \Omega^{k_1+k_2}(U)_{j_1+j_2} \tag{8.20}$$

and

$$\beta_1\beta_2 = (-1)^{k_1k_2+j_1j_2}\beta_2\beta_1. \tag{8.21}$$

For the general definition of $\beta_1\beta_2$ see [**50**].

11. One defines the *exterior derivative* (or *de Rham differential*)

$$d\colon \Omega^0(U)\to\Omega^1(U), \quad f\mapsto df, \tag{8.22}$$

by the condition

$$\langle\zeta \mid df\rangle = \zeta f \tag{8.23}$$

for $\zeta\in \mathrm{Der}(C^\infty(U)\otimes\Lambda(n))$ and $f\in\Omega^0(U) = C^\infty(U)\otimes\Lambda(n)$. By [**50**] and the definition of $\beta_1\beta_2$,

$$dr_i, \quad i=1,\dots,m, \qquad ds_j, \quad j=1,\dots,n, \tag{8.24}$$

defined by

$$\begin{aligned} &\Big\langle\frac{\partial}{\partial r_k} \mid dr_i\Big\rangle = \delta_{ik}, && \Big\langle\frac{\partial}{\partial s_j} \mid dr_i\Big\rangle = 0,\\ &\Big\langle\frac{\partial}{\partial r_k} \mid ds_l\Big\rangle = 0, && \Big\langle\frac{\partial}{\partial s_j} \mid ds_l\Big\rangle = \delta_{jl}, \end{aligned} \tag{8.25}$$

generate $\Omega(U)$ and any $\beta\in\Omega(U)$ can be uniquely written as

$$\beta = \sum_{\mu,\nu} dr_\mu ds^\nu f_{\mu,\nu}, \tag{8.26}$$

where

$$\begin{aligned} &\mu = (\mu_1,\dots,\mu_k), \quad 1\le\mu_1\le\dots\le\mu_k\le n, \quad l(\mu)=k,\\ &\nu = (\nu_1,\dots,\nu_n), \quad \nu_i\in\mathbb{N}=\mathbb{Z}_+\setminus\{0\},\\ &|\nu| = \sum_{i=1}^n \nu_i, \quad f_{\mu,\nu}\in C^\infty(U)\otimes\Lambda(n). \end{aligned} \tag{8.27}$$

Note in particular that by (8.21),

$$dr_idr_j = -dr_dr_i, \quad dr_ids_j = -ds_jdr_i, \quad ds_jds_k = ds_kds_j, \tag{8.28}$$

and by consequence

$$\underbrace{ds_j\dots ds_j}_{k\ \text{times}} \neq 0. \tag{8.29}$$

By means of (8.22) and (8.23), the operator $d\colon \Omega^0(U) \to \Omega^1(U)$ has the following explicit representation

$$df = \sum_{i=1}^{m} dr_i \frac{\partial f}{\partial r_i} + \sum_{j=1}^{n} ds_j \frac{\partial f}{\partial s_j}. \tag{8.30}$$

12. Since $\Omega(U)$ is a $(\mathbb{Z}_+, \mathbb{Z}_2)$-bigraded commutative algebra, the algebra $\mathrm{End}(\Omega(U))$ is bigraded too and if $u \in \mathrm{End}(\Omega(U))$ is of bidegree $(b, j) \in (\mathbb{Z}_+, \mathbb{Z}_2)$, then

$$u(\Omega^a(U)_i) \in \Omega^{a+b}(U)_{i+j}. \tag{8.31}$$

Now, an element $u \in \mathrm{End}(\Omega(U))$ of bidegree (b, j) is a *bigraded derivation* of $\Omega(U)$, if for any $\alpha \in \Omega^a(U)_i$, $\beta \in \Omega(U)$ one has the *Leibniz rule*

$$u(\alpha\beta) = u(\alpha)\beta + (-1)^{ab+ij} \alpha u(\beta). \tag{8.32}$$

There exists a *unique* derivation, the *exterior differentiation*,

$$d\colon \Omega(U) \to \Omega(U) \tag{8.33}$$

of bidegree $(1, 0)$, such that $d\big|_{\Omega^0(U)}$ is defined by (8.22), (8.30), and

$$d^2 = 0. \tag{8.34}$$

If $\beta \in \Omega(U)$,

$$\beta = \sum_{\mu,\nu} dr_\mu \, ds^\nu f_{\mu,\nu}, \tag{8.35}$$

then

$$d\beta = \sum_{\mu,\nu} (-1)^{l(\mu)+|\nu|} dr_\mu \, ds^\nu df_{\mu,\nu}. \tag{8.36}$$

Other familiar operations on ordinary manifolds have their counterparts in the graded case too.

13. If $\zeta \in \mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$, *inner differentiation* by ζ, or *contraction* by ζ, i_ζ is defined by

$$\langle \zeta_1, \ldots, \zeta_b \mid \mathrm{i}_\zeta \beta \rangle = (-1)^{|\zeta| \sum_{i=1}^{b} |\zeta_i|} \langle \zeta, \zeta_1, \ldots, \zeta_b \mid \beta \rangle \tag{8.37}$$

for $\zeta, \zeta_1, \ldots, \zeta_b \in \mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$ and $\beta \in \Omega^{b+1}(U)$. Moreover $\mathrm{i}_\zeta\colon \Omega(U) \to \Omega(U)$, $\beta \in \Omega^{b+1}(U)$, $\mathrm{i}_\zeta \beta \in \Omega^b(U)$, is a *derivation* of bidegree $(-1, |\zeta|)$.

Bigraded derivations on $\Omega(U)$ can be shown to constitute a *bigraded Lie algebra* $\mathrm{Der}\,\Omega(U)$ by the following Lie bracket. If $u_1, u_2 \in \mathrm{Der}\,\Omega(U)$ of bidegree (b_i, b_j), $i = 1, 2$, then

$$[u_1, u_2] = u_1 u_2 - (-1)^{b_1 b_2 + j_1 j_2} u_2 u_1 \in \mathrm{Der}\,\Omega(U). \tag{8.38}$$

14. From (8.38) we have that *Lie derivative* by the vector field ζ defined by

$$\mathrm{L}_\zeta = d\mathrm{i}_\zeta + \mathrm{i}_\zeta d \tag{8.39}$$

is a derivation of $\Omega(U)$ of bidegree $(0, |\zeta|)$.

The fact that exterior differentiation d, inner differentiation by ζ, i_ζ, and Lie derivative by ζ, L_ζ are *derivations*, has been used to implement them on the computer system starting from the representation of vector fields and differential forms (8.15) and (8.35).

15. If one has a graded manifold $(U, C^\infty(U) \otimes \Lambda(n))$, the exterior derivative is easy to be represented as an *odd* vector field in the following way

$$d = \sum_{i=1}^{m} dr_i \wedge \frac{\partial}{\partial r_i} + \sum_{j=1}^{n} ds_j \wedge \frac{\partial}{\partial s_j}, \tag{8.40}$$

where now the initial system has been augmented by n *even* variables $ds_1, \ldots, ds_n$ and m *odd* variables $dr_1, \ldots, dr_m$. The implementation of the supercalculus package is based on the theorem proved in [**50**] that locally a supermanifold, or a graded manifold, is represented as $U, C^\infty(U) \otimes \Lambda(n)$, $U \subset \mathbb{R}^n$, from which it is now easy to construct the differential geometric operations.

Suppose we have a supermanifold of dimension $(m|n)$. Local variables are given by $(r, s) = (r_i, s_j)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Associated to these coordinates, we have (dr_i, ds_j), $i = 1, \ldots, m$, $j = 1, \ldots, n$. We have to note that dr_i, $i = 1, \ldots, m$, are *odd* while ds_j, $j = 1, \ldots, n$, are *even*.

So the exterior algebra is

$$C^\infty(\mathbb{R}^m) \otimes \mathbb{R}[ds] \otimes \Lambda(n) \otimes \Lambda(m), \tag{8.41}$$

where in (8.41) a specific element is given by

$$f = \sum ds_1^{k_1} \ldots ds_m^{k_m} dr_{\mu_1} \ldots dr_{\mu_r} f_{k,\mu} \tag{8.42}$$

while in (8.42) $k_i \geq 0$, $i = 1, \ldots, m$, $1 \leq \mu_1 < \cdots < \mu_r \leq n$, while $f_{k,\mu} \in C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$.

2. Classical differential geometry

We shall describe here how classical differential geometric objects are realised in the graded setting of the previous section. We start at a superalgebra A on n even elements, $r_1, \ldots, r_n$, and n odd elements $s_1, \ldots, s_n$, i.e.,

$$A = C^\infty(\mathbb{R}^n) \otimes \Lambda(n), \tag{8.43}$$

where $\Lambda(n)$ is the exterior algebra on n elements, $s_1, \ldots, s_n$. A particular element $f \in A = C^\infty(\mathbb{R}^n) \otimes \Lambda(n)$ is represented as

$$f = \sum_\mu f_\mu s_\mu, \tag{8.44}$$

where μ is a multi-index $\mu \in M_n = \{\mu = (\mu_1, \ldots, \mu_k) \mid \mu_i \in \mathbb{N}, 1 \leq \mu_1 \leq \cdots \leq \mu_k \leq n\}$ and

$$s_\mu = s_{\mu_1} s_{\mu_2} \ldots s_{\mu_k}, \quad f_\mu \in C^\infty(\mathbb{R}^n), \tag{8.45}$$

where we in effect formally assume:

$$s_i = dr_i, \quad i = 1, \ldots, n. \tag{8.46}$$

1. **Functions** are represented as elements of the algebra $A_0 = C^\infty(\mathbb{R}^n)$.
2. **Derivations** of A_0 can be identified with *vector fields*
$$V = V_1 \frac{\partial}{\partial r_1} + \cdots + V_n \frac{\partial}{\partial r_n}, \tag{8.47}$$
where $V_i \in C^\infty(\mathbb{R}^n)$, $i = 1, \ldots, n$.
3. **Differential forms** are just specific elements of A.
4. **Exterior derivative** is a derivation of A which is *odd* and can be represented as the vector field
$$d = dr_1 \frac{\partial}{\partial r_1} + \cdots + dr_n \frac{\partial}{\partial r_n}, \quad dr_i = s_i. \tag{8.48}$$
5. **Contraction** by a V, where V is given by (8.47), can be represented as an *odd* derivation of A by
$$V \lrcorner \alpha = \left(V_1 \frac{\partial}{\partial s_1} + \cdots + V_n \frac{\partial}{\partial s_n} \right)(\alpha). \tag{8.49}$$
6. The **Lie derivative** by V can be easily implemented by the formula
$$\mathrm{L}_V(\alpha) = V \lrcorner d(\alpha) + d(V \lrcorner \alpha). \tag{8.50}$$

3. Overdetermined systems of PDE

In construction of classical and higher symmetries, nonlocal symmetries and deformations or recursion operators, one is always left with an overdetermined system of partial differential equations for a number of so-called generating functions (or sections). The final result is obtained as the general solution to this resulting system.

In Section 3.1 we shall describe how by the procedure which is called here `solve_equation`, written in the symbolic language LISP, one is able to solve the major part of the construction of the general solution of the overdetermined system of partial differential equations resulting from the symmetry condition (2.29) on p. 72 or the deformation condition (6.42) on p. 266.

It should be noted that each specific equation or system of equations arising from mathematical physics has its own specifics, e.g., the sine-Gordon equation is not polynomial but involves the sine function, similar to the Harry Dym equations, where radicals are involved.

In Subsection 3.2 we discuss, as an application, symmmetries of the Burgers equation, while finally in Subection 3.3 we shall devote some words to the polynomial and graded cases.

3.1. General case. Starting at the symmetry condition (2.29), one arrives at an *overdetermined system of homogeneous linear partial differential equations* for the generating functions F_i, $i = 1, \dots, m$. First of all, one notes that in case one deals with a differential equation[1] $\mathcal{E}^k \subset J^k(x, u)$, $x = (x_1, \dots, x_n)$, $u = (u_1, \dots, u_m)$, then the r-th prolongation $\mathcal{E}^{k+r}$ is always *polynomial* with respect to the higher jet variables in the fibre $\mathcal{E}^{k+r} \to \mathcal{E}^k$.

The symmetry condition (2.29) is also polynomial with respect to these variables, cf. Subsection 3.2. So the overdetermined system of partial differential equations can always be splitted with respect to the highest variables leading to a new system of equations.

These equations are stored in the computer system memory as right-hand sides of operators $\mathtt{equ(1)}, \dots, \mathtt{equ(te)}$, where the variable `te` stands for the `Total_Number_of_Equations` involved.

If at a certain stage, the computer system constructs new expressions which have to vanish in order to generate the general solution to the system of equations (for instance, the derivative of an equation is a consequence, which might be easier to solve). These new equations are added to the system as $\mathtt{equ(te+1)}, \dots$ and the value of `te` is adjusted automatically to the new situation.

In the construction of solutions to the system of equations we distinguish between a number of different cases:

1. **<u>CASE A</u>**: A partial differential equation is of a polynomial type in one (or more) of the variables, the functions F_* appearing in this equation are independent of this (or these) variable(s). By consequence, each of the coefficients of the polynomial has to be zero, and the partial differential equation decomposes into some new additional and smaller equations.

   EXAMPLE 8.1. The partal differential equation is

   $$\mathtt{equ(.)} := x_1^2 (F_1)_{x_2} + x_1 F_2, \tag{8.51}$$

   where in (8.51) the functions F_1, F_2 are independent of x_1.

   By consequence, the coefficients of the polynomial in x_1 have to be zero, i.e., $(F_1)_{x_2}$ and F_2. So equation (8.51) is equivalent to the system

   $$\begin{aligned} &\mathtt{equ(.)} := (F_1)_{x_2}, \\ &\mathtt{equ(.)} := F_2 \end{aligned} \tag{8.52}$$

[1] We use the notation $J^k(x, u)$ as a synonim for $J^k(\pi)$, where $\pi\colon (x, u) \mapsto (x)$.

2. **CASE B**: The partial differential equations `equ(.)` represents a derivative of a function F_*. In general

$$\texttt{equ}(.) := (F_*)_{x_{i_1}^{k_1},\dots,x_{i_r}^{k_r}}, \tag{8.53}$$

is a mixed $(k_1 + \cdots + k_r)$-th order derivative.

The general solution of (8.53) is

$$F_* := \sum_{s=1}^{r} \sum_{t=0}^{k_s-1} F_{i_s,t} x_{i_s}^t, \tag{8.54}$$

whereas in (8.54) $F_{i_s,t}$ depends on the same variables as F_*, except for x_{i_s}, $t = 0, \dots, k_s - 1$, $s = 1, \dots, r$.

EXAMPLE 8.2.

$$\texttt{equ}(.) := (F_1)_{x_1,x_2}. \tag{8.55}$$

The general solution to this equation is given by

$$F_1 := F_2 + F_3, \tag{8.56}$$

where F_2 depends on the same variables as F_1, except for x_1, while F_3 depends on the same variables as F_1, except for x_2.

3. **CASE C**: The partial differential equation `equ(.)` contains a function F_*, depending on all variables present as arguments of some other function(s) F_{**}, occuring in this equation, whereas there is no derivative of a function F_* present in the equation.

The partial differential equation can then be solved for the function F_*.

EXAMPLE 8.3.

$$\texttt{equ}(.) := x_1 F_1 + x_2 (F_2)_{x_1}, \tag{8.57}$$

where in (8.57) F_1, F_2 are dependent on x_1, x_2, x_3. The solution is

$$F_1 := (-x_2 (F_2)_{x_1}) / x_1 \tag{8.58}$$

We have to make a remark here. There is a switch in the system that checks for the coefficient for the function F_* to be a number. In case the switch `coefficient_check` is on, `equ(.)` will not be solved. In case the switch `coefficient_check` is off, the result is given as in (8.58).

4. **CASE D**: In the partial differential equation there is a derivative of a function F_* with respect to variables which are not present as argument of any other function F_{**}, while the coefficient of F_* is a number. By the *assumption that* $x_1, \dots, x_n$ *appear as polynomials*, the partial differential equation can be integrated.

EXAMPLE 8.4. Let the partial differential equation be given by

$$\texttt{equ}(.) := (F_1)_{x_3} + x_2 F_2, \tag{8.59}$$

where F_1 depends on x_1, x_2, x_3 and F_2 depends on x_1, x_2.

The solution to (8.59) is

$$F_1 := -x_2 x_3 F_2 + F_3, \tag{8.60}$$

whereas F_3 depends on x_1, x_2 and is *independent* of x_3.

5. **CASE E**: In the partial differential equation a specific variable x_i is present *just once* as argument of some function F_*. By appropriate differentiation, one may arrive at a simple equation, which can be solved.

   Evaluation of the original equation can result in an equation which can be solved too.

   EXAMPLE 8.5.

$$\texttt{equ}(.) := x_2 (F_1)_{x_2, x_3} + x_3 F_2, \tag{8.61}$$

   where F_1 depends on x_1, x_2, x_3 and F_2 depends on x_1, x_2.

   Differentiation with respect to x_3 twice results in

$$\texttt{equ}(.) := x_2 (F_1)_{x_2, x_3^3}. \tag{8.62}$$

   The solution to (8.62) is CASE B:

$$F_1 := F_3 x_3^2 + F_4 x_3 + F_5 + F_6, \tag{8.63}$$

   where F_1, F_4, F_5 are dependent on x_1, x_2, F_6 depends on x_1, x_3.

   Substitution of the result (8.63) into the original equation (8.61) leads to

$$\texttt{equ}(.) := 2 x_2 x_3 (F_3)_{x_2} + x_2 (F_4)_{x_2} + x_3 F_2. \tag{8.64}$$

   Due to CASE A, the procedure `solve_equation` constructs two new equations

$$\begin{aligned} \texttt{equ}(.) &:= 2 x_2 (F_3)_{x_2} + F_2, \\ \texttt{equ}(.) &:= x_2 (F_4)_{x_2} \end{aligned} \tag{8.65}$$

   The complete result of the procedure `solve_equation` will in this case be (8.63) and (8.65).

   Now the procedure `solve_equation` is then useful to solve the last two equations (8.65) constructed before; this last step is not carried through automatically.

   For this case there is a switch "`differentiation`" too, similar to the previous case.

In practical situations, one is able to solve the overdetermined system of partial differential equations, using the methods described in the CASES A, B, C, D, E and some additional considerations, which are specific for the problem at hand.

3.2. The Burgers equation. We shall discuss the construction of higher symmetries of order three of the Burgers equation in order to demonstrate the facilities of the `INTEGRATION` package, in effect the procedure `solve_equation` described in the previous subsection.

The Burgers equation is given by the following partial differential equation

$$u_t = uu_1 + u_2, \tag{8.66}$$

where partial derivatives with respect to x are given by indices $1, 2, \dots$ We start this example by introduction of the vector fields D_x, D_t in the jet bundle where local coordinates are given by x, t, u, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 and a generating function F_1, which is dependent on the jet variables x, t, u, u_1, u_2, u_3.

So the representation of the vector fields D_x, D_t is given by

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + u_4\frac{\partial}{\partial u_3} + u_5\frac{\partial}{\partial u_4} + u_6\frac{\partial}{\partial u_5}\\
&\quad + u_7\frac{\partial}{\partial u_6} + u_8\frac{\partial}{\partial u_7},\\
D_t &= \frac{\partial}{\partial t} + (u_t)\frac{\partial}{\partial u} + (u_t)_1\frac{\partial}{\partial u_1} + (u_t)_2\frac{\partial}{\partial u_2} + (u_t)_3\frac{\partial}{\partial u_3} + (u_t)_4\frac{\partial}{\partial u_4}\\
&\quad + (u_t)_5\frac{\partial}{\partial u_5} + (u_t)_6\frac{\partial}{\partial u_6},
\end{aligned}\tag{8.67}$$

where $(u_t), \dots, (u_t)_6$ are given by

$$\begin{aligned}
(u_t) &= uu_1 + u_2,\\
(u_t)_1 &= uu_2 + u_1^2 + u_3,\\
(u_t)_2 &= uu_3 + 3u_1u_2 + u_4,\\
(u_t)_3 &= uu_4 + 4u_1u_3 + 3u_2^2 + u_5,\\
(u_t)_4 &= uu_5 + 5u_1u_4 + 10u_2u_3 + u_6,\\
(u_t)_5 &= uu_6 + 6u_1u_5 + 15u_2u_4 + 10u_3^2 + u_7,\\
(u_t)_6 &= uu_7 + 7u_1u_6 + 21u_2u_5 + 35u_3u_4 + u_8.
\end{aligned}\tag{8.68}$$

In the remaining part of this section we shall present in effect a computer session and give some comments on the construction and use of the procedure `solve_equation`. We shall stick as close as possible to the real output of the computer system. `Boldtext` will refer to real input to the system, while the rest is just the output on screen.

Now from the symmetry condition (2.29) we obtain

$$2:\ \mathtt{equ(1)} = \mathtt{D_t(F_1)} - \mathtt{D_x(D_xF_1)} - \mathtt{uD_x(F_1)} - \mathtt{u_1F_1}; \tag{8.69}$$

where the solution is to be determined in such a way that the right-hand side in (8.69) has to vanish.

The resulting equation is now given by

$$\begin{aligned}\texttt{equ(1)} = {} & (F_1)_t - 2(F_1)_{u,u_1}u_1u_2 - 2(F_1)_{u,u_2}u_1u_3 - 2(F_1)_{u,u_3}u_1u_4 \\ & - 2(F_1)_{u,x}u_1 - (F_1)_{u^2}u_1^2 - 2(F_1)_{u_1,u_2}u_2u_3 - 2(F_1)_{u_1,u_3}u_2u_4 \\ & - 2(F_1)_{u_1,x}u_2 - (F_1)_{u_1^2}u_2^2 + (F_1)_{u_1}u_1^2 - 2(F_1)_{u_2,u_3}u_3u_4 \\ & - 2(F_1)_{u_2,x}u_3 - (F_1)_{u_2^2}u_3^2 + 3(F_1)_{u_2}u_1u_2 - 2(F_1)_{u_3,x}u_4 - (F_1)_{u_3^2}u_4^2 \\ & + 4(F_1)_{u_3}u_1u_3 + 3(F_1)_{u_3}u_2^2 - (F_1)_{x^2} - (F_1)_x u - F_1u_1\$ \end{aligned} \tag{8.70}$$

The dependency of the function F_1 is stored on a `depl!*` (dependency list):

$$\begin{aligned} & 3 : \texttt{lisp depl!}*; \\ & \quad (((f\ 1)\ u_3\ u_2\ u_1\ u\ t\ x)) \end{aligned} \tag{8.71}$$

`Equ(1)` is an equation, which is a polynomial with respect to the variable u_4, so in order to be 0, its coefficients should be zero.

These coefficients will be detected by the procedure `solve_equation(*)`, i.e., CASE A:

$4 :$ `solve_equation(1);`

`equ(1) breaks into equ(2),...,equ(4) by` u_4, u_5, u_6, u_7, u_8

$5 :$ `print_equations(2,4);`

$\texttt{equ(2)} := -\ (F_1)_{u_3^2}\$$

`Functions occurring :`

$F_1(u_3, u_2, u_1, u, t, x)$

$\texttt{equ(3)} := -\ 2((F_1)_{u,u_3}u_1 + (F_1)_{u_1,u_3}u_2 + (F_1)_{u_2,u_3}u + (F_1)_{u_3,x})\$$

`Functions occurring :`

$F_1(u_3, u_2, u_1, u, t, x)$

$$\begin{aligned}\texttt{equ(4)} := {} & (F_1)_t - 2(F_1)_{u,u_1}u_1u_2 - 2(F_1)_{u,u_2}u_1u_3 - 2(F_1)_{u,x}u_1 \\ & - (F_1)_{u^2}u_1^2 - 2(F_1)_{u_1,u_2}u_2u_3 - 2(F_1)_{u_1,x}u_2 - (F_1)_{u_1^2}u_2^2 + (F_1)_{u_1}u_1^2 \\ & - 2(F_1)_{u_2,x}u_3 - (F_1)_{u_2^2}u_3^2 + 3(F_1)_{u_2}u_1u_2 + 4(F_1)_{u_3}u_1u_3 + 3(F_1)_{u_3}u_2^2 \\ & - (F_1)_{x^2} - (F_1)_x u - F_1u_1\$ \end{aligned}$$

`Functions occurring :`

$$F_1(u_3, u_2, u_1, u, t, x) \tag{8.72}$$

We now are left with a system of three partial differential equations for the function F_1.

`Equ(2)` can now be solved, the result being a polynomial of degree 1 with respect to the variable u_3, while coefficients are functions still dependent on x, t, u, u_1, u_2.

The construction of this solution, as in CASE B, is performed by the procedure `solve_equation(*)` too, i.e.:

$$
\begin{aligned}
&6 : \texttt{solve\_equation}(2);\\
&\quad \texttt{equ}(2) :\ \texttt{Homogeneous integration of } (F_1)_{u_3^2}\\
&7 : \mathtt{F_1} := \mathtt{F_1};\\
&\quad F_1 \ := F_3 u_3 + F_2\$
\end{aligned} \tag{8.73}
$$

Substitution of this result into the third equation leads to:

$$
\begin{aligned}
&8 : \texttt{equ}(3) := \texttt{equ}(3);\\
&\quad \texttt{equ}(3) := -2((F_3)_u u_1 + (F_3)_{u_1} u_2 + (F_3)_{u_2} u_3 + (F_3)_x)\$
\end{aligned} \tag{8.74}
$$

and this equation splits up, since it is a polynomial of degree 1 with respect to u_3, CASE A:

$$
\begin{aligned}
&9 : \texttt{solve\_equation}(3);\\
&\quad \texttt{equ}(3)\ \texttt{breaks into equ}(5), ..., \texttt{equ}(6)\ \texttt{by}\ u_3, u_4, u_5, u_6, u_7, \cdots\\
&10 : \texttt{print\_equations}(5, 6);\\
&\quad \texttt{equ}(5) \ := -2(F_3)_{u_2}\$\\
&\quad \texttt{Functions occurring :}\\
&\quad F_3(u_2, u_1, u, t, x)\\
&\quad \texttt{equ}(6) \ := -2((F_3)_u u_1 + (F_3)_{u_1} u_2 + (F_3)_x)\$\\
&\quad \texttt{Functions occurring :}\\
&\quad F_3(u_2, u_1, u, t, x)
\end{aligned} \tag{8.75}
$$

Now the procedure can be repeated, since $\texttt{equ}(5)$ indicates that F_3 is independent of u_2, in effect a polynomial of degree 0, and $\texttt{equ}(6)$ can be splitted with respect to u_2:

$$
\begin{aligned}
&11 : \texttt{solve\_equation}(5);\\
&\quad \texttt{equ}(5) :\ \texttt{Homogeneous integration of } (F_3)_{u_2}\\
&12 : \texttt{solve\_equation}(6);\\
&\quad \texttt{equ}(6)\ \texttt{breaks into equ}(7), \ldots, \texttt{equ}(8)\ \texttt{by}\ u_2, u_3, u_4, u_5, u_6, \ldots\\
&13 : \texttt{print\_equations}(7, 8);\\
&\quad \texttt{equ}(7) \ := -2(F_4)_{u_1}\$\\
&\quad \texttt{Functions occurring :}\\
&\quad F_4(u_1, u, t, x)\\
&\quad \texttt{equ}(8) \ := -2((F_4)_u u_1 + (F_4)_x)\$\\
&\quad \texttt{Functions occurring :}
\end{aligned}
$$

$$F_4(u_1, u, t, x) \tag{8.76}$$

From equ(7) we have that F_4 is independent of u_1 and combination with equ(8) then results in the fact that F_4 is independent of u and x too:

14 : solve_equation(7);

equ(7) : Homogeneous integration of $(F_4)_{u_1}$

15 : solve_equation(8);

equ(8) breaks into equ(9), ..., equ(10) by $u_1, u_2, u_3, u_4, u_5, \ldots$

16 : print_equations(9, 10);

equ(9) $:= -2(F_5)_u\$$

Functions occurring :

$F_5(u, t, x)$

equ(10) $:= -2(F_5)_x\$$

Functions occurring :

$F_5(u, t, x)$

17 : solve_equation(9);

equ(9) : Homogeneous integration of $(F_5)_u$

18 : solve_equation(10);

equ(10) : Homogeneous integration of $(F_6)_x$ (8.77)

Summarising the results obtained thusfar, we are left with an expression for the function F_1 in terms of F_2 and F_7 and one equation, equ(4), which is polynomial with respect to u_3:

19 : f(1) := f(1);

$F_1 := F_7 u_3 + F_2\$$

20 : print_equations(1, te);

$$\begin{aligned}\text{equ}(4) := {}& (F_7)_t u_3 + (F_2)_t - 2(F_2)_{u,u_1} u_1 u_2 - 2(F_2)_{u,u_2} u_1 u_3 \\ & - 2(F_2)_{u,x} u_1 - (F_2)_{u^2} u_1^2 - 2(F_2)_{u_1,u_2} u_2 u_3 - 2(F_2)_{u_1,x} u_2 - (F_2)_{u_1^2} u_2^2 \\ & + (F_2)_{u_1} u_1^2 - 2(F_2)_{u_2,x} u_3 - (F_2)_{u_2^2} u_3^2 + 3(F_2)_{u_2} u_1 u_2 - (F_2)_{x^2} \\ & - (F_2)_x u + 3F_7 u_1 u_3 + 3F_7 u_2^2 - F_2 u_1\$\end{aligned}$$

Functions occurring :

$F_2(u_2, u_1, u, t, x)$

$F_7(t)$

21 : solve_equation(4);

equ(4) breaks into equ(11), ..., equ(13) by $u_3, u_4, u_5, u_6, u_7, \ldots$

$22:$ print_equations(11,13);

equ(11) $:= -(F_2)_{u_2^2}\$$

Functions occurring :

$F_2(u_2,u_1,u,t,x)$

equ(12) $:= (F_7)_t - 2(F_2)_{u,u_2}u_1 - 2(F_2)_{u_1,u_2}u_2 - 2(F_2)_{u_2,x} + 3F_7u_1\$$

Functions occurring :

$F_2(u_2,u_1,u,t,x)$

$F_7(t)$

equ(13) $:= (F_2)_t - 2(F_2)_{u,u_1}u_1u_2 - 2(F_2)_{u,x}u_1 - (F_2)_{u^2}u_1^2$
$- 2(F_2)_{u_1,x}u_2 - (F_2)_{u_1^2}u_2^2 + (F_2)_{u_1}u_1^2 + 3(F_2)_{u_2}u_1u_2 - (F_2)_{x^2}$
$- (F_2)_xu + 3F_7u_2^2 - F_2u_1\$$

Functions occurring :

$F_7(t)$

$F_2(u_2,u_1,u,t,x)$ (8.78)

The remaining system, equ(11), equ(12), equ(13), can be handled in a similar way as before, leading to an expression for the function F_2:

$23:$ solve_equation(11);

equ(11) : Homogeneous integration of $(F_2)_{u_2^2}$

$24:$ equ(12) := equ(12);

equ(12) $:= -2(F_9)_uu_1 - 2(F_9)_{u_1}u_2 - 2(F_9)_x + (F_7)_t + 3F_7u_1\$$

$25:$ solve_equation(12);

equ(12) breaks into equ(14), ..., equ(15) by $u_2, u_3, u_4, u_5, u_6, \ldots$

$26:$ equ(14);

$- 2(F_9)_{u_1}\$$

$27:$ solve_equation(14);

equ(14) : Homogeneous integration of $(F_9)_{u_1}$

$28:$ equ(15);

$- 2(F_{10})_uu_1 - 2(F_{10})_x + (F_7)_t + 3F_7u_1\$$

$29:$ solve_equation(15);

equ(15) breaks into equ(16), ..., equ(17) by $u_1, u_2, u_3, u_4, u_5, \ldots$

$30:$ print_equations(16,17);

$$\begin{aligned}&\texttt{equ(16)} \ := -2(F_{10})_u + 3F_7\$\\ &\texttt{Functions occurring:}\\ &F_7(t)\\ &F_{10}(u,t,x)\end{aligned} \tag{8.79}$$

and

$$\begin{aligned}&\texttt{equ(17)} \ := -2(F_{10})_x + (F_7)_t\$\\ &\texttt{Functions occurring:}\\ &F_7(t)\\ &F_{10}(u,t,x)\end{aligned}$$

$31:$ `solve_equation(16);`
`CASE C:`
`equ(16): Inhomogeneous integration of` $(F_{10})_u$

$32:$ `solve_equation(17);`
`equ(17): Inhomogeneous integration of` $(F_{11})_x$

$33:$ `f(2) := f(2);`

$$F_2 := ((F_7)_t u_2 x + 2F_{12}u_2 + 2F_8 + 3F_7 u u_2)/2\$ \tag{8.80}$$

while the original defining function F_1, and the remaining equation, `equ(13)`, are given by:

$34:$ `f(1) := f(1);`

$$F_1 := ((F_7)_t u_2 x + 2F_{12}u_2 + 2F_8 + 3F_7 u u_2 + 2F_7 u_3)/2\$$$

$35:$ `print_equations(1,te);`

$$\begin{aligned}&\texttt{equ(13)} := (2(F_{12})_t u_2 + 2(F_8)_t - 4(F_8)_{u,u_1} u_1 u_2 - 4(F_8)_{u,x} u_1\\ &- 2(F_8)_{u^2} u_1^2 - 4(F_8)_{u_1,x} u_2 - 2(F_8)_{u_1^2} u_2^2 + 2(F_8)_{u_1} u_1^2 - 2(F_8)_{x^2}\\ &- 2(F_8)_x u + (F_7)_{t^2} u_2 x + 2(F_7)_t u u_2 + 2(F_7)_t u_1 u_2 x + 4F_{12} u_1 u_2\\ &- 2F_8 u_1 + 6F_7 u u_1 u_2 + 6F_7 u_2^2)/2\$\\ &\texttt{Functions occurring:}\\ &F_7(t)\\ &F_8(u_1,u,t,x)\\ &F_{12}(t)\end{aligned} \tag{8.81}$$

`Equ(13)` is a polynomial with respect to the variable u_2, and the result is again a system of three equations, the first two of them can be solved in exactly the same way as before, leading to an expression for F_8:

$36:$ `solve_equation(13);`
`equ(13) breaks into equ(18),...,equ(20) by` $u_2, u_3, u_4, u_5, u_6, \ldots$

$37:$ `print_equations(18,19);`

$\texttt{equ(18)} := 2(-(F_8)_{u_1,2} + 3F_7)\$$

`Functions occurring :`

$F_7(t)$

$F_8(u_1, u, t, x)$

$\texttt{equ(19)} := 2(F_{12})_t - 4(F_8)_{u,u_1}u_1 - 4(F_8)_{u_1,x} + (F_7)_{t^2}x + 2(F_7)_t u$
$+ 2(F_7)_t u_1 x + 4F_{12}u_1 + 6F_7 u u_1\$$

`Functions occurring :`

$F_7(t)$

$F_8(u_1, u, t, x)$

$F_{12}(t)$

$38:$ `solve_equation(18);`

`equ(18) :  Inhomogeneous integration of` $(F_8)_{u_1^2}$

$39:$ `print_equations(19,19);`

$\texttt{equ(19)} := -4(F_{14})_u u_1 - 4(F_{14})_x + 2(F_{12})_t + (F_7)_{t^2}x + 2(F_7)_t u$
$+ 2(F_7)_t u_1 x + 4F_{12}u_1 + 6F_7 u u_1\$$

`Functions occurring :`

$F_7(t)$

$F_{12}(t)$

$F_{14}(u, t, x)$

$40:$ `solve_equation(19);`

`equ(19) breaks into equ(21),...,equ(22) by` $u_1, u_2, u_3, u_4, u_5, \ldots$

$41:$ `equ(21);`

$2(-2(F_{14})_u + (F_7)_t x + 2F_{12} + 3F_7 u)\$$

$42:$ `solve_equation(21);`

`equ(21) :  Inhomogeneous integration of` $(F_{14})_u$

$43:$ `print_equations(22,22);`

$\texttt{equ(22)} := -4(F_{15})_x + 2(F_{12})_t + (F_7)_{t^2}x\$$

`Functions occurring :`

$F_7(t)$

$F_{12}(t)$

$F_{15}(t, x)$

$44:$ `solve_equation(22);`

`equ(22) : Inhomogeneous integration of` $(F_{15})_x$

$45:$ `f(8) := f(8);`

$$F_8 := (4(F_{12})_t u_1 x + (F_7)_{t^2} u_1 x^2 + 4(F_7)_t u u_1 x + 8F_{16} u_1 + 8F_{13} + 8F_{12} u u_1 + 6F_7 u^2 u_1 + 12F_7 u_1^2)/8\$ \tag{8.82}$$

while F_1 and the remaining equation, `equ(20)`, are given as:

$46:$ `f(1) := f(1);`

$$F_1 := (4(F_{12})_t u_1 x + (F_7)_{t^2} u_1 x^2 + 4(F_7)_t u u_1 x + 4(F_7)_t u_2 x + 8F_{16} u_1 + 8F_{13} + 8F_{12} u u_1 + 8F_{12} u_2 + 6F_7 u^2 u_1 + 12F_7 u u_2 + 12F_7 u_1^2 + 8F_7 u_3)/8\$$$

$47:$ `print_equations(1,te);`

$$\texttt{equ(20)} := (8(F_{16})_t u_1 + 8(F_{13})_t - 16(F_{13})_{u,x} u_1 - 8(F_{13})_{u^2} u_1^2 - 8(F_{13})_{x^2} - 8(F_{13})_x u + 4(F_{12})_{t^2} u_1 x + 4(F_{12})_t u u_1 + (F_7)_{t^3} u_1 x^2 + 2(F_7)_{t^2} u u_1 x - 2(F_7)_{t^2} u_1 + 2(F_7)_t u^2 u_1 + 4(F_7)_t u_1^2 - 8F_{13} u_1)/4\$$$

`Functions occurring :`

$F_7(t)$

$F_{12}(t)$

$F_{13}(u,t,x)$

$F_{16}(t)$ (8.83)

The remaining equation can then be solved in a straightforward way leading to the final result:

$48:$ `solve_equation(20);`

`equ(20) breaks into equ(23),...,equ(25) by` $u_1, u_2, u_3, u_4, u_5, \ldots$

$49:$ `equ(23);`

$$4(-2(F_{13})_{u^2} + (F_7)_t)\$$$

$50:$ `solve_equation(23);`

`equ(23) : Inhomogeneous integration of` $(F_{13})_{u^2}$

$51:$ `f(13) := f(13);`

$$F_{13} := ((F_7)_t u^2 + 4F_{18} u + 4F_{17})/4\$$$

$52:$ `print_equations(24,24);`

$$\texttt{equ(24)} := -16(F_{18})_x + 8(F_{16})_t + 4(F_{12})_{t^2} x + 4(F_{12})_t u + (F_7)_{t^3} x^2 + 2(F_7)_{t^2} u x - 2(F_7)_{t^2} - 8F_{18} u - 8F_{17}\$$$

$$
\begin{aligned}
&\texttt{Functions occurring :}\\
&F_{17}(t,x)\\
&F_7(t)\\
&F_{12}(t)\\
&F_{16}(t)\\
&F_{18}(t,x)\\
53:\ &\texttt{solve\_equation(24);}\\
&\texttt{equ(24) breaks into equ(26)},\ldots,\texttt{equ(27) by } u,u_1,u_2,u_3,u_4,\ldots\\
54:\ &\texttt{print\_equations(26,27);}\\
&\texttt{equ(26)} := 2(2(F_{12})_t + (F_7)_{t^2}x - 4F_{18})\$\\
&\texttt{Functions occurring :}\\
&F_{18}(t,x)\\
&F_7(t)\\
&F_{12}(t)\\
&\texttt{equ(27)} := -16(F_{18})_x + 8(F_{16})_t + 4(F_{12})_{t^2}x + (F_7)_{t^3}x^2\\
&+2(F_7)_{t^2} - 8F_{17}\$\\
&\texttt{Functions occurring :}\\
&F_{17}(t,x)\\
&F_7(t)\\
&F_{12}(t)\\
&F_{16}(t)\\
&F_{18}(t,x)\\
55:\ &\texttt{solve\_equation(26);}\\
&\texttt{equ(26) : Solved for } F_{18}\\
56:\ &\texttt{solve\_equation(27);}\\
&\texttt{equ(27) : Solved for} F_{17}\\
57:\ &\texttt{print\_equations(1,te);}\\
&\texttt{equ(25)} := 8(F_{16})_{t^2} + 4(F_{12})_{t^3}x\\
&+(F_7)_{t^4}x^2 - 8(F_7)_{t^3}\$\\
&\texttt{Functions occurring :}\\
&F_7(t)\\
&F_{12}(t)\\
&F_{16}(t)
\end{aligned}
\tag{8.84}
$$

and

$58:$ `solve_equation(25);`

`equu(25) breaks into equ(28),...,equ(30)` *by* $x, u, u_1, u_2, u_3, \ldots$

$59:$ `print_equations(28,30);`

`equ(28)` $:= (F_7)_{t^4}\$$

`Functions occurring :`

$F_7(t)$

`equ(29)` $:= 4(F_{12})_{t^3}\$$

`Functions occurring :`

$F_{12}(t)$

`equ(30)` $:= 8((F_{16})_{t^2} - (F_7)_{t^3})\$$

`Functions occurring :`

$F_7(t)$

$F_{16}(t)$

$60:$ `solve_equation(28);`

`equ(28) : Homogeneous integration of` $(F_7)_{t^4}$

$61:$ `f(7) := f(7);`

$F_7 := c(4)t^3 + c(3)t^2 + c(2)t + c(1)\$$

$62:$ `solve_equation(29);`

`equ(29) : Homogeneous integration of` $(F_{12})_{t^3}$

$63:$ `f(12) := f(12);`

$F_{12} := c(7)t^2 + c(6)t + c(5)\$$

$64:$ `equ(30) := equ(30);`

`equ(30)` $:= 8(-6c(4) + (F_{16})_{t^2})\$$

$65:$ `solve_equation(30);`

`equ(30) : Inhomogeneous integration of` $(F_{16})_{t^2}$

$66:$ `f(16) := f(16);`

$F_{16} := c(9)t + c(8) + 3c(4)t^2\$$

$67:$ `factor t,x;` (8.85)

and

$68:$ `f(1) := f(1);`

$$\begin{aligned}
&F_1 := (t^3c(4)(3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3)\\
&+ 6t^2xc(4)(uu_1 + u_2)\\
&+ t^2(4c(7)uu_1 + 4c(7)u_2 + 3c(4)u^2 + 12c(4)u_1 + 3c(3)u^2u_1\\
&+ 6c(3)uu_2 + 6c(3)u_1^2 + 4c(3)u_3)\\
&+ 3tx^2c(4)u_1\\
&+ 2tx(2c(7)u_1 + 3c(4)u + 2c(3)uu_1 + 2c(3)u_2)\\
&+ t(4c(9)u_1 + 4c(7)u + 4c(6)uu_1 + 4c(6)u_2 + 6c(4)\\
&+ 2c(3)u^2 + 3c(2)u^2u_1 + 6c(2)uu_2 + 6c(2)u_1^2 + 4c(2)u_3)\\
&+ x^2(3c(4) + c(3)u_1)\\
&+ 2x(2c(7) + c(6)u_1 + c(3)u + c(2)uu_1 + c(2)u_2)\\
&+ 4c(9) + 4c(8)u_1 + 2c(6)u + 4c(5)uu_1 + 4c(5)u_2 - 6c(3)\\
&+ c(2)u^2 + 3c(1)u^2u_1 + 6c(1)uu_2 + 6c(1)u_1^2 + 4c(1)u_3)/4\$
\end{aligned} \tag{8.86}$$

and

$$\begin{aligned}
&69 : \texttt{for i} := \texttt{1 : 9 do write vec(i)} := \texttt{df(f(1), c(i))};\\
&\quad \texttt{vec(1)} := (3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3)/4\$\\
&\quad \texttt{vec(2)} := (t(3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3) + 2x(uu_1 + u_2) + u^2)/4\$\\
&\quad \texttt{vec(3)} := (t^2(3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3)\\
&\quad + 4tx(uu_1 + u_2) + 2tu^2 + x^2u_1 + 2xu - 6)/4\$\\
&\quad \texttt{vec(4)} := (t^3(3u^2u_1 + 6uu_2 + 6u_1^2 + 4u_3)\\
&\quad + 6t^2x(uu_1 + u_2) + 3t^2(u^2 + 4u_1) + 3tx^2u_1 + 6txu + 6t + 3x^2)/4\$\\
&\quad \texttt{vec(5)} := uu_1 + u_2\$\\
&\quad \texttt{vec(6)} := (2t(uu_1 + u_2) + xu_1 + u)/2\$\\
&\quad \texttt{vec(7)} := t^2(uu_1 + u_2) + txu_1 + tu + x\$\\
&\quad \texttt{vec(8)} := u_1\$\\
&\quad \texttt{vec(9)} := tu_1 + 1\$\\
&70 :
\end{aligned} \tag{8.87}$$

The previous application demonstrates in a nice way how calculations concerning symmetries and other invariants of partial differential equations are performed.

We finish this section with the remark that it is possible to run the program automatically on this system (8.66). Doing this, the complete construction does take 0.3 seconds. Most problems need however the researcher as operator in the construction of the general solution.

3.3. Polynomial and graded cases. A very often arising situation is the construction of symmetries and of conservation laws for equations admitting scaling symmetry.

Let us take for example:

EXAMPLE 8.6. The KdV equation is given by:

$$u_t = uu_x + u_{xxx}, \tag{8.88}$$

which as we have seen in Section 5 of Chapter 3 admits a scaling symmetry

$$S = -x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + \cdots \tag{8.89}$$

This means that in physical terms all variables are of appropriate dimensions, whereas in mathematical terms it means that all variables are graded[2], i.e.,

$$\text{degree}(x) \equiv [x] = -1,\ [t] = -3,\ [u] = 2,\ [u_x] = 3,\ [u_t] = 5, \dots. \tag{8.90}$$

This grading means that all objects are graded too, and for the generating functions of symmetries and conservation laws only those functions are of interest which are of a specified degree in the variables.

EXAMPLE 8.7. Suppose that in the previous example we are interested to have the most general functions F and G of degree 5 and 7 respectively, with respect to the graded variables u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx} which are of degree 2, 3, 4, 5, 6, 7 respectively. The result will be:

$$F = c_1u_{xxx} + c_2uu_x, \quad G = c_3u_{xxxxx} + c_4uu_{xxx} + c_5u_xu_{xx} + c_6u^2u_x. \tag{8.91}$$

If, however, we are in the situation that F is of degree 5 with respect to the graded variables p_1, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx} which are of degree 1, 2, 3, 4, 5, 6, 7 respectively, then the result will be:

$$F = c_1u_{xxx} + c_2p_1u_{xx} + (c_3u + c_4p_1^2)u_x + c_5p_1^3u + c_6p_1^5, \tag{8.92}$$

while for G we have the general presentation

$$\begin{aligned} G = {} & c_1u_{xxxxx} + c_2p_1u_{xxxx} + (c_3u + c_4p_1^2)u_{xxx} \\ & + (c_5u_x + c_6p_1u + c_7p_1^3)u_{xx} + c_8p_1u_x^2 \\ & + (c_9u^2 + c_{10}p_1^2u + c_{11}p_1^4)u_x + c_{12}p_1u^3 \\ & + c_{13}p_1^3u^2 + c_{14}p_1^5u + c_{15}p_1^7. \end{aligned} \tag{8.93}$$

[2]The term *graded* here means that some weights can be assigned to all variables in such a way that the equation becomes homogeneous with respect to these weights.

Procedures are availabe to construct the most general presentation of a function of a specified degree, with respect to a specified list of graded variables.

Once one knows that all objects are graded, the conditions (1.37) do lead to *polynomial* equations with respect to the jet variables, the coefficients of which have to vanish. This process does lead to just algebraic linear equations for the constants in the original expressions (8.92) and (8.93).

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